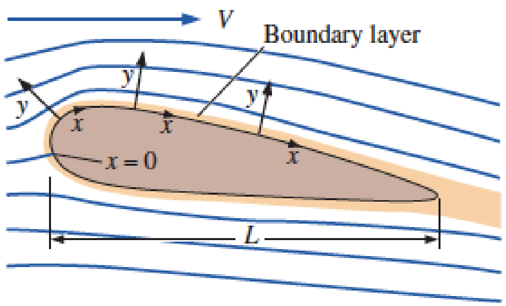
Boundary layer equations

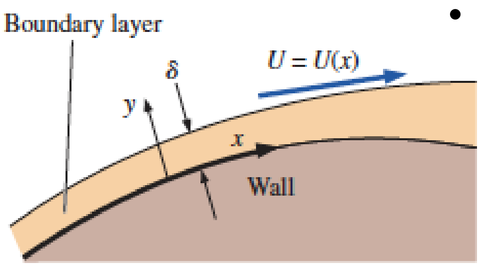
- We consider steady, two-dimensional flow in the xy-plane in Cartesian coordinates. The methodology can be extended to axisymmetric boundary layers or to three-dimensional boundary layers in any coordinate system.
- We neglect gravity since we are not dealing with free surfaces or with buoyancy-driven flows (free convection flows), where gravitational effects dominate.
- We consider laminar boundary layers; turbulent boundary layer equations are beyond the scope of this course.
- For a boundary layer along a solid wall, we adopt a coordinate system in which x is everywhere parallel to the wall and y is everywhere normal to the wall.
- When we solve the boundary layer equations, we do so at one x-location at a time, using this coordinate system locally, and it is locally orthogonal.



The nondimensionalized Navier–Stokes equation is

$$(\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[\mathrm{Eu}]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*$$

- The Euler number is of order 1, since pressure differences outside the boundary layer are determined by the Bernoulli equation and $\Delta P \sim \rho V^2$.
- V is a characteristic velocity of the outer flow, typically the free-stream velocity for bodies immersed in a uniform flow.
- The characteristic length is L, some characteristic size of the body. For boundary layers, x is of order o L, and Reynolds number is Re_x, usuallly very high.



Redo the nondimensionalization of the equations based on appropriate scales within the boundary layer.

- Since $x \sim L$, we use L as the scale for distances in the streamwise direction and for derivatives with respect to x. However, this scale is too large for derivatives with respect to y. We use δ for distances in the direction normal to the streamwise direction and for derivatives with respect to y.

- Similarly, we use U as the characteristic velocity, where U is the magnitude of the velocity component parallel to the wall at a location just above the boundary layer. U is in general a function of x.

- Thus, within the boundary layer at some value of x, the orders of magnitude are

$$u \sim U \qquad P - P_\infty \sim \rho U^2 \qquad \frac{\partial}{\partial x} \sim \frac{1}{L} \qquad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

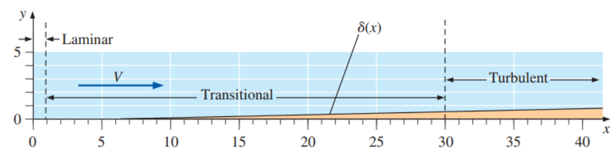
- The order of magnitude of velocity component v is obtained from the continuity equation

$\nabla \cdot \vec{u} = 0$

$$\underbrace{\frac{\partial u}{\partial x}}_{\sim U/L} + \underbrace{\frac{\partial v}{\partial y}}_{\sim v/\delta} = 0 \quad \rightarrow \quad \frac{U}{L} \sim \frac{v}{\delta}$$

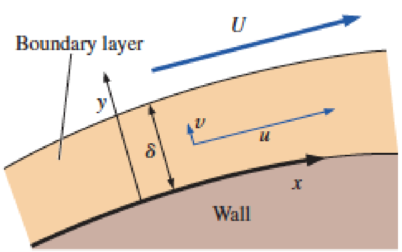
- Since the two terms have to balance each other, they must be of the same order of magnitude. Thus we obtain the order of magnitude of velocity component v**,**

$$v \sim \frac{U\delta}{L}$$



On scale

- Since δ/L << 1 in a boundary layer, we conclude that v << u, and the adimensional variables are



$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{\delta} \qquad u^* = \frac{u}{U} \qquad v^* = \frac{vL}{U\delta} \qquad P^* = \frac{P - P_\infty}{\rho U^2}$$

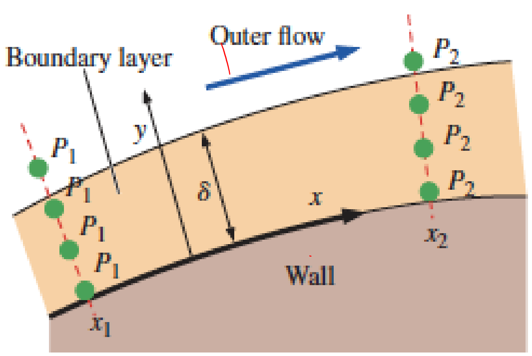
We now consider the x- and y-components of the Navier–Stokes equation. We substitute these nondimensional variables into the y-momentum equation, giving

$$\underbrace{u}_{u^*U}\underbrace{\frac{\partial v}{\partial x}}_{\frac{\partial}{\partial x^*}\frac{v^*U\delta}{L^2}} + \underbrace{v}_{v^*\frac{U\delta}{L}}\underbrace{\frac{\partial v}{\partial y}}_{\frac{\partial}{\partial y^*}\frac{v^*U\delta}{L\delta}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial y}}_{\frac{1}{\rho}\frac{\partial}{\partial y^*}\frac{P^*\rho U^2}{\delta}} + \underbrace{\nu\frac{\partial^2 v}{\partial x^2}}_{\nu\frac{\partial^2}{\partial x^{*2}}\frac{v^*U\delta}{L^3}} + \underbrace{\nu\frac{\partial^2 v}{\partial y^2}}_{\nu\frac{\partial^2}{\partial y^{*2}}\frac{v^*U\delta}{L\delta^2}}$$

After some algebra

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\left(\frac{L}{\delta}\right)^2\frac{\partial P^*}{\partial y^*} + \left(\frac{\nu}{UL}\right)\frac{\partial^2 v^*}{\partial x^{*2}} + \left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v^*}{\partial y^{*2}}$$

The middle term on the r.h.s. is clearly smaller than any other term since $Re_L = UL/\nu \gg 1$. For the same reason, the last term on the right is much smaller than the first term on the right. Neglecting these two terms leaves the two terms on the left and the first term on the right. However, since $L \gg \delta$, the pressure gradient is orders of magnitude greater than the advective terms on the left of the equation. Thus, the only term left is the pressure term. Since no other term in the equation can balance that term, we have no choice but to set it to zero. Thus, the nondimensional y-momentum equation is



$$\frac{\partial P^*}{\partial y^*} \cong 0$$

The pressure across a boundary layer (y-direction) is nearly constant.

Since P is not a function of y, we replace ∂P/∂x by dP/dx, where P is the pressure calculated from the outer flow approximation (using either continuity plus Euler, or the potential flow equations plus Bernoulli). The x-component of the Navier–Stokes equation becomes

$$\underbrace{u}_{u^* U}\underbrace{\frac{\partial u}{\partial x}}_{\frac{\partial}{\partial x^*}\frac{u^* U}{L}} + \underbrace{v}_{v^*\frac{U\delta}{L}}\underbrace{\frac{\partial u}{\partial y}}_{\frac{\partial}{\partial y^*}\frac{u^* U}{\delta}} = \underbrace{-\frac{1}{\rho}\frac{dP}{dx}}_{\frac{1}{\rho}\frac{\partial}{\partial x^*}\frac{P^*\rho U^2}{L}} + \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\nu\frac{\partial^2}{\partial x^{*2}}\frac{u^* U}{L^2}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\nu\frac{\partial^2}{\partial y^{*2}}\frac{u^* U}{\delta^2}}$$

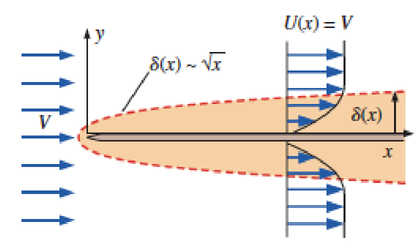
or

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{dP^*}{dx^*} + \left(\frac{\nu}{UL}\right)\frac{\partial^2 u^*}{\partial x^{*2}} + \left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2\frac{\partial^2 u^*}{\partial y^{*2}}$$

The middle term on the right side is orders of magnitude smaller than the terms on the left. What about the last term on the right? If we neglect this term, we throw out all the viscous terms and are back to the Euler equation. Clearly this term must remain. Furthermore, since all the remaining terms are of order unity, the combination of parameters in parentheses in the last term on the right side must also be of order 1,

$$\left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2 \sim 1$$

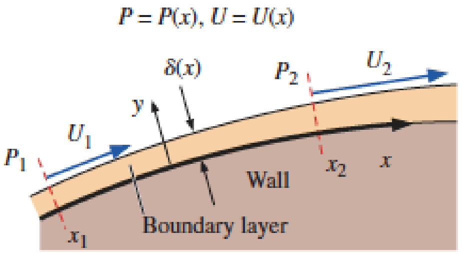
$$\frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}_L}}$$



x-momentum boundary layer equation: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Finally, since we know from the y-momentum equation analysis that the pressure across the boundary layer is the same as that outside the boundary layer, we apply the Bernoulli equation to the outer flow region. Differentiating with respect to x we get

$$\frac{P}{\rho} + \frac{1}{2}U^2 = \text{constant} \quad \rightarrow \quad \frac{1}{\rho}\frac{dP}{dx} = -U\frac{dU}{dx}$$



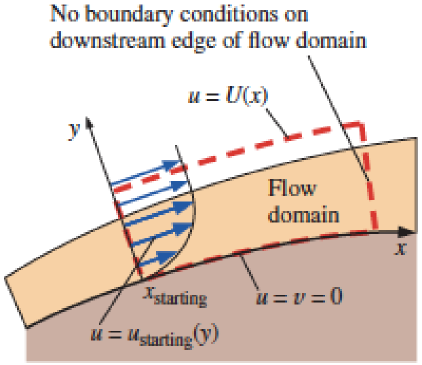
Substitution yields

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Boundary layer equations:

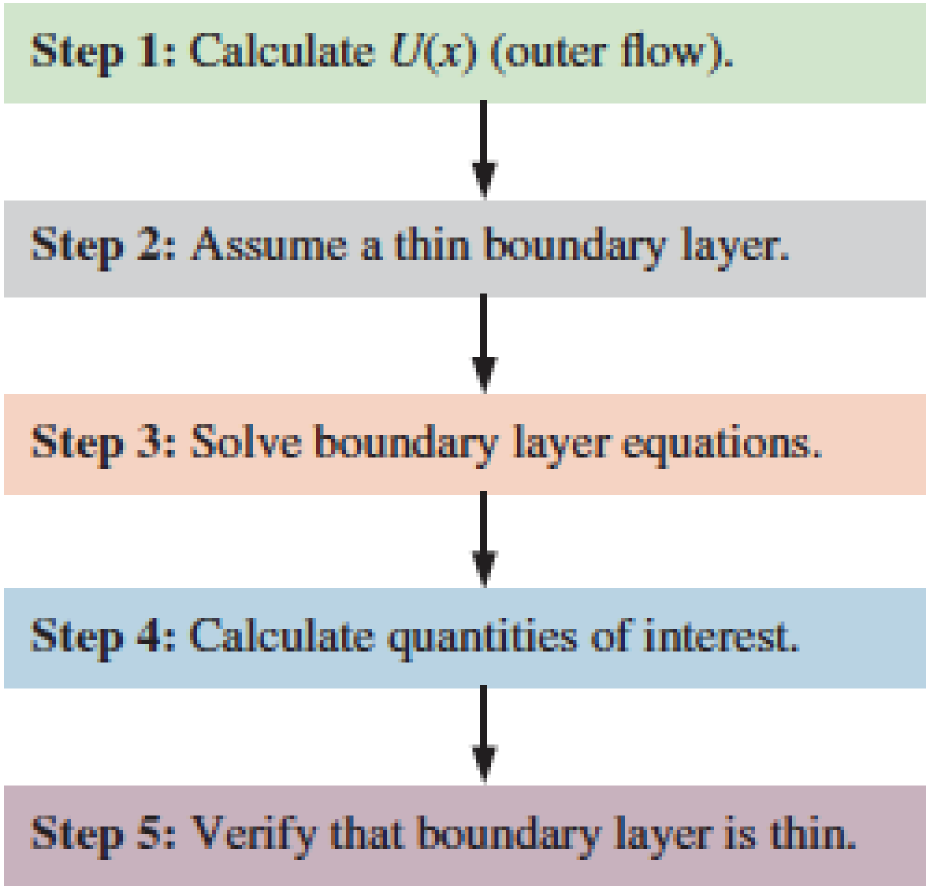
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$



$$\nu = \frac{\mu}{\rho}$$

- For a typical boundary layer problem along a wall, we specify the no-slip condition at the wall (u = v = 0 at y = 0), the outer flow condition at the edge of the boundary layer and beyond [u = U(x) as y → ∞], and a starting profile at some upstream location [u = $u_{starting}$(y) at x = $x_{starting}$, where $x_{starting}$ may or may not be zero]. With these boundary conditions, we simply march downstream in the x-direction, solving the boundary layer equations as we go.



Example: Flat plate

Outer flow: $U(x) = V = \text{constant}$

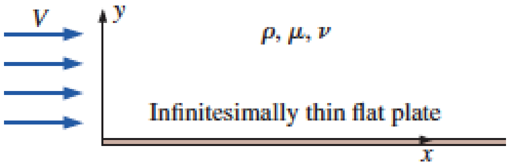
$\partial U/\partial x = 0$

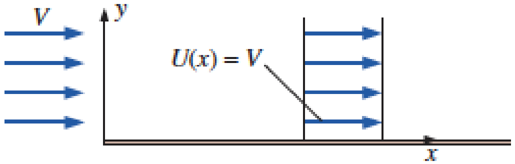
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

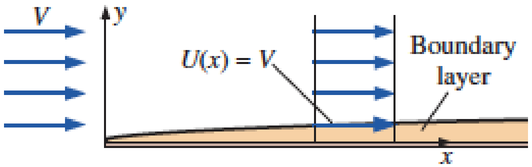
$$u = 0 \text{ at } y = 0 \qquad u = U \text{ as } y \to \infty$$

$$v = 0 \text{ at } y = 0 \qquad u = U \text{ for all } y \text{ at } x = 0$$

No convenient analytical solution is available. However, a series solution was obtained in 1908 by Blasius.







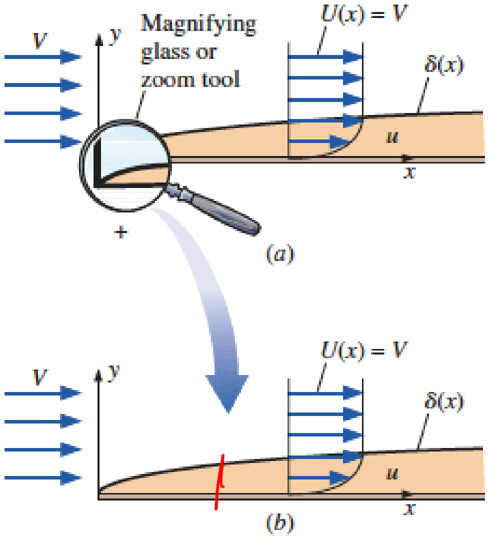
Blasius similarity solution

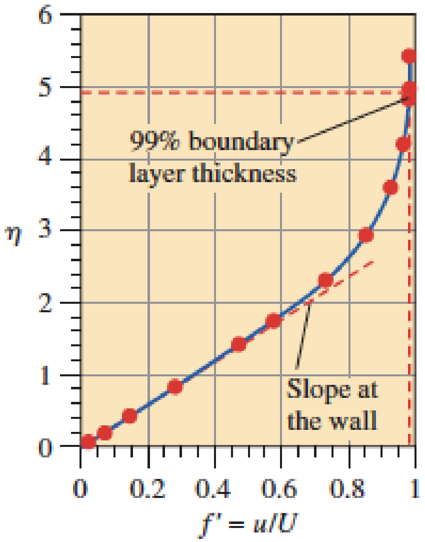
Blasius introduced a **similarity variable** η that combines independent variables x and y into one nondimensional independent variable,

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

and he solved for a nondimensionalized form of the x-component of velocity,

$$f' = \frac{u}{U} = \text{function of } \eta$$





$$\eta = 4.91 = \sqrt{\frac{U}{\nu x}}\delta \rightarrow \frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$$

Shear stress in physical variables: $\tau_w = 0.332\frac{\rho U^2}{\sqrt{Re_x}}$

Blasius solution

Non-linear third order ODE.

Solved numerically or by a series expansion.

Similarity Variable

$$\eta = \frac{y}{\delta} = \frac{y}{\sqrt{\nu x / U_0}}$$

Streamfunction

$$f(\eta) = \frac{\psi}{\delta U_0} = \frac{\psi}{\sqrt{\nu x U_0}}$$

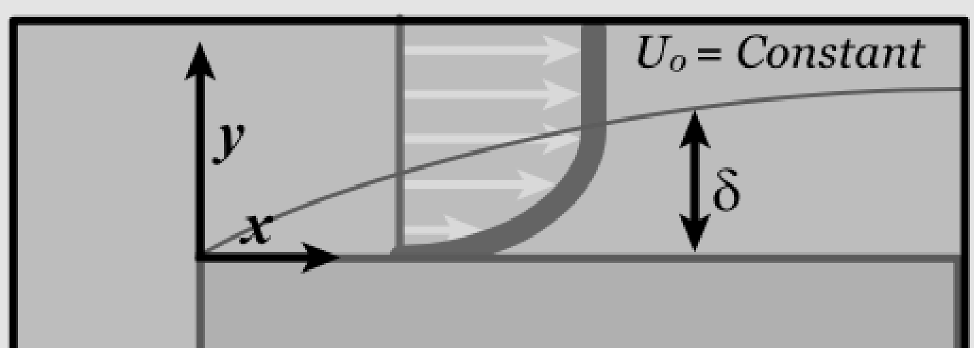
Blasius Equation

$$f''' + ff'' = 0$$

Boundary Conditions

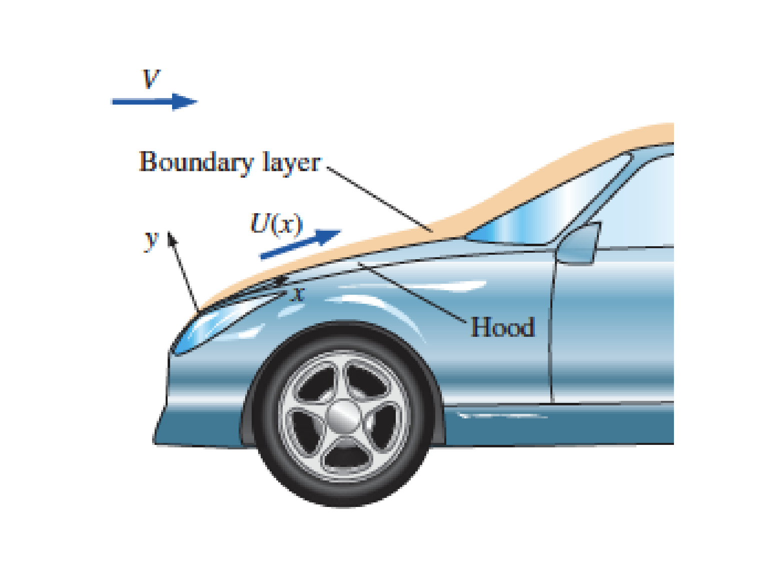
wall: $\eta = 0 \qquad f = f' = 0$

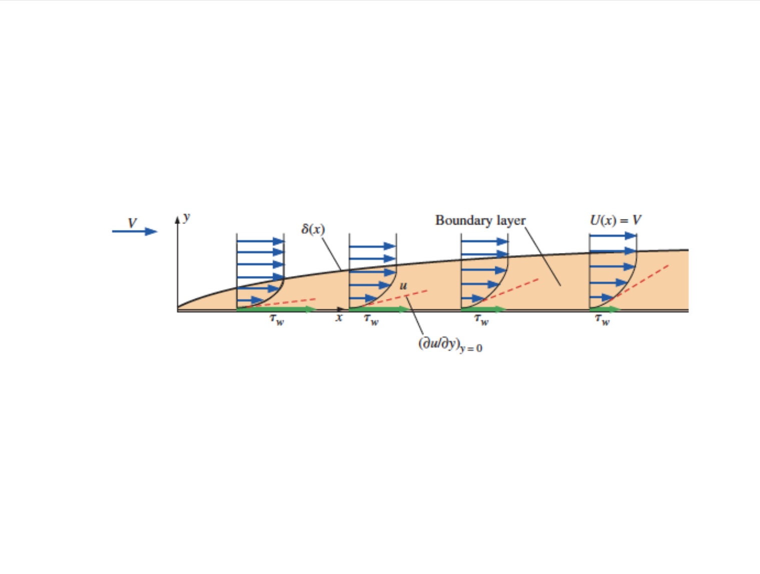
freestream: $\eta \rightarrow \infty \qquad f' = 1$



- Discussion: The Blasius boundary layer solution is valid only for flow over a flat plate perfectly aligned with the flow.

- However, it is often used as a quick approximation for the boundary layer developing along solid walls that are not necessarily flat nor exactly parallel to the flow, as in a car hood.





Summary of expressions for laminar and turbulent boundary layers on a smooth flat plate aligned parallel to a uniform stream*			
		(a)	(b)
Property	Laminar	Turbulent(†)	Turbulent(‡)
Boundary layer thickness	$\frac{\delta}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta}{x} \cong \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta}{x} \cong \frac{0.38}{(\mathrm{Re}_x)^{1/5}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} \cong \frac{0.020}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta^*}{x} \cong \frac{0.048}{(\mathrm{Re}_x)^{1/5}}$
Momentum thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta}{x} \cong \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\theta}{x} \cong \frac{0.037}{(\mathrm{Re}_x)^{1/5}}$
Local skin friction coefficient	$C_{f,x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_{f,x} \cong \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$	$C_{f,x} \cong \frac{0.059}{(\mathrm{Re}_x)^{1/5}}$

* Laminar values are exact and are listed to three significant digits, but turbulent values are listed to only two significant digits due to the large uncertainty affiliated with all turbulent flow fields.

† Obtained from one-seventh-power law.

‡ Obtained from one-seventh-power law combined with empirical data for turbulent flow through smooth pipes.

Local friction coefficient, laminar flat plate: $$C_{f,x} = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$