

UNIVERSO PRIMITIVO
Mestrado em Física Astronomia 2026-2027

Exercise Sheet 1

1. In a spatially flat FLRW universe, fundamental observers follow the Hubble flow and have fixed comoving coordinates. Their proper separation is $\mathbf{r}(t) = a(t) \mathbf{x}$, where $a(t)$ is the scale factor and $\mathbf{x}$ is their comoving separation.
 - 1.1. Differentiate this expression to obtain the Hubble law, $\mathbf{v}(t) = H(t) \mathbf{r}(t)$, where $H = \dot{a}/a$.
 - 1.2. Derive the corresponding expression for two non-comoving observers with relative comoving coordinate velocity $\dot{\mathbf{x}}$. Identify their physical peculiar velocity.
2. Consider the mixed-index stress-energy tensor, T^μ_ν , for a homogeneous and isotropic perfect fluid. Take ρ to be its mass-equivalent density.
 - 2.1. Apply the conservation law $\nabla_\mu T^\mu_\nu \equiv T^\mu_{\nu;\mu} = 0$ for $\nu = 0$ to obtain the energy conservation equation $\dot{\rho} = -3H(\rho + p)$, where $H = \dot{a}/a$ is the *Hubble function*.
 - 2.2. Use this equation to show that $dE = -pdV$, where $dE = d(\rho a^3 L^3)$ is the energy inside a volume element, $dV = d(a^3 L^3)$ and L^3 is an arbitrary comoving volume.
 - 2.3. For constant equation of state (EoS) parameter, $w = p/\rho c^2$, integrate the energy conservation equation in 2.1 to show that $\rho(t) = \rho_i \left(\frac{a(t)}{a_i} \right)^{-3(1+w)}$, where ρ_i is the energy density at $a = a_i$.
 - 2.4. Use the expression in 2.3 to derive the time dependence of the scale factor for the following components: radiation ($w = 1/3$); collisionless matter ($w = 0$); and cosmological constant ($w = -1$) assuming the conditions (1), (2) and (3) at the bottom of slide 12 of chapter 2 of the course notes, respectively.
3. Consider the FLRW dynamic equations discussed in class.
 - 3.1. Use the Friedmann and the acceleration equations to derive the energy conservation equation in 2.1.
 - 3.2. Use the definition of the cosmological density parameters to re-write Friedmann equation as follows (the subscript 'm' refers to all forms of matter, i.e. includes baryonic and dark matter):

$$H^2(t) = \frac{8\pi G}{3} (\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

$$= H_0^2 \left[\Omega_{r0} \left(\frac{a_0}{a} \right)^4 + \Omega_{m0} \left(\frac{a_0}{a} \right)^3 + \Omega_{k0} \left(\frac{a_0}{a} \right)^2 + \Omega_{\Lambda 0} \right]$$
 - 3.3. For a concordance model with $\Omega_{r0} = 9.4 \times 10^{-5}$, $\Omega_{m0} = 0.32$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0.68$, estimate the redshifts of radiation–matter and matter–dark-energy equality epochs.
4. Use the Friedmann equation in 3.2 to calculate the Age of the universe for:
 - 4.1. A critical density universe ($\Omega_{r0} = 0$, $\Omega_{m0} = 1$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0$), with $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$
 - 4.2. A flat Λ universe with $\Omega_{r0} \simeq 0$, $\Omega_{m0} = 0.32$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0.68$, and $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$
[Hint: integrate the Friedmann equation with respect to the scale factor].