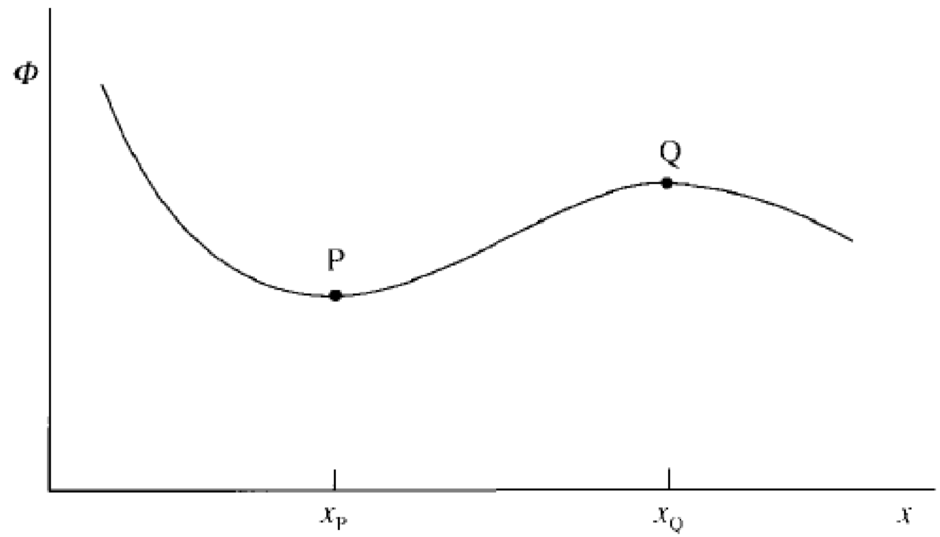


Figure 8.1 Positions of stable (P) and unstable (Q) equilibrium for a part whose potential energy Φ varies with x in the manner shown.

Stability, instability and marginal stability

Faber Chap. 8

As every physicist knows, a dynamical system which is in equilbrium may be *stable* or *unstable*. The simplest case of the distinction is that of a particle of mass m which can move only in one dimension, in circumstances where the particle's potential energy Φ varies with its position x in the manner suggested by fig. 8.1. The particle experiences no force when it is situated at the minimum, P, or at the maximum, Q, and in principle it can remain at rest indefinitely in either of these positions. However, if it is slightly displaced from P it accelerates towards P, whereas if it is slightly displaced from Q it accelerates away from Q; in the first position the particle is stable and in the second it is unstable. Near any minimum such as P the restoring force $\partial\Phi/\partial x$ can normally be expanded as a Taylor series in powers of displacement $\xi_P = x - x_P$. Since it is zero at P itself, an adequate approximation for small values of ξ_P is

$F = -\frac{\partial \phi}{\partial x}$

$$\frac{\partial \Phi}{\partial x} \approx \left(\frac{\partial^2 \Phi}{\partial x^2}\right)_P \xi_P,$$

$F = m\vec{a}$

in which case the equation of motion of the particle is linear in ξ_P,

$$m\,\frac{\partial^2 \xi_P}{\partial t^2} = -\left(\frac{\partial^2 \Phi}{\partial x^2}\right)_P \xi_P.$$

Oscilador harmônico

The oscillations which it describes are then simple harmonic, with angular frequency ω_P such that

$$\omega_P^2 = \frac{1}{m}\left(\frac{\partial^2 \Phi}{\partial x^2}\right)_P.$$

An equation of motion similar to (8.1) applies in the neighbourhood of Q, but since $(\partial^2\Phi/\partial x^2)_Q$ is negative the roots for ω are necessarily imaginary, $\omega_Q = \pm i s_Q$ with s_Q real. Hence the displacement $\xi_Q = x - x_Q$ of a particle which starts at rest at $t = 0$ from a position such that $\xi_Q = \xi_o$ is given at later times by

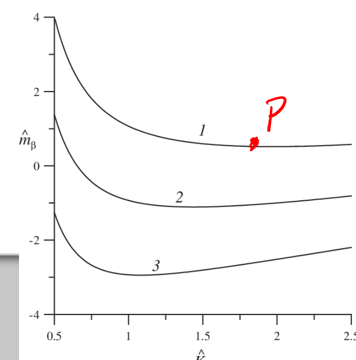
$$\xi_Q \approx \frac{1}{2}\,\xi_o\,(e^{s_Q t} + e^{-s_Q t}),$$

as long as it remains small. If ξ_o is infinitesimal, then by the time the displacement becomes apparent $\exp(s_Q t)$ must be very much greater than unity, in which case $\exp(-s_Q t)$ must be negligible. When a particle leaves a position of unstable equilibrium, therefore, its displacement normally grows in an exponential fashion.

Suppose now that $(\partial\Phi/\partial x)_{\mathrm{P}}$ is necessarily always zero – perhaps because of some symmetry requirement – while $(\partial^2\Phi/\partial x^2)_{\mathrm{P}}$ can be reduced in magnitude and ultimately reversed in sign by altering the external constraints which determine Φ. In that case P is always an equilibrium position, but the equilibrium is stable in one range of the constraints and unstable in an adjacent range. Where the changeover occurs one has

$$\omega_{\mathrm{P}}^2 = \frac{1}{m}\left(\frac{\partial^2\Phi}{\partial x^2}\right)_{\mathrm{P}} = 0,$$

and this is the condition for what is called *marginal stability*. When it is satisfied, the force experienced by a particle near P is normally determined by $(\partial^3\Phi/\partial x^3)_{\mathrm{P}}$ or, if $(\partial^3\Phi/\partial x^3)_{\mathrm{P}}$ is zero for symmetry reasons, by $(\partial^4\Phi/\partial x^4)_{\mathrm{P}}$; it is then proportional to ξ_{P}^2 or ξ_{P}^3 rather than to ξ_{P}.



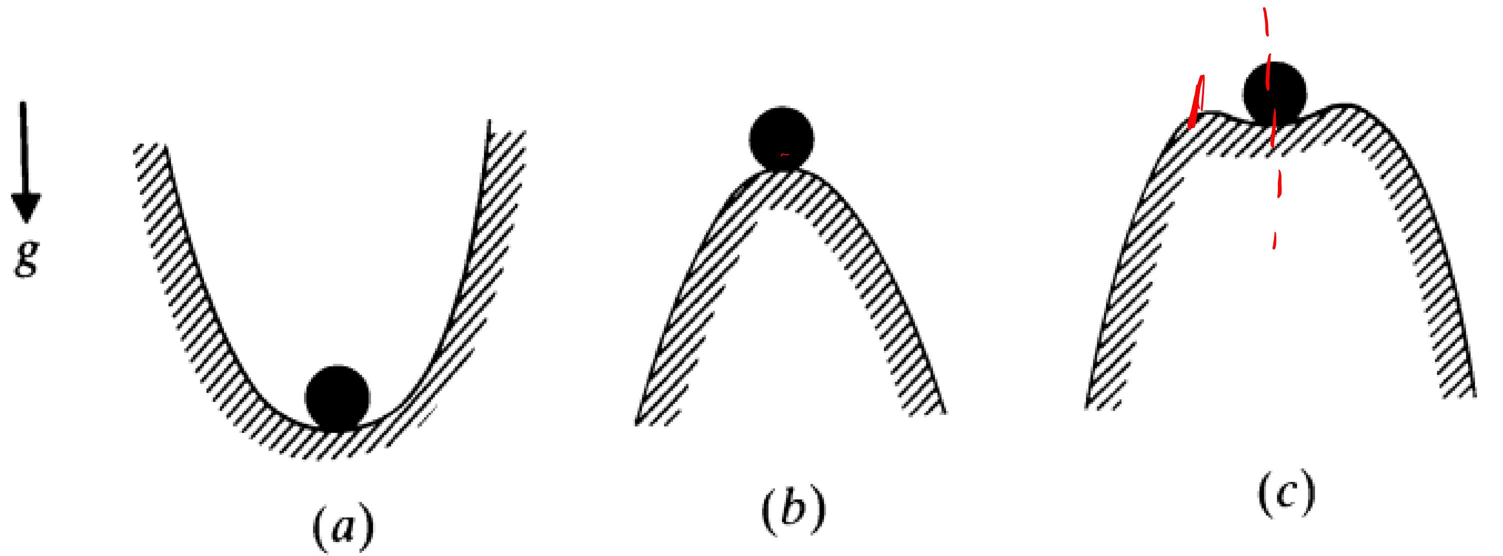


Fig. 9.3. (*a*) A stable state. (*b*) An unstable state. (*c*) A state which is stable to infinitesimal disturbances but unstable to disturbances which exceed some small threshold amplitude.

Similar results apply, of course, to any mechanical system for which energy is conserved. If the system is a complicated one, a full description of its state requires specification of a great many different coordinates of position. There always exists a set of normal coordinates ζ_n, however, such that for small ζ_n the potential energy Φ and kinetic energy T of the system may be expressed in the form

$$\Phi = \Phi_o + \sum_n \frac{1}{2} m_n \omega_n^2 \zeta_n^2,$$

Parábola em N dimensões

$$T = \sum_n \frac{1}{2} m_n \left(\frac{\partial \zeta_n}{\partial t}\right)^2,$$

where Φ_o is the potential energy of the equilibrium state for which all ζ_n are zero, and the equilibrium is stable if and only if $m_n \omega_n^2 > 0$ for *all* values of n. In continuous systems the normal coordinates often describe periodic modes of distortion of the system as a whole, rather than displacements of isolated parts of the system.

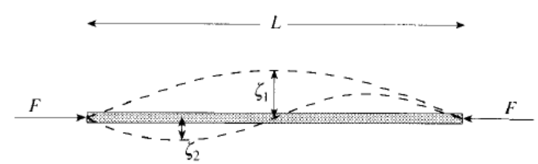


Figure 8.2 A strut under compression. The broken curves suggest two normal modes of flexural distortion, corresponding to $n = 1$ and $n = 2$.

Dissipative systems

$\vec{F} = -3\pi\mu D\vec{V}$

In so far as the above remarks apply to conservative systems they may seem to have little relevance to viscous fluids, which are inherently dissipative. If, however, a particle moving in the potential of fig. 8.1 is subject to a dissipative retarding force proportional to its velocity, the principal effect of this is merely to damp – and perhaps overdamp – oscillations in ξ_P and to slow down the exponential rate of growth of ξ_Q. That does not invalidate the conclusion that P and Q represent states of stable and unstable equilibrium respectively. Indeed, the fluctuations which always accompany dissipation in thermal equilibrium now make it impossible in principle, as well as in practice, for a particle to remain indefinitely at Q. Nor does the existence of dissipation invalidate the conclusion that when, as a result of a continuous change in the form of $\Phi\{x\}$, the equilibrium at P changes from being stable to being unstable, this equilibrium passes through a state of marginal stability.

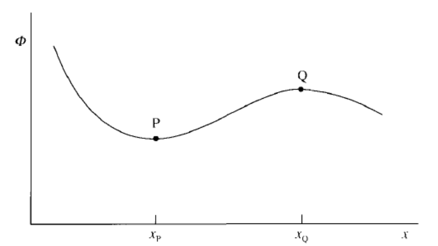


Figure 8.1 Positions of stable (P) and unstable (Q) equilibrium for a part
whose potential energy Φ varies with x in the manner shown.

The general procedure for investigating the stability or otherwise of patterns of fluid flow involves perturbing the pattern in various ways and calculating whether the amplitude – say $\zeta_n\{t\}$ – of each perturbation mode decreases or increases with time; the amplitude may well describe a velocity rather than a displacement, but that is a rather trivial distinction in this context. The modes must be consistent with the boundary conditions to which the fluid is subject, and they should form, like the periodic normal modes of the Euler strut, a complete set in terms of which any possible perturbation may be expanded. The exact equations of motion of the fluid are always non-linear in ζ_n, and one cannot achieve a detailed understanding of what happens once an instability has developed without taking non-linear terms into account. As a first step, however, it may suffice to establish the condition for a state of marginal stability to exist; having done that, one may confidently assert that true stability lies on one side of this condition and instability on the other.

Since marginal stability requires

$$\frac{\partial \zeta_n}{\partial t} = 0 \tag{8.2}$$

to first order only in ζ_n, the condition for its existence may be established using approximate equations of motion from which all terms which are non-linear in ζ_n have been deleted. If, as is often the case, there are several competing modes of instability, the first to develop once the condition for marginal stability has been exceeded is normally the one for which s_n $(= \zeta_n^{-1}\partial\zeta_n/\partial t)$ is largest. Linearised equations of motion suffice to settle this question as well.

Oscillatory instabilities

A characteristic of dissipative systems, which cannot readily be illustrated by reference to simple mechanical models of point masses moving in variable potentials but which is familiar in the context of electrical circuits, is that they may spontaneously oscillate. To do so they must incorporate a source of power, of course, and some feedback mechanism which selectively amplifies an oscillatory component in the thermal fluctuations of the system. At the onset of an oscillatory instability a system is said to become *overstable* (a potentially misleading term for which Eddington was responsible). If the amplitude $\zeta_n\{t\}$ of the overstable mode is taken to vary with time like $\exp\{-i(\omega_n + is_n)t\}$ for small ζ_n, where ω_n ($\neq 0$) and s_n are both real, overstability requires s_n to be positive whereas stability requires it to be negative. The condition for marginal overstability is $s_n = 0$, i.e.

$$\frac{\partial \zeta_n}{\partial t} = -\, i\omega_n \zeta_n$$

to first order in ζ_n.