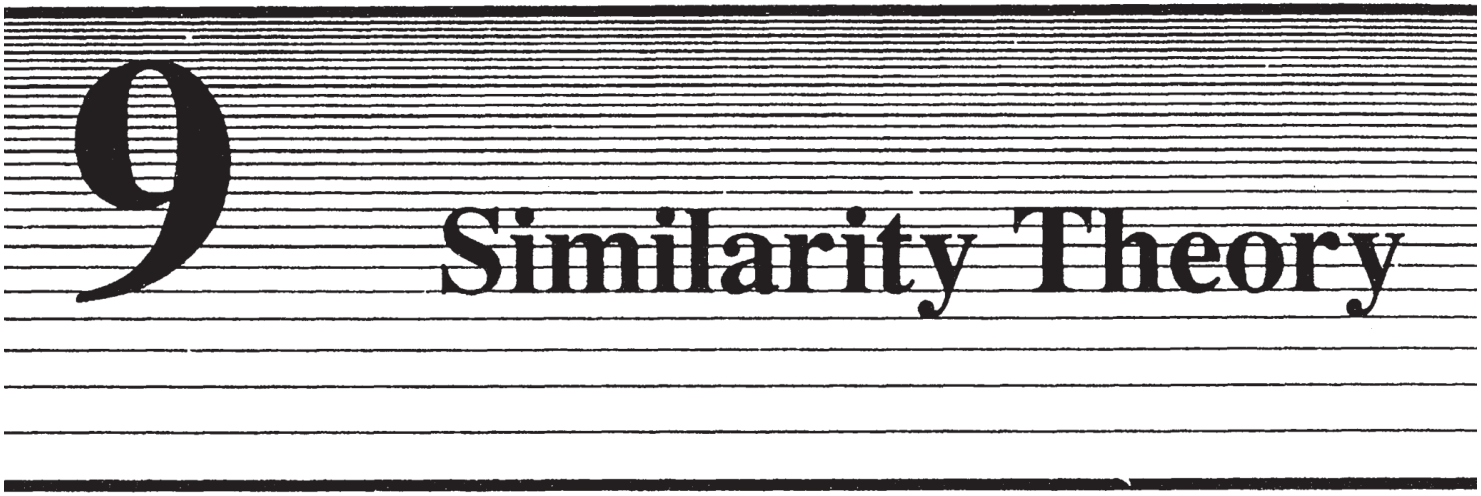
CLP

Lecture 8



Homework: no lecture next week

9.1 – Sara

9.2 – Cátia

9.4 – Diogo

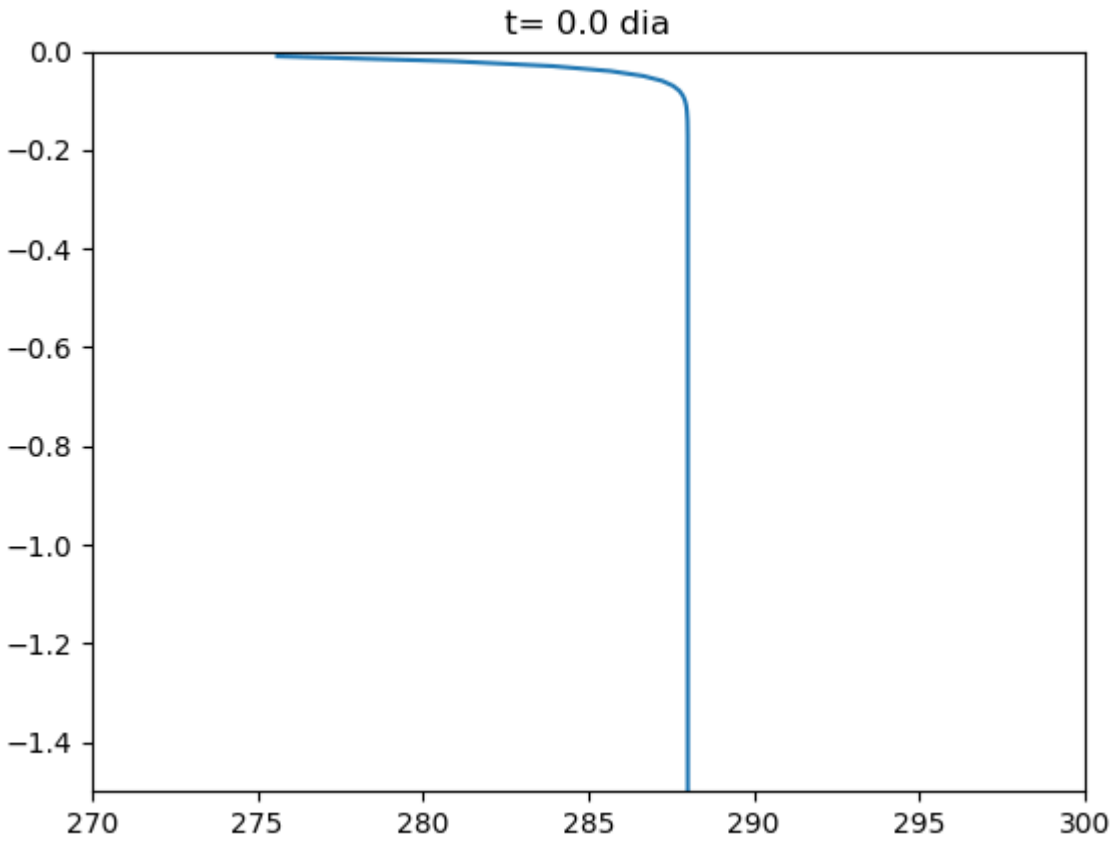
9.6 – Florian

9.7 – Jason

9.9 – Maria

9.16 – Mariana

Heat conduction
in the soil



$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot (-\chi \nabla T) + \dot{q}_V$$

Without internal heat sources, the unidimensional Fourier law is:

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial}{\partial z}\left(-\chi \frac{\partial T}{\partial z}\right)$$

Representing the evolution of temperature in a profile of soil, with thermal conductivity χ, given the initial and spatial boundary conditions ($at\ z = [0, L_z]$).

The solution requires numerical methods.

The simplest approach does not work…

Making $\chi = const, \lambda = \frac{\chi}{\rho c_p}$ (thermal diffusivity), the explicit solution

$$\frac{T_k^{n+1} - T_k^n}{\Delta t} = \lambda \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta z^2}$$

(where T_k^{n+1} is the temperature in position **k**, instant **n+1**)

Is numerically unstable: its error grows exponentially in time…

We can rewrite in a way that imposes the mean value theorem

$$\frac{T_k^{n+1} - T_k^n}{\Delta t} = \left(\frac{T_{k-1}^{n+1/2} + T_{k+1}^{n+1/2} - 2T_k^{n+1/2}}{\Delta z^2} \right)$$
$$= \lambda \left((1-\alpha) \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta z^2} + \alpha \frac{T_{k-1}^{n+1} + T_{k+1}^{n+1} - 2T_k^{n+1}}{\Delta z^2} \right)$$

with $\alpha = \frac{1}{2}$ (if $\alpha = 0$ we go back to the explicit method. Rewriting:

$$-\alpha \frac{\lambda \Delta t}{\Delta z^2} T_{k-1}^{n+1} + \left(1 + \frac{\alpha \lambda \Delta t}{\Delta z^2} \right) T_k^{n+1} - \alpha \frac{\lambda \Delta t}{\Delta z^2} T_{k+1}^{n+1}$$
$$= T_k^n + (1-\alpha) \Delta t \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta z^2}$$

or

$$M \vec{T}^{n+1} = \vec{B}^n$$

$$M\,\vec{T}^{n+1} = \vec{B}^{n}$$

This is an iterative algorithm to compute the distribution of temperature in successive time steps.

If $\alpha = 0$, the system is explcit, an unstable.

If $\alpha = 1$, the system is fully implicit.

If $\alpha = \frac{1}{2}$, if imposes the mean value theorem, it is semi-implicit, and is named as the Crank-Nicholson method.

Spatial boundary conditions

$$-\alpha\frac{\lambda\Delta t}{\Delta z^2}T_{k-1}^{n+1}+\left(1+\frac{2\alpha\lambda\Delta t}{\Delta z^2}\right)T_k^{n+1}-\alpha\frac{\lambda\Delta t}{\Delta z^2}T_{k+1}^{n+1}$$
$$=T_k^n+(1-\alpha)\Delta t\frac{T_{k-1}^n-2T_k^n+T_{k+1}^n}{\Delta z^2}$$

Can only be applied in interior grid points: $k\in[1,N_z-2]$

At the boundaries $(T_0^{n+1},T_{N_z-1}^{n+1})$ temperature is computed according with the boundary conditions: a la Dirichlet:

$$T_0^{n+1}=T_b\big((n+1)\Delta t\big)$$

or a la von Neumann:

$$T_{N_z-1}^{n+1}=T_{N_{z-2}}^{n+1}+\left(\frac{\partial T_b}{\partial z}\right)\Delta z$$

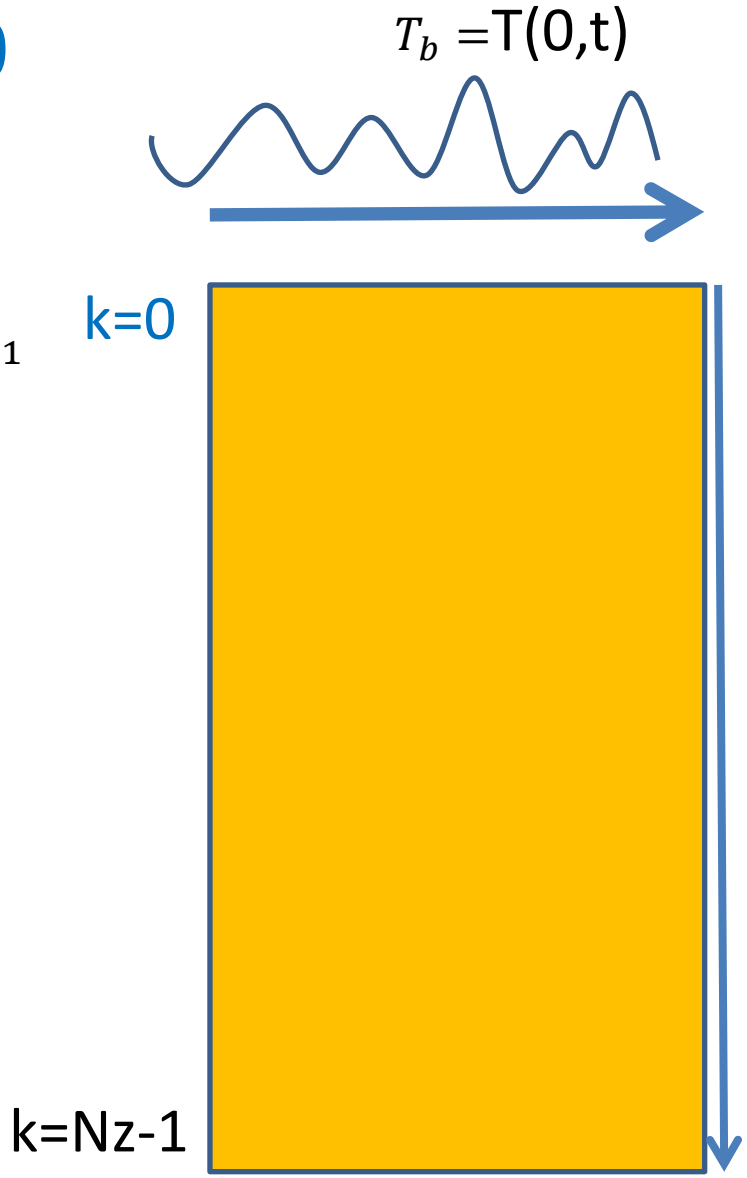
Boundary condition at $z = 0$

$k = 0$: Dirichlet (T specified)

$$-\alpha \frac{\lambda \Delta t}{\Delta z^2} T_b^{n+1} + \left(1 + \frac{\alpha \lambda \Delta t}{\Delta z^2}\right) T_0^{n+1} - \alpha \frac{\lambda \Delta t}{\Delta z^2} T_1^{n+1}$$
$$= T_0^n + (1-\alpha)\lambda \Delta t \frac{T_b^n - 2T_0^n + T_1^n}{\Delta z^2}$$

Or

$$\left(1 + \frac{\alpha \lambda \Delta t}{\Delta z^2}\right) T_0^{n+1} - \alpha \frac{\lambda \Delta t}{\Delta z^2} T_1^{n+1}$$
$$= T_0^n + (1-\alpha)\Delta t \lambda \frac{T_b^n - 2T_0^n + T_1^n}{\Delta z^2}$$
$$+ \alpha \frac{\lambda \Delta t}{\Delta z^2} T_b^{(n+1)}$$



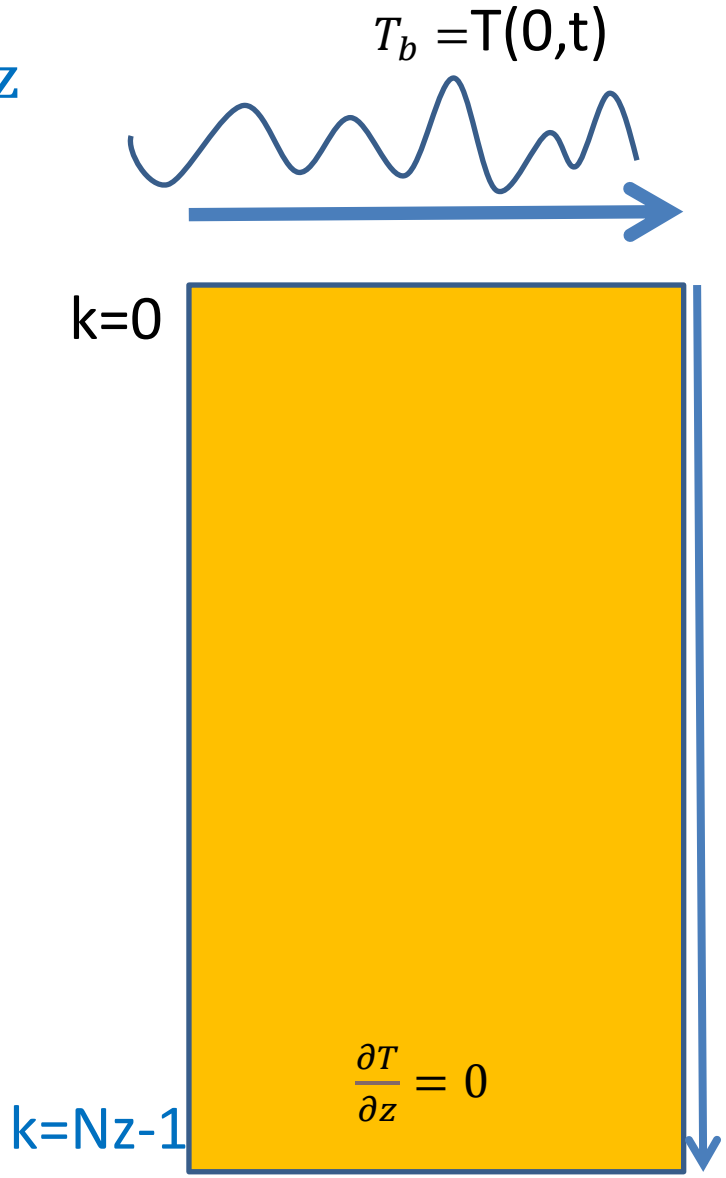
Boundary condition at $z = L_z$

$k = N_z - 1$, NO HEAT FLUX: $\frac{\partial T}{\partial z} = 0$ (von Neumann):

$$-\alpha \frac{\lambda \Delta t}{\Delta z^2} T_{N_z-2}^{n+1} + \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta z^2}\right) T_{N_z-1}^{n+1} - \alpha \frac{\lambda \Delta t}{\Delta z^2} T_{N_z-1}^{n+1} = T_{N_z-1}^{n} + (1-\alpha)\lambda\Delta t \frac{T_{N_z-2}^{n} - 2T_{N_z-1}^{n} + T_{N_z-1}^{n}}{\Delta z^2}$$

Or

$$-\alpha \frac{\lambda \Delta t}{\Delta z^2} T_{N_z-2}^{n+1} + \left(1 + \frac{\alpha\lambda\Delta t}{\Delta z^2}\right) T_{N_z-1}^{n+1} = T_{N_z-1}^{n} + (1-\alpha)\lambda\Delta t \frac{T_{N_z-2}^{n} - T_{N_z-1}^{n}}{\Delta z^2}$$



$$M\vec{x} = \vec{b}$$

$$\begin{bmatrix} \left(1+\frac{2\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & 0 & 0 & 0 & \cdots & 0 \\ \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(1+\frac{2\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & & & & & & \\ 0 & 0 & & & & \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(1+\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) \end{bmatrix} \begin{bmatrix} T_0^{n+1} \\ T_1^{n+1} \\ \\ \\ \\ \\ \\ \\ \\ T_{N_z-2}^{n+1} \\ T_{N_z-1}^{n+1} \end{bmatrix} = \vec{b}$$

Tridiagonal matrix.

$$M\vec{x} = \vec{b}$$

$$M\vec{x} = \begin{bmatrix} T_0^n + \frac{(1-\alpha)\lambda\Delta t}{\Delta z^2}(T_b^n - 2T_0^n + T_1^n) + \frac{\alpha\lambda\Delta t}{\Delta z^2}T_b^{n+1} \\ T_k^n + \frac{(1-\alpha)\lambda\Delta t}{\Delta z^2}(T_{k-1}^n - 2T_k^n + T_{k+1}^n) \\ T_{N_z-1}^n + \frac{(1-\alpha)\lambda\Delta t}{\Delta z^2}(T_{N_z-2}^n - T_{N_z-1}^n) \end{bmatrix}$$

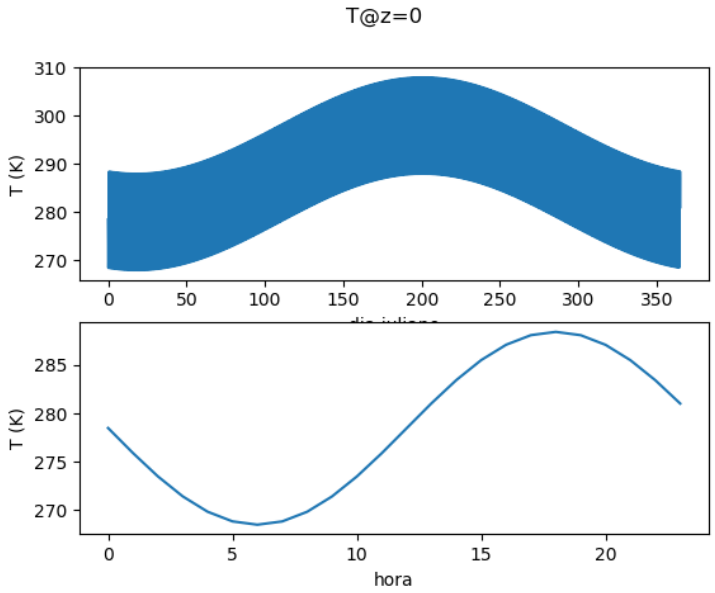
$$M\vec{T}^{n+1} = \vec{b}^{n}$$

The implicit solution is stable for all Δt, but accuracy will be better for small Δt.

Crank-Nicholson $(\alpha = 0.5)$ is more accurate.

Note a problem in the implicit method: information propagates throughout the domain instantaneously…

Python




```
import numpy as np
import matplotlib.pyplot as plt
alpha=0.5 #Crank-Nicholson
Nz=500;Lz=5.;dz=Lz/Nz; #dz=1cm
z=np.arange(dz,Lz,dz)
TimeSpan=365*24*3600. #1 ano
dt=3600.;tempo=np.arange(0.,TimeSpa
ddia=24*3600;dano=365*ddia
lam=0.25/1600/890 #difusividade térmica
Tmed=288;AmpD=10;AmpA=10
lev=np.array([0,2,4,9,19,29,39,59],int);nlev=len(lev)
Tz=np.ones((nt,nlev))*Tmed
T0=Tmed+AmpD*np.sin(2*np.pi*tempo/ddia+np.pi)\
   +AmpA*np.sin(2*np.pi*tempo/dano-np.pi*3./5);
plt.subplot(2,1,1);plt.plot(tempo/3600/24,T0)
plt.ylabel('T (K)');plt.xlabel('dia juliano')
plt.subplot(2,1,2);plt.plot(tempo[:24]/3600,T0[:24])
plt.ylabel('T (K)');plt.xlabel('hora')
plt.suptitle('T@z=0')
plt.figure()
```

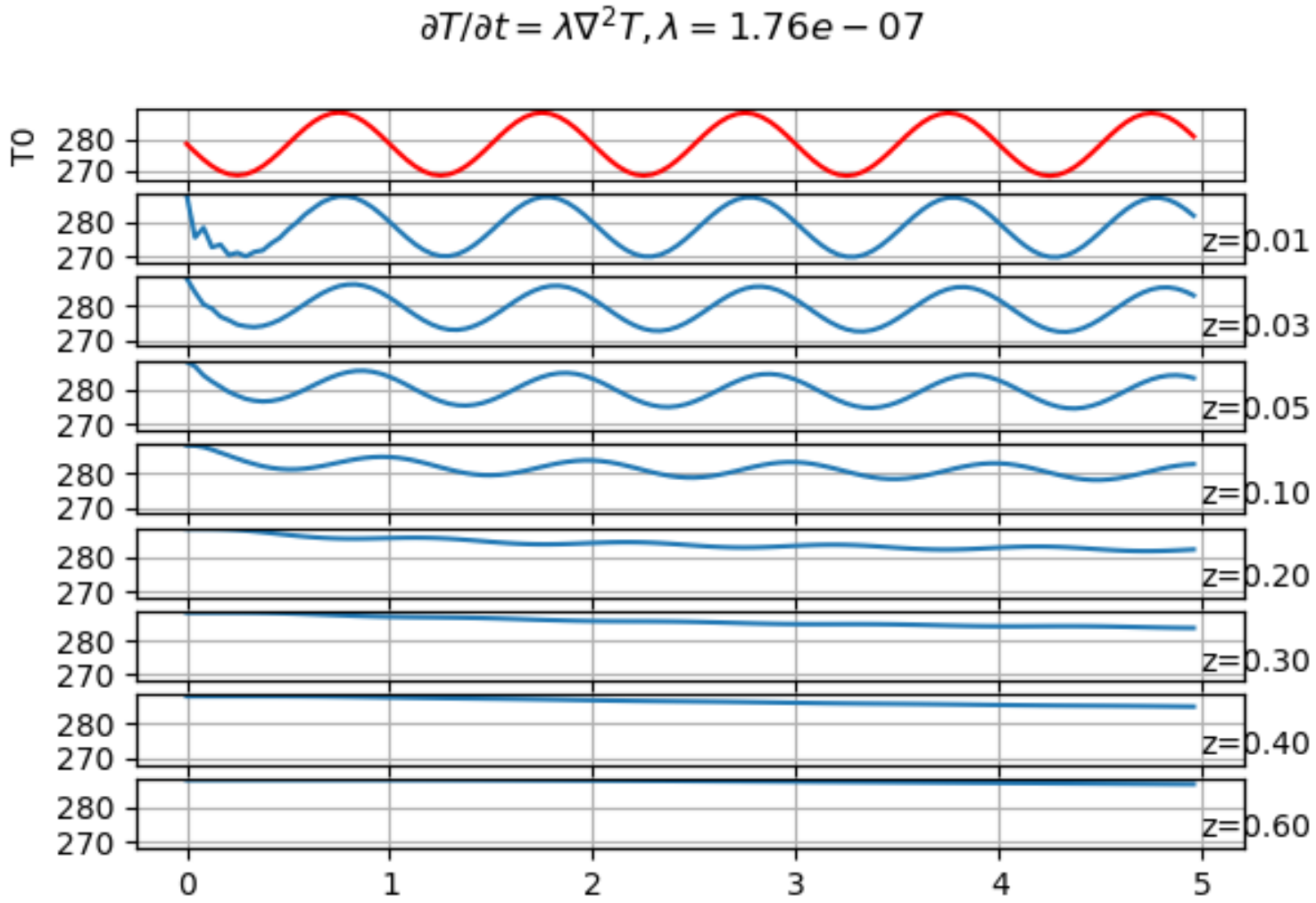
Cálculos preliminares: a matriz M é constante!

```
Tmin=np.min(T0);Tmax=np.max(T0);
zmin=-np.max(z);zmax=0;
T=Tmed*np.ones((Nz)) #perfil inicial de T
beta=alpha*lam*dt/dz**2
zeta=(1-alpha)*lam*dt/dz**2
M=np.zeros((Nz,Nz),float);b=np.zeros((Nz),float)
M[0,0]=1+2*beta;M[0,1]=-beta
for k in range(1,Nz-1):
    M[k,k-1]=-beta
    M[k,k]=1+2*beta
    M[k,k+1]=-beta
M[Nz-1,Nz-2]=-beta
M[Nz-1,Nz-1]=1+beta
Minv=np.linalg.inv(M)
```

Integration

```
for it in range(1,nt):
    b[0]=T[0]+zeta*(T0[it-1]-2*T[0]+T[1])+beta*T0[it]
    for iz in range(1,Nz-1):
        b[iz]=T[iz]+zeta*(T[iz-1]-2*T[iz]+T[iz+1])
    b[Nz-1]=T[Nz-1]+zeta*(T[Nz-2]-T[Nz-1])
    T=np.matmul(Minv,b)
    for klev in range(nlev):
        Tz[it,klev]=T[lev[klev]]
plt.subplot(nlev+2,1,1)
tempoh=tempo/3600/24;plt.plot(tempoh,T0,color='red')
plt.grid();plt.ylabel('T0')
for klev in range(nlev):
    ax=plt.subplot(nlev+2,1,klev+2)
    plt.plot(tempoh,Tz[:,klev]); plt.grid()
    ax2=ax.twinx(); ax2.set_yticks([])
    ax2.set_ylabel('z=%3.2f' % (z[lev[klev]]),rotation=0)
plt.suptitle(r'$\partial T /\partial t = \lambda \nabla^2 T,
\lambda=%4.2e$' %(lam))
```

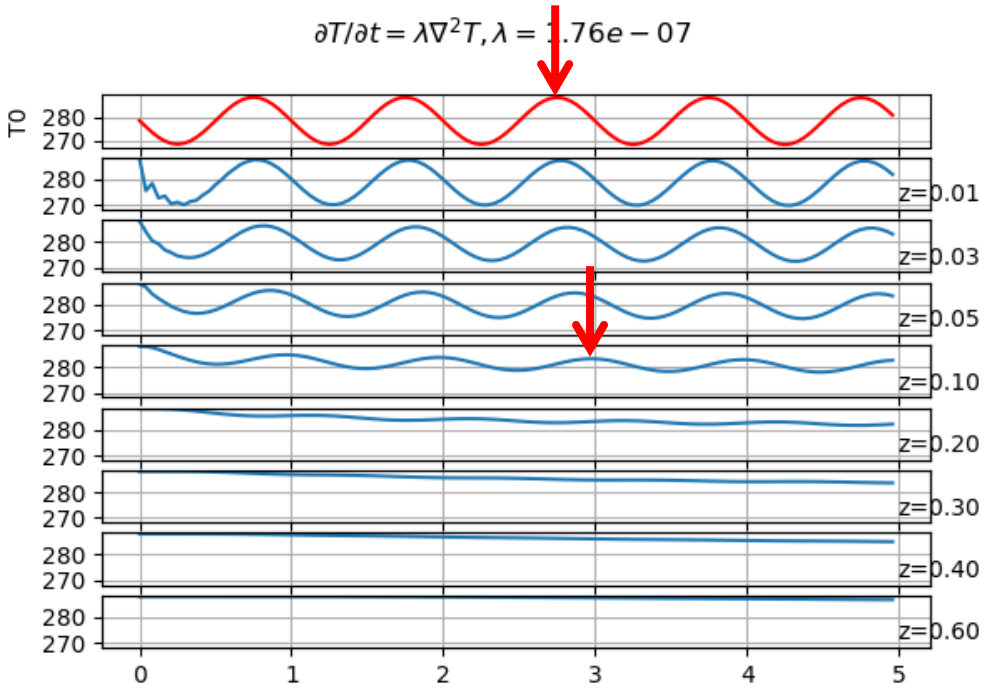
Diurnal cycle



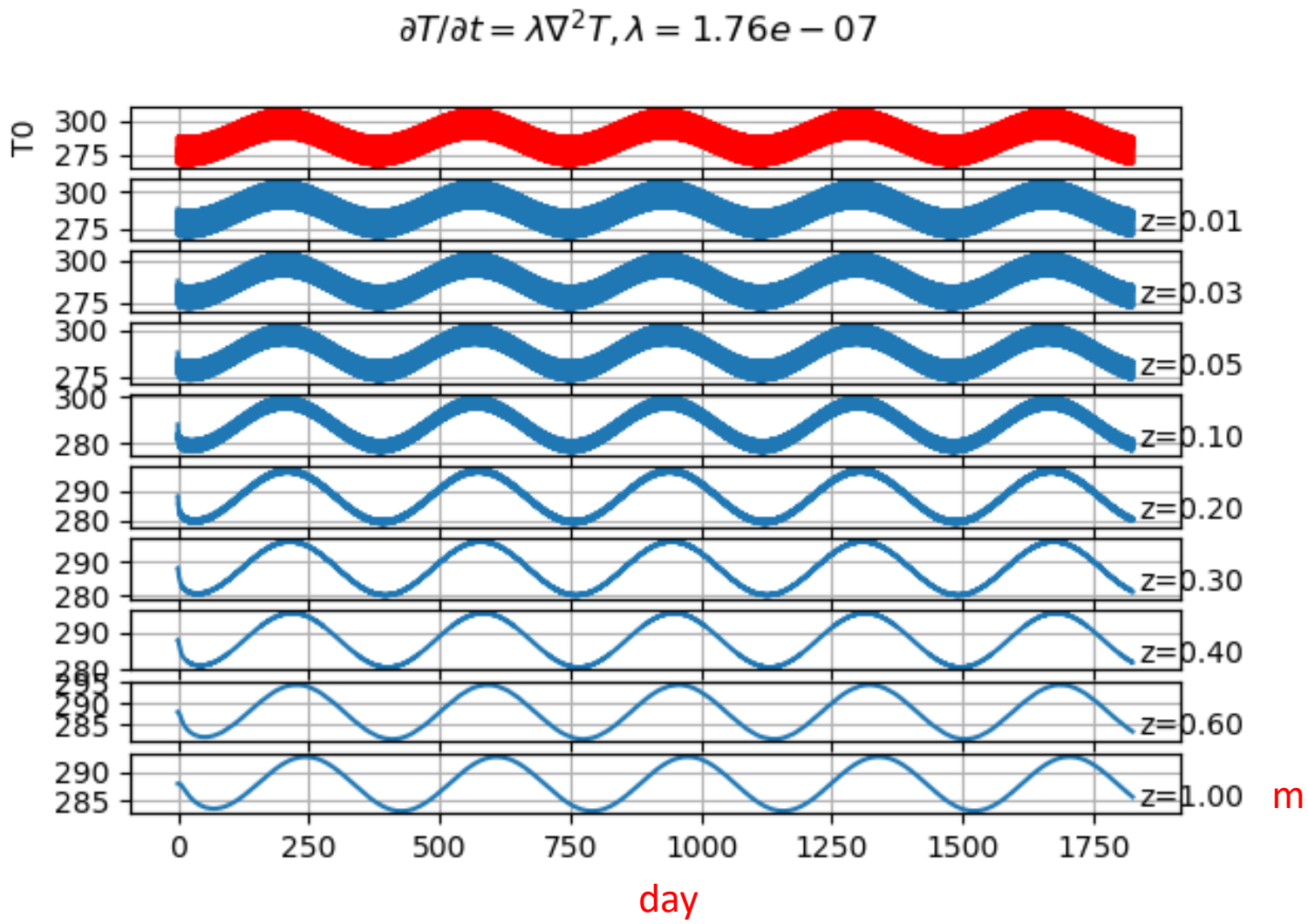
At the surface (z=1cm) the soil temperature is close the the air's.

In depth the cycle has less amplitude and lags in phase.

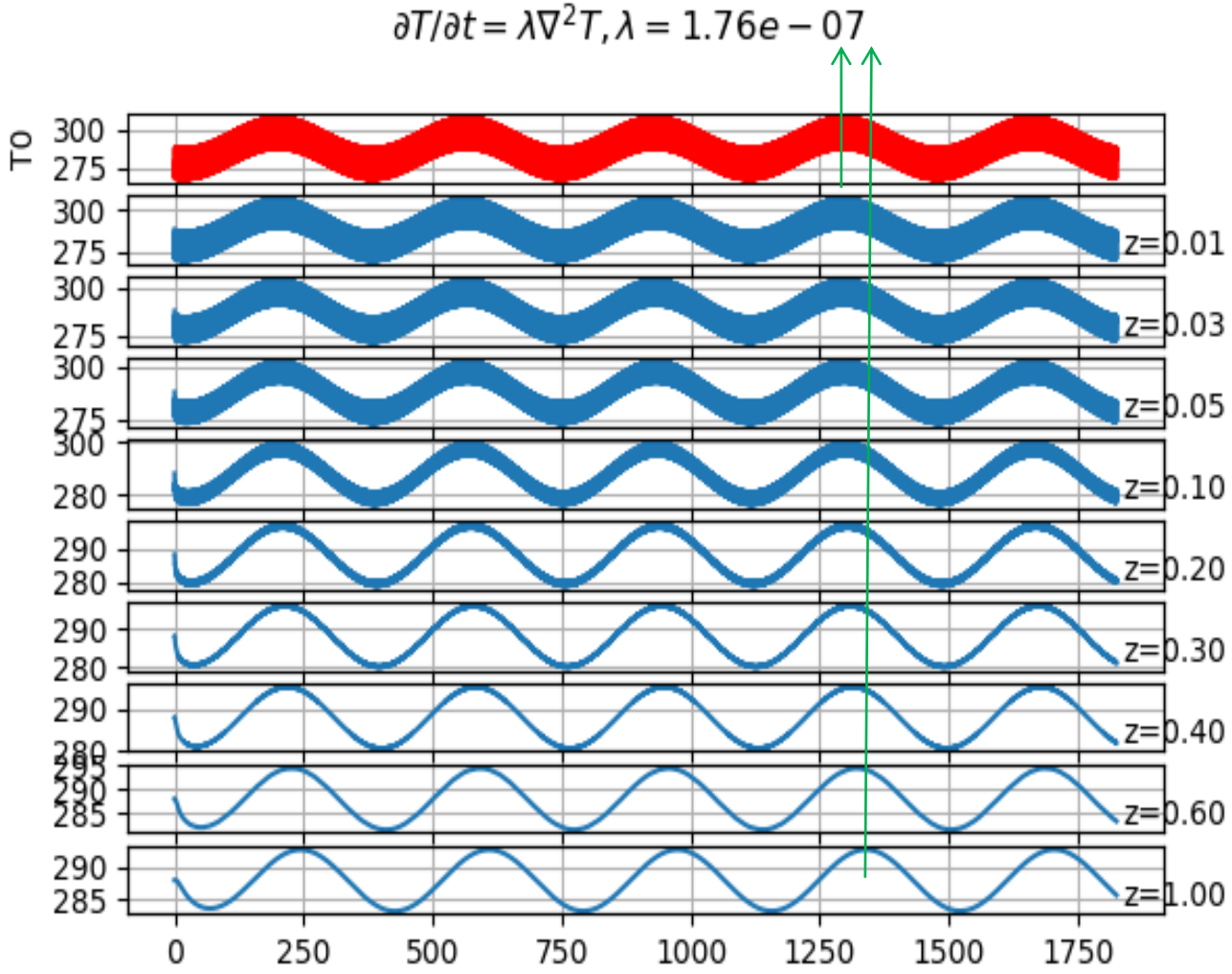
Note that the temperature in depth is still drifting because it started with an unbalanced initial state.



5 years


$\partial T/\partial t = \lambda \nabla^2 T, \lambda = 1.76e-07$
T0
z=0.01
z=0.03
z=0.05
z=0.10
z=0.20
z=0.30
z=0.40
z=0.60
z=1.00
m
0
250
500
750
1000
1250
1500
1750
day

In depth we only see an annual cycle with a phase lag



Similarity theory

For a number of boundary layer situations, our knowledge of the governing physics is insufficient to derive laws based on first principles. Nevertheless, boundary layer observations frequently show consistent and repeatable characteristics, suggesting that we could develop empirical relationships for the variables of interest.

Similarity theory provides a way to organize and group the variables to our maximum advantage, and in turn provides guidelines on how to design experiments to gain the most information.

The mathematical basis of similarity theory: dimensional analysis

The equations of Physics must be independent of the system of units.

In other words, they must be invariant under a change in such system.

This imposes some restrictions on the equations, and allows us to get some generic results when we cannot do better.

This invariance reminds us of relativity theory which is also based in a requirement of invariance of the laws under a change of the coordinate frame.

Buckingham PI theorem

Similarity theory is based on the organization of variables into ***dimensionless groups***. Fortunately, there is a ***dimensional-analysis*** procedure called ***Buckingham Pi theory*** that aids us in forming dimensionless groups from selected variables. It is hoped that the proper choice of groups will allow empirical relationships between these groups that are "***universal***" — namely, that work everywhere all the time for the situation studied.

The four steps in developing a similarity theory are:

(1) select (guess) which variables are relevant to the situation,

(2) organize the variables into dimensionless groups,

(3) perform an experiment, or gather the relevant data from previous experiments, to determine the values of the dimensionless groups,

(4) fit an empirical curve or regress an equation to the data in order to describe the relationship between groups.

The result of this four-step process is an empirical equation or a set of curves which show the same shape — in other words the curves look self similar. Hence, the name ***similarity theory***. If this empirical result is indeed universal, then we can use it on days and locations other than those of the experiment itself. Such expectations should be tested with an independent data set, before the results are disseminated to the rest of the scientific community.

If we selected in step (1) more variables than were necessary, the data will "tell us" of our mistake by indicating no change of the other dimensionless groups with respect to the group that is irrelevant. If we selected too few variables, or excluded an important variable, the data will also indicate our error by showing a large scatter or no repeatable patterns between the dimensionless groups.

Similarity theory does not tell us the form of the equation or the relationship between the dimensionless groups. Instead, we must use trial and error, physical insight, or automated techniques to select the form that qualitatively "looks the best". For example, we might express one group as a power law function of another group, as a logarithmic relationship, or as a constant that is not a function of other groups. The chosen equation usually contains unknown coefficients, which can then be solved by regression against the observed data.

Similarity relationships are usually designed to apply to ***equilibrium*** (steady-state) situations. They are frequently used to yield equilibrium profiles of mean variables and turbulence statistics as a function of height or position. Rarely is time included as one of the relevant variables. Some variables, such as depth of the boundary layer, are so strongly dependent on time that no successful similarity expressions have been found to diagnose them. Instead, boundary layer depth must be calculated or measured using other techniques. This depth is used as input into dimensionless groups to diagnose other variables that do reach a quasi-steady state.

Problem. Find a similarity relationship for the buoyancy flux, $\overline{w'\theta_v'}$, as a function of height in the convective mixed layer.

Solution. First (step 1), guess the relevant variables. Based on the problem statement, we already know that two of the variables of interest are $\overline{w'\theta_v'}$ and z. The depth of the mixed layer, z_i, and the strength of the heat flux near the surface, $\overline{w'\theta_v'}_s$, might also influence the flux within the interior of the mixed layer. Thus, we will use four variables for this analysis.

Step (2), group these four variables into dimensionless groups. By inspection, we can easily produce two dimensionless groups: $[z/z_i]$ and $[\overline{w'\theta_v'} / \overline{w'\theta_v'}_s]$. We have thus reduced our degrees of freedom from four to two.

In performing our experiment for step (3), dimensional analysis tells us that we need not measure all combinations of z, z_i, $\overline{w'\theta_v'}$, and $\overline{w'\theta_v'}_s$. Instead, we need only measure various combinations of the two groups: $[z/z_i]$ and $[\overline{w'\theta_v'} / \overline{w'\theta_v'}_s]$. This greatly simplifies the design and conduct of our experiment.

Similarity:

$$\frac{\overline{w'\theta_v'}}{\overline{w'\theta_v'}_s} = 1 - 1.2\left[\frac{z}{z_i}\right]$$

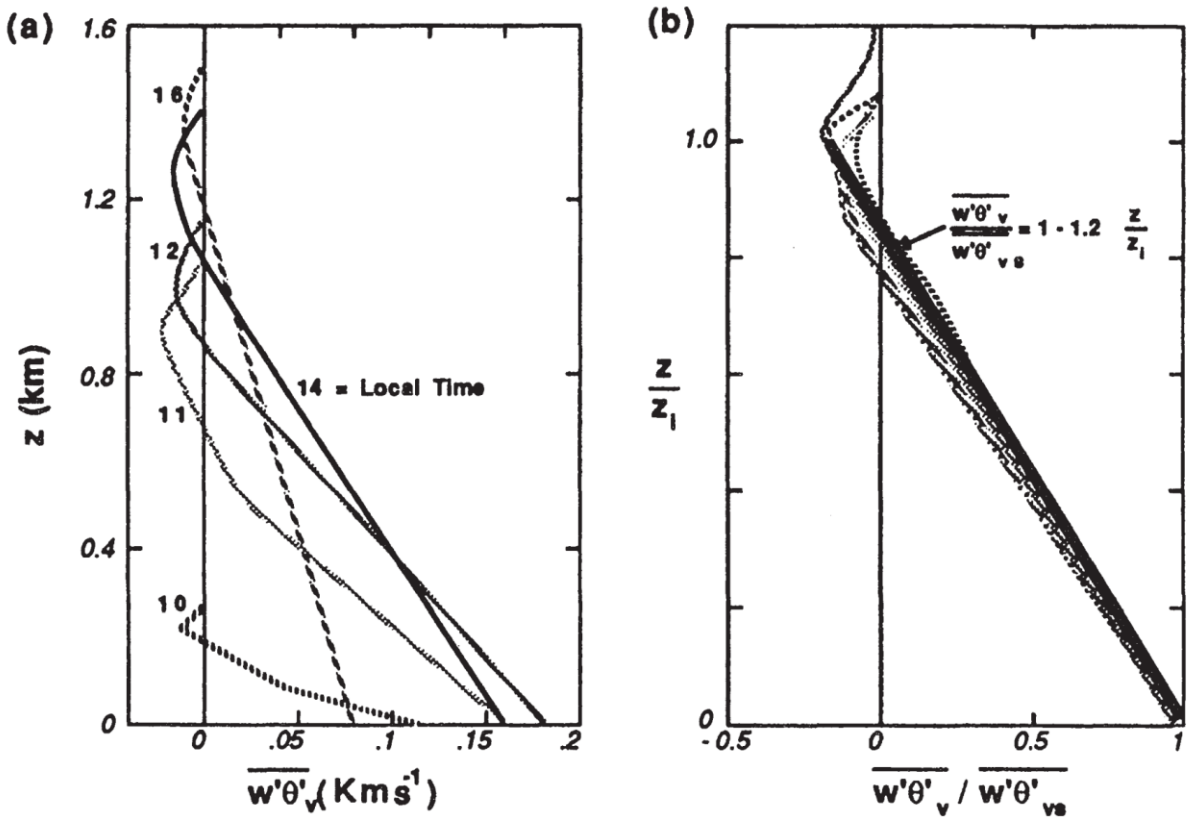


Fig. 9.1 Raw heat flux data from a simulation of Wangara Day 33 (a) replotted in a dimensionless framework (b). The empirical straight line estimate from similarity theory is also shown in (b).

Buckingham Pi Dimensional Analysis Methods

The example is that of fluid flow through a pipe, and involves the question: "How does the shear stress, τ, vary?"

Buckingham Pi Dimensional Analysis Methods

Step 1. Hypothesize which variables could be important to the flow.
Example: stress, density, viscosity, velocity, pipe diameter, pipe roughness

Step 2. Find the dimensions of each of the variables in terms of the fundamental dimensions. The fundamental dimensions are:

L = length
M = mass
T = time
K = temperature
A = electric current
I = luminous intensity

The dimensions of any variable can be broken into these fundamental dimensions.

Example:

variable	name	fundamental dimensions
ρ	fluid density	$M L^{-3}$
μ	dynamic viscosity	$M L^{-1} T^{-1}$
U	velocity	$L T^{-1}$
τ	shear stress	$M L^{-1} T^{-2}$
D	pipe diameter	L
z_o	pipe roughness length	L

The first two variables describe fluid characteristics, the next two describe flow characteristics, and the last two describe pipe characteristics.

Step 3: Count the number of fundamental dimensions in our problem.

Example: There are 3 dimensions: L, M, T.

Step 4: Pick a subset of your original variables to become "key variables", subject to the following restrictions:

(a) The number of key variables must equal the number of fundamental dimensions.

(b) All fundamental dimensions must be represented in the key variables.

(c) No dimensionless group must be possible from any combination of these key variables.

Example: Pick 3 variables: ρ, D, and U to be the key variables.

Note that there are many other equally valid choices for key variables, such as: ρ, z_o, U; or τ, μ, D, etc. It does not matter which three are picked, assuming that all of the above restrictions are satisfied. An invalid set would be U, D, z_o, because D/z_o is dimensionless, and also because the fundamental dimension M is not represented. Another invalid set is τ, ρ, U, because $\tau/(\rho U^2)$ is dimensionless.

Step 5. Form dimensionless equations of the remaining variables in terms of the key variables.

Example: $\tau = (\rho)^a (D)^b (U)^c$

$\mu = (\rho)^d (D)^e (U)^f$

$z_o = (\rho)^g (D)^h (U)^i$

where a-i are unknown powers.

Step 6. Solve for the powers a, b, c, . . . to yield dimensionally consistent equations.

Example: Solve each equation independently. For the first equation:

$$\tau = (\rho)^a (D)^b (U)^c$$

or $$M L^{-1} T^{-2} = (M L^{-3})^a (L)^b (L T^{-1})^c$$

or $$M L^{-1} T^{-2} = M^a \quad L^{-3a+b+c} \quad T^{-c}$$

The dimensions on the left hand side must equal the dimensions on the right. Thus:

M: $1 = a$

L: $-1 = -3a + b + c$

T: $-2 = -c$

These three equations can be solved for the three unknowns, yielding:

$a = 1 \qquad b = 0 \qquad c = 2$.

Thus, a dimensionally consistent equation is: $\tau = (\rho)^1 (D)^0 (U)^2$, or $\tau = \rho U^2$.

Similarly, we find that $d = 1$, $e = 1$, $f = 1$: yielding $\mu = \rho U D$.

Also: $g = 0$, $h = 1$, $i = 0$: yielding $z_o = D$.

Step 7. For each equation, divide the left hand side by the right hand side to give dimensionless (Pi) groups. The number of Pi groups will always equal the number of variables minus the number of dimensions.

Example:

$$\pi_1 = \frac{\tau}{\rho U^2} \qquad \pi_2 = \frac{\mu}{\rho U D} \qquad \pi_3 = \frac{z_o}{D}$$

We started with 6 variables in our example, and reduced our degrees of freedom down to 3 dimensionless groups.

Step 8. (Optional) If desired, alternative Pi groups can be formed from the ones derived in the previous step, as long as: the total number of Pi groups does not change, all variables are represented, and no one Pi group can be formed from any combination of the remaining groups.

Example: One alternative set of Pi groups might be: π_1 , π_4 $(=\pi_2/\pi_3)$, π_5 $(=1/\pi_3)$.

This new set is:

$$\pi_1 = \frac{\tau}{\rho U^2} \qquad \pi_4 = \frac{\mu}{\rho U z_o} \qquad \pi_5 = \frac{D}{z_o}$$

In fact, regardless of which set of primary variables were chosen, we can always arrive at the same set of Pi groups via this Pi-group manipulation process.

This is the end of the formal cookbook procedure for Buckingham Pi Theory. Of course, it is really only the second step of the overall similarity procedure. The next step would be to perform the necessary experiments to discover the relationships between the Pi groups. An example of laboratory pipe flow data is shown in Fig 9.2.

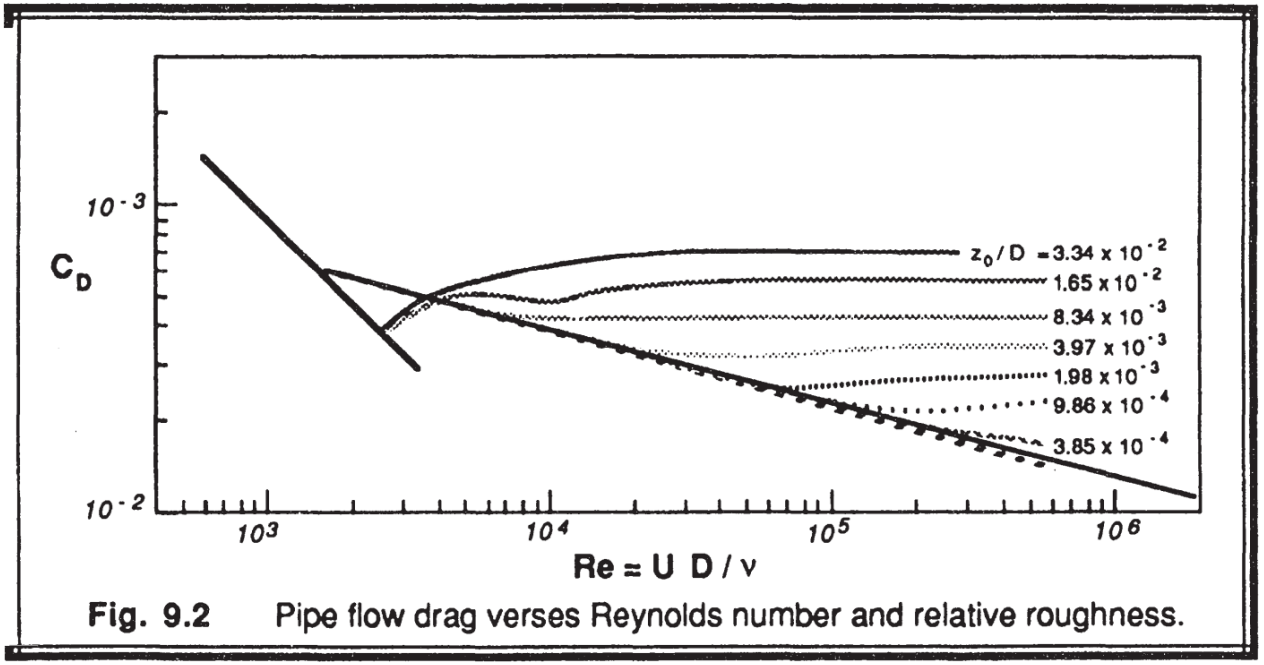


Fig. 9.2 Pipe flow drag verses Reynolds number and relative roughness.

Discussion: Several very important facts can be learned from this data. First, the stress decreases as the Reynolds number increases, until a critical Reynolds number of about 2100 is reached. This critical Reynolds number marks the transition from laminar to turbulent flow. At lower Reynolds number (laminar flow), the stress is NOT dependent on the relative roughness. As suggested in Section 9.1, the data is telling us that pipe roughness is not relevant for laminar flow.

Second, at Reynolds numbers just larger than critical, the stress increases again. Third, as Reynold's number increases further, the stress again decreases with Reynolds number, independent of the pipe roughness. Fourth, this roughness independence fails when some ***roughness Reynolds number*** (given by π_4) is reached. Fifth, at even larger Reynolds number, the stress is a constant depending only on relative roughness and not on the Reynolds number itself.

This last observation is of important consequence for the atmosphere. As previously discussed, the Reynolds number for the atmosphere is very large, on the order of 10^6 to 10^8, even within the boundary layer. Fig 9.2 shows us that ***large Reynolds number flow*** is independent of the Reynolds number! Hence, we can usually ignore molecular viscosity and the associated Reynolds number in descriptions of the boundary layer. However, for the very smallest size eddies and in the very thin microlayer near the surface, molecular viscosity continues to be important for TKE dissipation and transport across the surface, respectively.

Homework

7.3 – Mariana

7.5 – Sara

7.6 – Cátia

7.7 – Diogo

7.12 – Florian

7.15 – Jason

7.18 – Maria