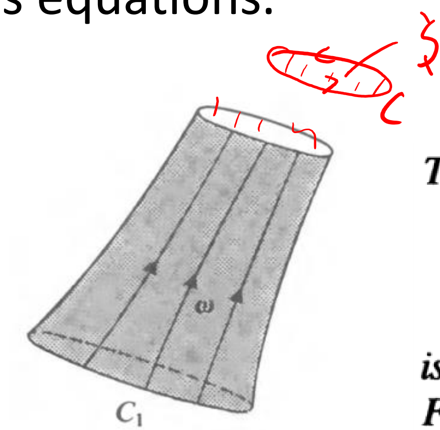
# Vorticity and lines of vorticity

$$\nabla \cdot (\underbrace{\nabla \times \vec{V}}_{\omega}) = 0$$

- Since $\vec{\Omega} = \nabla \times \vec{V}$ its divergence is zero, i.e. $\boxed{\nabla \cdot \vec{\Omega} = 0}$.

- The vorticity is a solenoidal field with lines of vorticity (like streamlines) parallel to its direction and density proportional to its magnitude.
- Dynamics of the lines of vorticity differs in the Euler and Navier-Stokes equations.




*The quantity*

$$\Gamma = \int_S \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_c \vec{u} \cdot d\vec{l} \quad (5.6)$$

*is the same for all cross-sections $S$ of a vortex tube. Furthermore, $\Gamma$ is independent of time.*

# Euler fluid

Chap. 5, Acheson

Euler equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \chi\right)$$

Identity

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} + \nabla(\tfrac{1}{2}\boldsymbol{u}^2)$$

$$\nabla \times \left[ \frac{\partial \boldsymbol{u}}{\partial t} + \underbrace{(\nabla \wedge \boldsymbol{u})}_{\vec{\omega}} \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right) \right]$$

Application of the curl

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \wedge (\boldsymbol{\omega} \wedge \boldsymbol{u}) = 0$$

Identity

$$\nabla \wedge (\boldsymbol{F} \wedge \boldsymbol{G}) = (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + \boldsymbol{F}(\nabla \cdot \boldsymbol{G}) - \boldsymbol{G}(\nabla \cdot \boldsymbol{F})$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}\underbrace{\nabla \cdot \boldsymbol{u}}_{=0} - \boldsymbol{u}\underbrace{\nabla \cdot \boldsymbol{\omega}}_{\nabla \cdot \nabla \times (\ldots) = 0} = 0$$

We have

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}}_{\frac{D\vec{\omega}}{Dt}} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$$

Vorticity equation

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}}$$

$$\omega_x \frac{\partial}{\partial x}\hat{x} + \omega_y \frac{\partial}{\partial y}\hat{y} + \omega_z \frac{\partial}{\partial z}\hat{z}$$

If the flow is 2D

$$\boldsymbol{u} = [u(x, y, t), v(x, y, t), 0]$$

$$\boldsymbol{\omega} = (0, 0, \omega)$$

$u_z = cte$

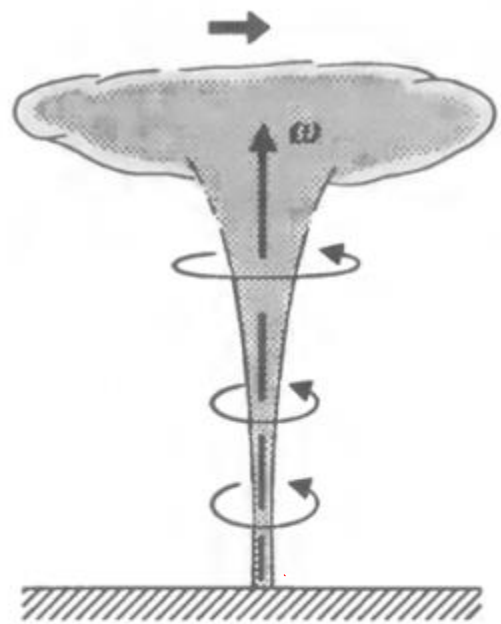
Then

$$(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \omega \frac{\partial \boldsymbol{u}}{\partial z} = 0$$

It follows that

$$\frac{D\omega}{Dt} = 0$$

*In the two-dimensional flow of an ideal fluid subject to a conservative body force* **g** *the vorticity ω of each individual fluid element is conserved.*




Vortex lines in the z-direction

$$\boldsymbol{\omega} \doteq \omega \boldsymbol{k}$$

Z-component:

$$\frac{D\boldsymbol{\omega}}{Dt} \doteq \omega \frac{\partial \boldsymbol{u}}{\partial z}$$

$$\frac{D\omega}{Dt} \doteq \omega \frac{\partial w}{\partial z}$$

$u_z(z+\Delta z)$

$\neq$

$u_z(z)$

The vorticity of a fluid element increases with time if $\partial w/\partial z > 0$ .

If the fluid elements are being stretched in the z-direction, it leads to an intensification of the local vorticity field.

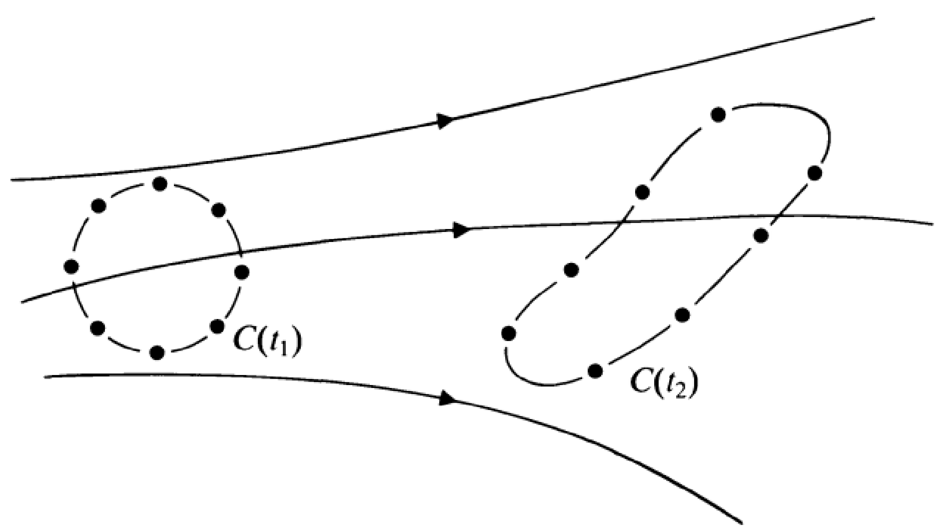
Kelvin circulation theorem

THEOREM. *Let an inviscid, incompressible fluid of constant density be in motion in the presence of a conservative body force* $\mathbf{g} = -\nabla\chi$ *per unit mass. Let* $C(t)$ *denote a closed circuit that consists of the same fluid particles as time proceeds* (*Fig*. 5.1). *Then the circulation*

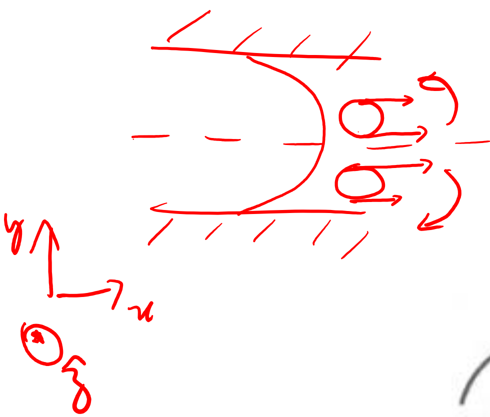
$$\Gamma = \int_{C(t)} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \tag{5.1}$$

*round* $C(t)$ *is independent of time.*

$$\frac{D\Gamma}{Dt} = 0$$



The inviscid equations of motion enter the proof only in helping to evaluate a line integral round $C$, so if viscous forces happened to be important elsewhere in the flow, i.e. off the curve $C$, this would not affect the conclusion that $\Gamma$ remains constant round $C$.




# Navier-Stokes: Viscosity drives the diffusion of vorticity

Faber, chap. 10

To find what difference viscosity makes, we need to repeat the above analysis using the Navier–Stokes equation as our starting point, rather than the Euler equation. The viscous term on the left-hand side of (6.25) is $-\eta\boldsymbol{\nabla}\wedge\boldsymbol{\Omega}$, and the curl of this, since $\boldsymbol{\nabla}\cdot\boldsymbol{\Omega}=0$, is $\eta\nabla^2\boldsymbol{\Omega}$. Hence we now have

$$\frac{\mathrm{D}\boldsymbol{\Omega}}{\mathrm{D}t}=(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}+\underbrace{\frac{\eta}{\rho}}_{=\nu}\nabla^2\boldsymbol{\Omega}.$$

Apart from the $(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ term, the effects of which are as described above, this is just a three-dimensional diffusion equation for each of the components of $\boldsymbol{\Omega}$; to be more precise, it becomes a three-dimensional equation in the co-moving frame for which $\mathrm{D}\boldsymbol{\Omega}/\mathrm{D}t$ and $\partial\boldsymbol{\Omega}/\partial t$ are the same. Thus vorticity is not permanently embedded if the fluid has viscosity; where $\nabla^2\boldsymbol{\Omega}$ is non-zero it spreads by diffusion, and its *diffusivity* is the kinematic viscosity, $\nu=\eta/\rho$. Since the process described by the diffusion equation always conserves the thing which is diffusing, whether it be dye or heat or whatever, the fact that vorticity is liable to diffuse does not affect our conclusion that lines of vorticity are conserved.

Diff. eq.

$$\frac{\partial\phi}{\partial t}=D\nabla^2\phi$$

# Example: Poiseuille flow

If, however, the vorticity is positive in region A and negative in an adjoining region B, diffusion from A to B and *vice versa* is bound to result in some degree of cancellation. The lines of vorticity in such situations tend to form closed loops which disappear by collapsing to a point. For example, consider the simple case of a fluid undergoing Poiseuille flow along a straight cylindrical pipe whose axis is the $x_3$ axis. In the plane $x_2 = 0$, say, $\Omega_1$ and $\Omega_3$ both vanish, while

$$\Omega_2 = -\frac{\partial u_3}{\partial x_1} = -\frac{x_1}{2\eta}\nabla_3 p$$

Thus $\boldsymbol{\Omega}$ changes sign on the axis in the plane $x_2 = 0$, and it also does so in the plane $x_1 = 0$ where $\Omega_1$ is the non-vanishing component; evidently the lines of vorticity are closed circular loops coaxial with the pipe. Now the direction in which the lines of vorticity diffuse is determined by the sign of $\partial\Omega/\partial r$. Because this is positive we should picture the loops as diffusing inwards, to smaller values of radius $r$, and ultimately collapsing on the axis. We should therefore picture the surface of the fluid, where it is in contact with the solid wall of the pipe, as a vorticity source at which new loops are continuously created to replace those which collapse.