# UNIVERSO PRIMITIVO

Mestrado em Física Astronomia 2020-2021

Exercise Sheet 4+5

1. Use the relation $\mathrm{T}_\nu = (4/11)^{1/3}\,\mathrm{T}_\gamma$ derived in classroom to obtain the cosmic neutrino background number and energy density expressions in slide 24 of Chapter 4. Compute these densities at present assuming an effective number of neutrino families equal to $\mathrm{N}_{\mathrm{eff}} = 3.046$ and a CMB present-day temperature $\mathrm{T}_\gamma = 2.725$ K.

2. Recall the Riccati equation derived in classroom for weakly interactive massive particles (WIMPs) written as ($Y \equiv N_X$):
$$\frac{dY}{dx} = -\frac{\lambda}{x^2}[Y^2 - Y_{eq}^2]$$
where $x = M_X/T$, $M_X$ is the mass of the WIMP particles, $T$ is the photon temperature and $\lambda$ can be treated as a constant. Let $\Delta \equiv Y - Y_{eq}$ be the variable that measures the deviation of $Y$ from its equilibrium value.

   2.1. Prove that:
$$\frac{d\Delta}{dx} = -\frac{dY_{eq}}{dx} - \frac{\lambda}{x^2}[\Delta^2 + 2Y_{eq}\Delta]$$

   2.2. Simplify this equation using the approximation $Y \simeq Y_{eq} \Rightarrow \Delta \simeq 0$ and $d\Delta/dx \simeq 0$, valid for the temperature range $1 < x < x_f$, where $x_f = M_X/T_f$ is the freeze-out temperature. [Hint: note that under these approximations, the first term inside the square brackets is smaller than the second term]

   2.3. Derive an expression for $\Delta$ assuming $Y_{eq} \approx e^{-x}$. How does it depend on $x$ and $\lambda$?

   2.4. Re-derive $\Delta$, now using $Y_{eq} = N_X^{eq} = n_X^{eq}/s$ [Hint: assume that the WIMP particles are already non-relativistic and write their equilibrium density, $n_X^{eq}$, and the specific entropy of the fluid, $s$, as a function of $x$].

3. Considering the equilibrium number density of protons, neutrons and a nuclear species with $Z$ protons and $A - Z$ neutrons (where $A$ is the nuclear atomic mass and $Z$ the charge of the nucleus) can be written as:
$$n_i = g_i\left(\frac{m_i T}{2\pi}\right)^{3/2}\exp\left(\frac{\mu_i - m_i}{T}\right)$$
where $i = \{p, n, A\}$, show that the number density of the nucleus $A$ is given by:
$$n_A = \frac{g_A}{2^A}A^{3/2}\left(\frac{m_B T}{2\pi}\right)^{3(1-A)/2} n_p^Z n_n^{A-Z}\exp\left(\frac{B_A}{T}\right)$$
where $B_A = Zm_p + (A - Z)m_n - m_A$ is the biding energy of the nucleon $A$. [Hint: Note that the chemical potential of the nucleus, $\mu_A$, is related with the chemical potentials of the protons, $\mu_p$, and neutrons, $\mu_n$, by $\mu_A = Z\mu_p + (A - Z)\mu_n$. Use also the approximations $m_A = Am_B$, with $m_B = m_p \approx m_n$].

4. The Lagrangian for a scalar field in a curved spacetime is

$$L = \sqrt{-g}\,[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor.

4.1. Evaluate the scalar field Lagrangian for a homogeneous field $\phi = \phi(t)$ in a flat FLRW spacetime.

4.2. Use the Euler-Lagrange equation to derive the equation of motion for the scalar field (the Klein-Gordon equation).

4.3. Use the Friedmann and the acceleration equations with $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and $p_\phi = \dot{\phi}^2/2 - V(\phi)$ to prove that $\dot{H} = -\dot{\phi}^2/2M_{PL}$. Determine the equation of state (EoS) parameter, $w = p_\phi/\rho_\phi$, and the scale factor, $a(t)$, assuming that the inflationary field is slow-rolling, $\dot{\phi} \sim 0$.

5. The power spectrum of curvature perturbations, $\Delta_R^2$, and the spectral index, $n_s$, are given by:

$$\Delta_R^2 = \left(\frac{1}{8\pi^2\epsilon}\frac{H^2}{M_{PL}^2}\right)_{k=aH} \quad ; \qquad n_s - 1 = -2\epsilon - \eta \; ,$$

where $\epsilon$ and $\eta$ are the inflation parameters. Derive expressions for $\Delta_R^2$ and $n_s - 1$ that put in evidence the dependence of these quantities on the inflationary potential and its derivatives, assuming slow-roll inflation.

6. The discovery of a Higgs like scalar particle at the LHC raises questions about the feasibility of having the Higgs scalar field driving cosmological inflation. To investigate this hypothesis, consider that the Higgs potential is given by $V(\phi) = \lambda\,(\phi^2 - v^2)^2$ where $\lambda$ is a constant proportional to the field's mass and $v = 246$ GeV.

6.1. Sketch the potential and compute the slow-roll parameters, $\epsilon_V = M_{PL}^2(V'/V)^2/2$ , and $\eta_V = M_{PL}^2 V''/V$.

6.2. Discuss if the slow-roll conditions can be satisfied simultaneously inside the field range $0 < \phi < v$. Is slow-roll inflation possible inside this range?

6.3. Now look at the regime, $\phi \gg v$ . Show that $\epsilon_V(\phi)$ and $\eta_V(\phi)$ become independent of $v$. For what field values does inflation occur? Determine the field values at the end of inflation, $\phi_E$, and at a number of e-foldings $N_* = 60$ before, $\phi_*$ (assume that $\phi_* \gg \phi_E$).

6.4. Compute the amplitude of the power spectrum of scalar fluctuations, $\Delta_R^2$, at $\phi_*$. Express your answer in terms of $N_*$ and the mass of the Higgs boson defined as $m_H^2 = V''(\phi = v)$.

6.5. Estimate the value of $m_H$ required to match the observed scalar power amplitude $\Delta_*^2 = 2 \times 10^{-9}$. Is this consistent with the LHC measurement of $m_H = 125$ GeV?