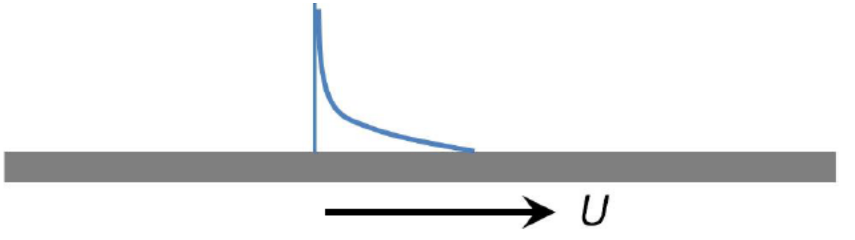
Sudden motion of an infinite flat plane (revisited)

Flow above a solid wall at y = 0. Initially, the fuid is at rest. At time t = 0, the boundary starts to move with velocity U in the x direction.



The velocity field is

$$\mathbf{u} = (u(y,t), 0, 0).$$

and the Navier-Stokes equation

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u},$$

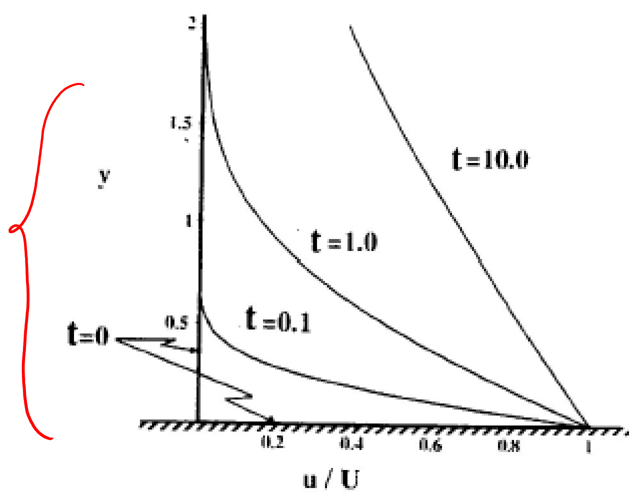
reduces to

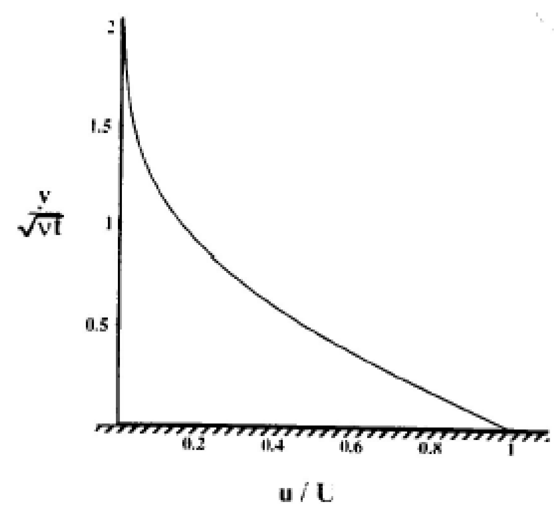
$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2},$$

- Boundary conditions: u = U on y = 0 and U → 0 as y → ∞.
- We also impose the initial condition: u = 0 at t = 0.
- The velocity u(x, t) thus satisfies the 1-D diffusion equation with diffusivity $\upsilon = \frac{\mu}{\varrho}$, where υ is the kinematic viscosity.
- Similarity solution is

$$u(y,t) = U\left[1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right].$$

The velocity $u(y,t)$ will be approximately zero wherever $y/2\sqrt{\nu t}$ is large. In addition, for a fixed value of y, the velocity will remain less than $0.01U$ until a time t such that $y \approx 4\sqrt{\nu t}$. Hence, at time t, the fluid is only moving within a narrow region of thickness $4\sqrt{\nu t}$. This narrow region is called the *viscous boundary layer*. Note that the boundary layer thickness is independent of U.





SOLUTION OF THE 1D DIFFUSION EQUATION

We seek a *similarity solution*:

$$u(y,t) = f(\eta), \text{ where } \eta = yt^a,$$

for some constant a. Using the chain rule:

$$\frac{\partial}{\partial y} = t^a \frac{d}{d\eta},$$
$$\frac{\partial}{\partial t} = ayt^{a-1}\frac{d}{d\eta},$$

so that equation (4.1) becomes:

$$ayt^{a-1}\frac{df}{d\eta} = \nu t^{2a}\frac{d^2 f}{d\eta^2},$$

and therefore:

$$\frac{d^2 f}{d\eta^2} - \frac{ayt^{-a-1}}{\nu}\frac{df}{d\eta} = 0.$$

For the similarity solution to exist, this equation must only contain y and t in the combination $\eta = yt^a$ and therefore $-a-1 = a$. We get: $a = -\frac{1}{2}$. Solutions thus exists for the similarity variable $\eta = y/\sqrt{t}$ and satisfy:

$$\frac{d^2 f}{d\eta^2} + \frac{\eta}{2\nu}\frac{df}{d\eta} = 0.$$

Substituting $v = df/d\eta$ we have:

$$\frac{dv}{d\eta} = -\frac{\eta}{2\nu}v,$$

which has general solution:

$$v = \frac{df}{d\eta} = A\exp\left(-\frac{\eta^2}{4\nu}\right).$$

Integrating again, we obtain:

$$f = A\int_0^\eta \exp\left(-\frac{\eta^2}{4\nu}\right)d\eta + B.$$

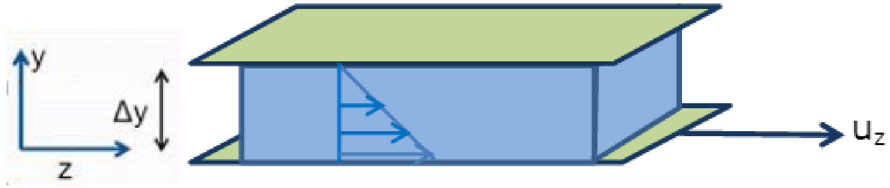
The above integral can be expressed in terms of the error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-x^2\right)dx.$$

Substituting $x = \eta/2\sqrt{\nu}$, we have:

$$f = A\sqrt{\nu\pi}\,\text{erf}\left(\frac{\eta}{2\sqrt{\nu}}\right) + B.$$

Start up of shear flow (parallel plates)



Let us now modify the previous problem by considering the start-up of a shear flow between two parallel plates located at $y = 0$ and $y = h$. Once again, we begin to move the lower plate with velocity U at $t = 0$. The problem is the same as that above except that the boundary condition at infinity is replaced by one at $y = h$. The velocity now satisfies:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{4.3}$$

together with the boundary conditions: $u(0,t) = U$ and $u(h,t) = 0$, and the initial condition $u(y,0) = 0$.

First, we observe that the steady solution $u_s = U\,(1 - y/h)$ satisfies the equation at any $t \neq 0$ and the boundary conditions. We then write:

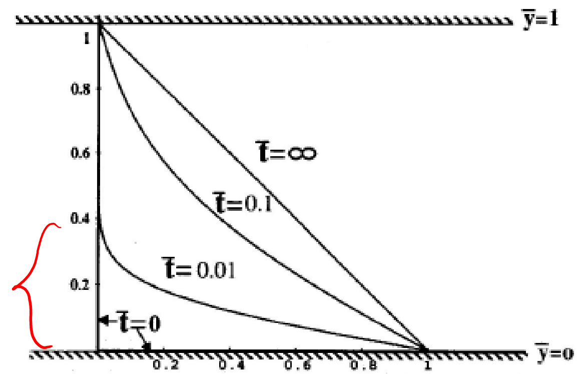
$$u(y,t) = u_s + v(y,t),$$

and seek a separable solution of the form:

$$v(y,t) = T(t)Y(y).$$

Hence, the solution is:

$$u(y,t) = U\left(1-\frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$



This flow resembles that of the unbounded plate until the boundary layer grows to the width of the channel. The solution then approaches the steady state u_s. Note that the slowest decaying exponential in the sum corresponds to $n = 1$. As a result, the flow reaches u_s on a time of order $h^2/(\nu\pi^2)$. For water in a 1cm channel, this time is about 10s and scales inversely with ν so that in a fluid of lower viscosity it becomes longer.

SOLUTION OF START UP OF SHEAR FLOW

This gives:

$$YT' = \nu T Y'',$$

so that:

$$\frac{Y''}{Y} = \frac{1}{\nu}\frac{T'}{T} = k,$$

where k is the constant of integration. Since u_s takes care of the moving boundary, we want to find solutions satisfying $Y(0) = Y(h) = 0$. We thus choose solutions of the form:

$$Y(y) = \sin\left(\frac{n\pi y}{h}\right),$$

so that:

$$\frac{Y''}{Y} = -\frac{n^2\pi^2}{h^2}.$$

It follows:

$$\frac{T'}{T} = -\frac{\nu n^2\pi^2}{h^2},$$

and so we have separable solutions of the form:

$$v_n = \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The general solution for u satisfying the boundary conditions is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) + \sum_{n=1}^{\infty} a_n \exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

The initial condition at $t = 0$ requires:

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{h}\right) = -U\left(1 - \frac{y}{h}\right),$$

for $0 < y < h$. We can determine the a_n using Fourier series properties:

$$a_n = \frac{2U}{h}\int_0^h \left(\frac{y}{h} - 1\right)\sin\left(\frac{n\pi y}{h}\right) dy = -\frac{2U}{n\pi},$$

Hence, the solution is:

$$u(y,t) = U\left(1 - \frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\nu n^2\pi^2}{h^2}t\right)\sin\left(\frac{n\pi y}{h}\right).$$

Diffusion of vorticity from the surface to the fluid

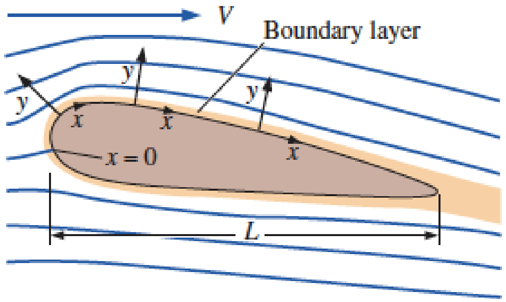
Let us now return to the case of the flow above a boundary that is set in motion at time $t = 0$. Initially, the vorticity is zero everywhere, except at $y = 0$ where the fluid velocity jumps from U to 0. At time t, the velocity is given by equation (4.2). The vorticity ω reads:

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi \nu t}} \exp\left(-\frac{y^2}{4\nu t}\right).$$

This is a Gaussian distribution of standard deviation $\sqrt{2\nu t}$. Hence, as times increases, the vorticity gradually spreads away from the boundary over a distance of order $\sqrt{2\nu t}$.

Boundary layer equations

- We consider steady, two-dimensional flow in the xy-plane in Cartesian coordinates. The methodology can be extended to axisymmetric boundary layers or to three-dimensional boundary layers in any coordinate system.
- We neglect gravity since we are not dealing with free surfaces or with buoyancy-driven flows (free convection flows), where gravitational effects dominate.
- We consider laminar boundary layers; turbulent boundary layer equations are beyond the scope of this course.
- For a boundary layer along a solid wall, we adopt a coordinate system in which x is everywhere parallel to the wall and y is everywhere normal to the wall.
- When we solve the boundary layer equations, we do so at one x-location at a time, using this coordinate system locally, and it is locally orthogonal.

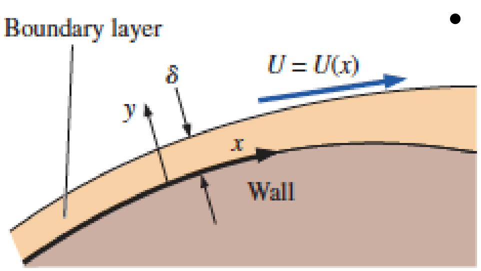


The nondimensionalized Navier–Stokes equation is

$$(\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[\mathrm{Eu}]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*$$

- The Euler number is of order 1, since pressure differences outside the boundary layer are determined by the Bernoulli equation and $\Delta P \sim \rho V^2$.
- V is a characteristic velocity of the outer flow, typically the free-stream velocity for bodies immersed in a uniform flow.
- The characteristic length is L, some characteristic size of the body. For boundary layers, x is of order o L, and Reynolds number is Re_x, usuallly very high.

Redo the nondimensionalization of the equations based on appropriate scales within the boundary layer.



- Since x ~ L, we use L as the scale for distances in the streamwise direction and for derivatives with respect to x. However, this scale is too large for derivatives with respect to y. We use δ for distances in the direction normal to the streamwise direction and for derivatives with respect to y.

- Similarly, we use U as the characteristic velocity, where U is the magnitude of the velocity component parallel to the wall at a location just above the boundary layer. U is in general a function of x.

- Thus, within the boundary layer at some value of x, the orders of magnitude are

$$u \sim U \qquad P - P_\infty \sim \rho U^2 \qquad \frac{\partial}{\partial x} \sim \frac{1}{L} \qquad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

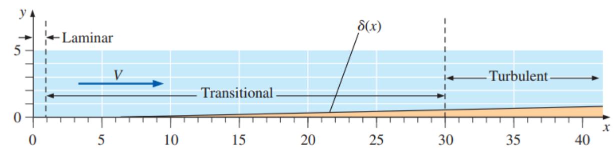
- The order of magnitude of velocity component v is obtained from the continuity equation

$\nabla \cdot \vec{u} = 0$

$$\underbrace{\frac{\partial u}{\partial x}}_{\sim U/L} + \underbrace{\frac{\partial v}{\partial y}}_{\sim v/\delta} = 0 \quad \rightarrow \quad \frac{U}{L} \sim \frac{v}{\delta}$$

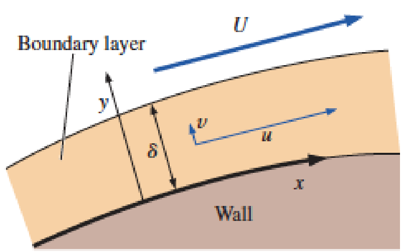
- Since the two terms have to balance each other, they must be of the same order of magnitude. Thus we obtain the order of magnitude of velocity component v**,**

$$v \sim \frac{U\delta}{L}$$



On scale

- Since δ/L << 1 in a boundary layer, we conclude that v << u, and the adimensional variables are



$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{\delta} \qquad u^* = \frac{u}{U} \qquad v^* = \frac{vL}{U\delta} \qquad P^* = \frac{P - P_\infty}{\rho U^2}$$

We now consider the x- and y-components of the Navier–Stokes equation. We substitute these nondimensional variables into the y-momentum equation, giving

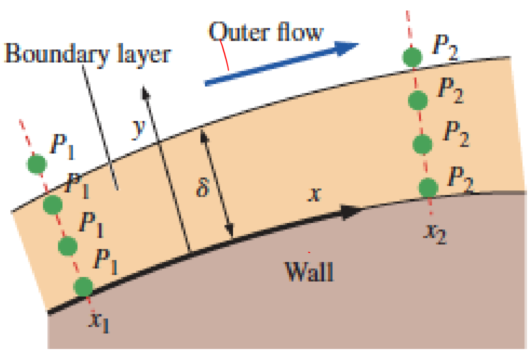
$$\underbrace{u}_{u^*U}\underbrace{\frac{\partial v}{\partial x}}_{\frac{\partial}{\partial x^*}\frac{v^*U\delta}{L^2}} + \underbrace{v}_{v^*\frac{U\delta}{L}}\underbrace{\frac{\partial v}{\partial y}}_{\frac{\partial}{\partial y^*}\frac{v^*U\delta}{L\delta}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial y}}_{\frac{1}{\rho}\frac{\partial}{\partial y^*}\frac{P^*\rho U^2}{\delta}} + \underbrace{\nu\frac{\partial^2 v}{\partial x^2}}_{\nu\frac{\partial^2}{\partial x^{*2}}\frac{v^*U\delta}{L^3}} + \underbrace{\nu\frac{\partial^2 v}{\partial y^2}}_{\nu\frac{\partial^2}{\partial y^{*2}}\frac{v^*U\delta}{L\delta^2}}$$

After some algebra

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\left(\frac{L}{\delta}\right)^2\frac{\partial P^*}{\partial y^*} + \left(\frac{\nu}{UL}\right)\frac{\partial^2 v^*}{\partial x^{*2}} + \left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v^*}{\partial y^{*2}}$$

$\gg 1$ $= Re^{-1} \ll 1$ $\gg 1$ $Re^{-1} \ll 1$

The middle term on the r.h.s. is clearly smaller than any other term since $Re_L = UL/\upsilon \gg 1$. For the same reason, the last term on the right is much smaller than the first term on the right. Neglecting these two terms leaves the two terms on the left and the first term on the right. However, since $L \gg \delta$, the pressure gradient is orders of magnitude greater than the advective terms on the left of the equation. Thus, the only term left is the pressure term. Since no other term in the equation can balance that term, we have no choice but to set it to zero. Thus, the nondimensional y-momentum equation is



$$\frac{\partial P^*}{\partial y^*} \cong 0$$

The pressure across a boundary layer (y-direction) is nearly constant.

Since P is not a function of y, we replace ∂P/∂x by dP/dx, where P is the pressure calculated from the outer flow approximation (using either continuity plus Euler, or the potential flow equations plus Bernoulli). The x-component of the Navier–Stokes equation becomes

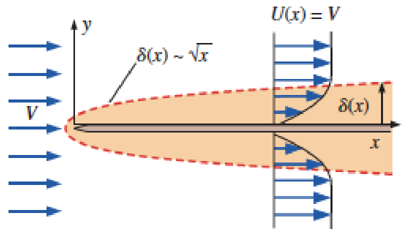
$$\underbrace{u}_{u^*U}\underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial}{\partial x^*}\frac{u^*U}{L}} + \underbrace{v}_{v^*\frac{U\delta}{L}}\underbrace{\frac{\partial u}{\partial y}}_{\frac{\partial}{\partial y^*}\frac{u^*U}{\delta}} = \underbrace{-\frac{1}{\rho}\frac{dP}{dx}}_{\frac{1}{\rho}\frac{\partial}{\partial x^*}\frac{P^*\rho U^2}{L}} + \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\nu\frac{\partial^2}{\partial x^{*2}}\frac{u^*U}{L^2}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\nu\frac{\partial^2}{\partial y^{*2}}\frac{u^*U}{\delta^2}}$$

or

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{dP^*}{dx^*} + \underbrace{\left(\frac{\nu}{UL}\right)}_{Re^{-1}}\frac{\partial^2 u^*}{\partial x^{*2}} + \underbrace{\left(\frac{\nu}{UL}\right)}_{Re^{-1}}\underbrace{\left(\frac{L}{\delta}\right)^2}_{\gg 1}\frac{\partial^2 u^*}{\partial y^{*2}}$$

The middle term on the right side is orders of magnitude smaller than the terms on the left. What about the last term on the right? If we neglect this term, we throw out all the viscous terms and are back to the Euler equation. Clearly this term must remain. Furthermore, since all the remaining terms are of order unity, the combination of parameters in parentheses in the last term on the right side must also be of order 1,

$$\left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2 \sim 1 \qquad\qquad \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}}$$



x-momentum boundary layer equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$