Rayleigh-Taylor instability

The Rayleigh–Taylor instability arises when a vessel which contains two fluids separated by a horizontal interface – one at least of the fluids must of course be a liquid – is suddenly inverted so that the heavier fluid lies above the lighter one. The gravitational potential energy of the system, which was at its minimum value before inversion, is now at its maximum, and although the system is still in equilibrium while the interface remains horizontal the equilibrium is clearly liable to be unstable. Whether or not it is actually unstable with respect to any particular perturbation depends upon whether the gravitational energy which this releases is greater or less than the increase in surface free energy. The system is marginally stable with respect to the perturbation when the two are equal.

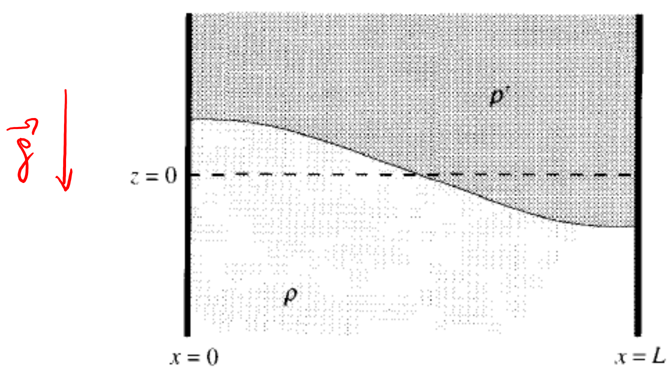
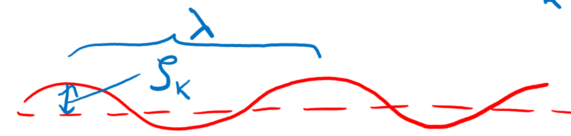


Figure 8.3 A layer of one fluid with a denser fluid above it, in a container of width L, is stabilised by surface tension against the perturbation suggested here provided that (8.6) is satisfied.

Further reading: Chandrasekhar, S. (2013). Hydrodynamic and hydromagnetic stability. Courier Corporation.

All possible small perturbations of the surface may be expressed in terms of their Fourier components, a typical Fourier component involving a vertical displacement of the interface

$K = \frac{2\pi}{\lambda}$



$$\zeta = \zeta_k\{t\}e^{ik\cdot r},$$

where $\boldsymbol{r}$ is a vector which lies in the $z = 0$ plane, i.e. the plane of the undisturbed interface. Per unit area of the interface, the reduction in gravitational potential energy associated with a single wave of this form, averaged over any integral number of wavelengths, is [(5.29)]

$E_p = \int_S \rho(\vec{r}) \cdot g \cdot h(\vec{r}) \, ds$

$$\frac{1}{4}(\rho' - \rho)g\zeta_k^2,$$

where ρ' and ρ are the densities of the heavier and lighter fluids respectively. The increase of the surface free energy, similarly averaged, is

Free energy

$\delta F = \sigma \delta A$

$$\sigma\left\langle\left\{1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right\}^{1/2} - 1\right\rangle \approx \frac{1}{4}\sigma k^2\zeta_k^2$$

J/m^2

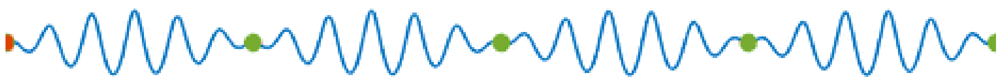
to second order in ζ_k, where σ is the interfacial surface tension. Marginal stability is therefore only possible for one wavevector k_c, such that

$$(\rho' - \rho)g = \sigma k_c^2. \qquad (8.4)$$

critical K

(λ_{MIN} or $K_{\cancel{MAX}\,c}$)

A dispersion relation relates the wavenumber of a wave to its frequency



Frequency dispersion of surface gravity waves on deep water. The ■ red square moves with the phase velocity, and the ● green dots propagate with the group velocity. In this deep-water case, the phase velocity is twice the group velocity. The ■ red square traverses the figure in the time it takes the ● green dot to traverse half.

Deep water waves [edit]

Further information: Dispersion (water waves) and Airy wave theory

The dispersion relation for deep water waves is often written as

$$\omega = \sqrt{gk},$$

where g is the acceleration due to gravity. Deep water, in this respect, is commonly denoted as the case where the water depth is larger than half the wavelength.[4] In this case the phase velocity is

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}},$$

and the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{1}{2} v_p.$$

Waves on a string [edit]

Further information: Vibrating string

For an ideal string, the dispersion relation can be written as

$$\omega = k\sqrt{\frac{T}{\mu}},$$

$$v = \lambda f = \frac{\cancel{2\pi}}{k} \cdot \frac{\omega}{\cancel{2\pi}} = \frac{\omega}{k}$$

$$\Rightarrow \boxed{\omega = v k}$$

where T is the tension force in the string, and μ is the string's mass per unit length. As for the case of electromagnetic waves in vacuum

Electromagnetic waves in a vacuum [edit]

For electromagnetic waves in vacuum, the angular frequency is proportional to the wavenumber:

$$\omega = ck.$$

This is a *linear* dispersion relation. In this case, the phase velocity and the group velocity are the same:

$$v = \frac{\omega}{k} = \frac{d\omega}{dk} = c;$$

they are given by c, the speed of light in vacuum, a frequency-independent constant.

(Source: https://en.wikipedia.org/wiki/Dispersion_relation)

In order to find the rate at which modes for which $k < k_c$ grow in amplitude, one needs to know how the velocities of each fluid depend upon $\partial\zeta_k/\partial t$. With that information at one's disposal, one may follow the routine procedure of evaluating the mean kinetic energy per unit area and hence the total energy, a sum of gravitational and surface terms proportional to ζ_k^2 and kinetic terms proportional to $(\partial\zeta_k/\partial t)^2$; by equating the time derivative of the total energy to zero one may then obtain, after cancellation of a factor $\partial\zeta_k/\partial t$, a linear equation of motion relating ζ_k to $\partial^2\zeta_k/\partial t^2$ which provides the required answer. We, however, can make use of a result already available as (5.40), which tells us, in the notation of §8.1, that the dispersion relation for waves on the interface is

angular frequency

$$\omega_k^2 = -\frac{\rho' - \rho}{\rho' + \rho} gk + \frac{\sigma k^3}{\rho' + \rho}, \quad \text{with } s_k = 0,$$

$\omega = \frac{2\pi}{\tau}$

as long as k is greater than the critical wavevector which (8.4) describes, which implies that when $k < k_c$ we have

See slide 30

$$s_k^2 = \frac{\rho' - \rho}{\rho' + \rho} gk - \frac{\sigma k^3}{\rho' + \rho}, \quad \text{with } \omega_k = 0.$$

$\exp\{-\mathrm{i}(\omega_n + \mathrm{i}s_n)t\}$

The value of k, say $k_{\max}$, which maximises s_k and hence the rate of growth is clearly such that

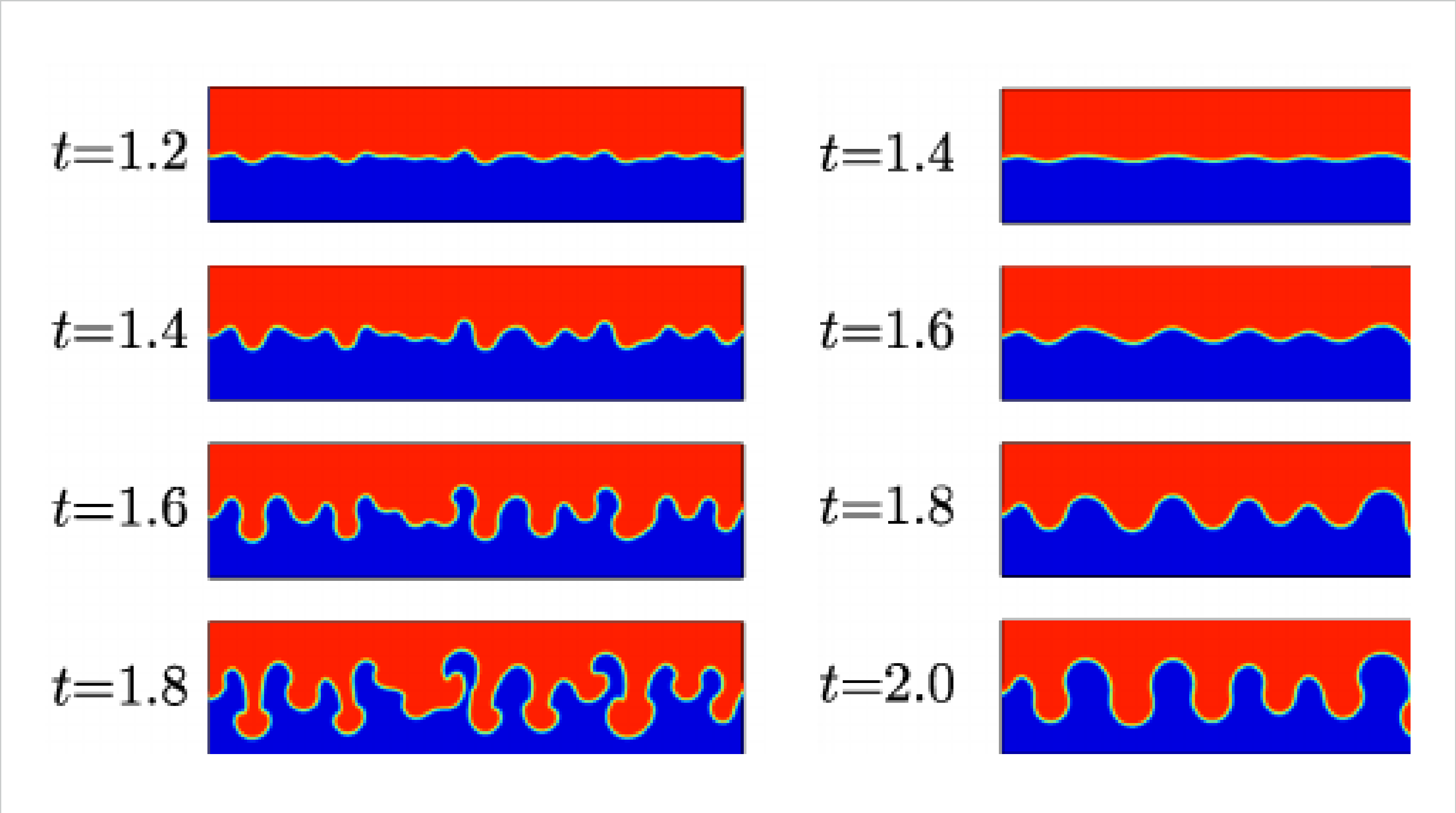
$$(\rho' - \rho)g = 3\sigma k_{\max}^2,$$

so

$$k_{max} = \frac{k_c}{\sqrt{3}}. \tag{8.5}$$

We may infer from the above results that if it were possible to invert almost instantaneously a *large* vessel containing two fluids, so large that the boundary conditions imposed virtually no limitations on the allowed values of $\boldsymbol{k}$, the contents would be inherently unstable. The interface would inevitably develop corrugations whose periodicity would be the wavelength associated with k_{max}, i.e. $2\pi\sqrt{3\sigma/(\rho' - \rho)g}$, which amounts to about 3 cm when the heavier fluid is water and the lighter one is air. In practice, however, rapid inversion is possible only with small vessels, and the fact that liquid inside an inverted bottle is stabilised by surface tension if the opening of the bottle is small enough must be familiar to every reader. For simplicity, suppose the vessel to be a rectangular one, with vertical sides and a cross-section in the $z = 0$ plane of which the larger dimension is L. The smallest non-zero value of k consistent with the boundary conditions [fig. 8.3 and some remarks about the boundary conditions applicable to water waves at the start of §5.8] is then π/L. In that case the inverted contents are stable provided that $\pi/L > k_c$, i.e. provided that

$$L < \pi\sqrt{\frac{\sigma}{(\rho' - \rho)g}}. \tag{8.6}$$



Rayleigh-Taylor

Rayleigh-Plateau

A free jet of water, emerging from a circular orifice, is liable to break up into a regular succession of drops, and according to Plateau's analysis of some observations by Savart the drops are separated by a distance λ which is about 8·8 times the radius a of the jet before it disintegrates. If a stationary cylinder of water could be obtained it would break up in the same way, and indeed the droplets of water which are to be seen on spiders' webs after a damp cold night are probably formed by accretion from layers of dew which are cylindrical when first deposited. The explanation lies in the fact that, volume for volume, spheres have smaller surface areas than cylinders.

Suppose an initially uniform cylinder of liquid to be subject to a small *varicose deformation*, which preserves rotational symmetry about the x axis (the axis of the cylinder) but alters its radius in a periodic fashion from a to

$$b = \langle b \rangle + \zeta_k \cos kx \quad (\zeta_k \ll a).$$

The volume of the cylinder per unit length, averaged over an integral number of wavelengths, is

πa^2

$$V = \langle \pi b^2 \rangle = \pi \langle b \rangle^2 + \frac{1}{2}\pi \zeta_k^2,$$

and since this must equal the initial volume per unit length, πa^2, we have

$$\langle b \rangle = \sqrt{a^2 - \tfrac{1}{2}\zeta_k^2} \approx a - \frac{\zeta_k^2}{4a}.$$

Thus the surface area of the cylinder per unit length, similarly averaged, is

$$A = \left\langle 2\pi b \sqrt{1 + \left(\frac{\mathrm{d}b}{\mathrm{d}x}\right)^2} \right\rangle$$

$$\approx 2\pi a + \frac{\pi \zeta_k^2}{2a}\{(ka)^2 - 1\}.$$

In this problem there is no gravitational term to consider, and it is the surface free energy per unit length, σA, which plays the role of the potential energy Φ of §8.1. The condition for marginal stability is $\partial^2 A/\partial \zeta_k^2 = 0$, equivalent to

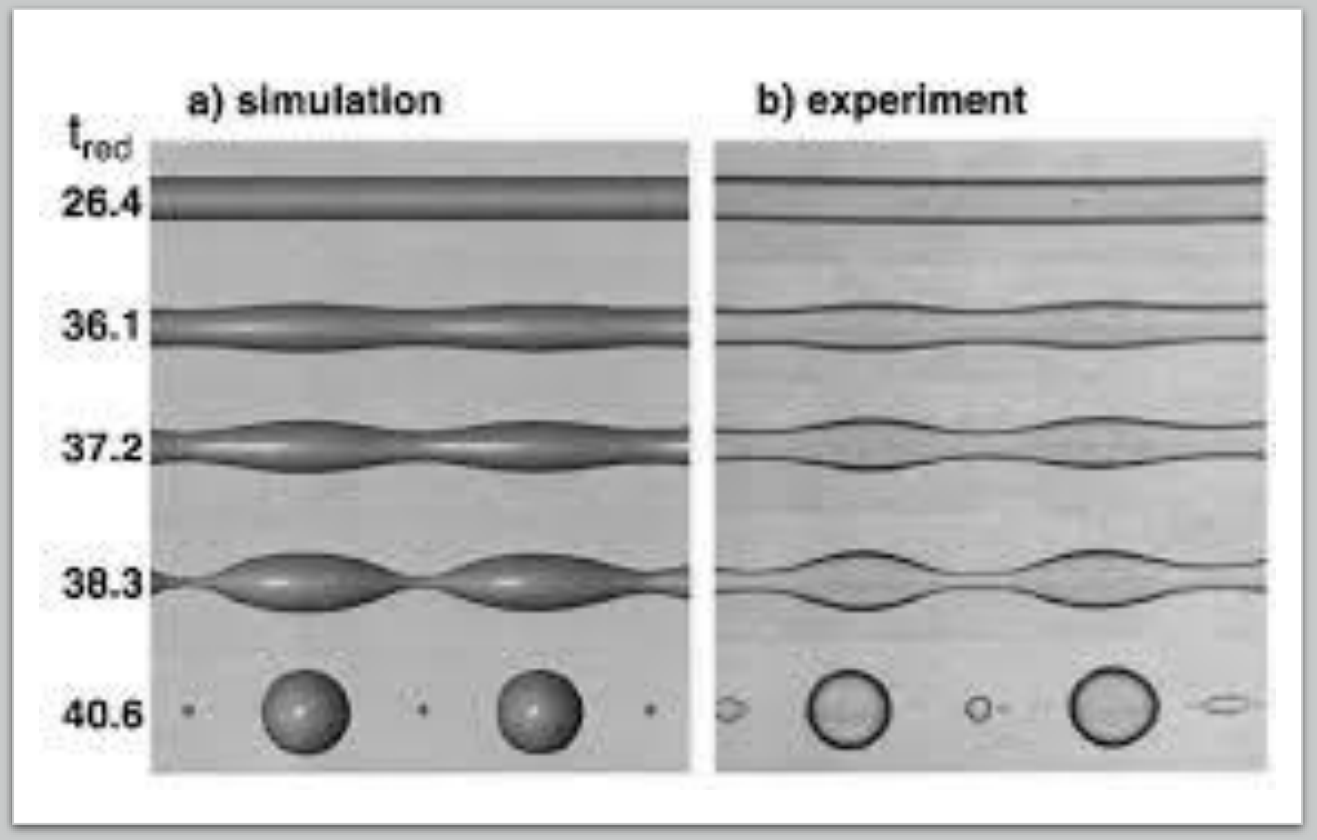
$$k = k_c = \frac{1}{a}. \Rightarrow \lambda_c = 2\pi a$$

$\phi = \sigma A$

$\frac{\partial^2 \phi}{\partial \zeta_k^2} \sim \frac{\partial^2 A}{\partial \zeta_k^2}$

The cylinder is inherently unstable, as Plateau was the first to note, to any periodic deformation for which k is less than k_c, i.e. for which the wavelength λ is greater than $2\pi a$.

To find the rate of growth of a mode for which $k < k_c$ one may follow the routine procedure outlined in §8.2. Provided that the viscosity of the liquid may be neglected, i.e. provided that potential theory may be employed, it is not difficult to calculate the fluid velocity $\boldsymbol{u}\{x, r\}$ associated with rate of change of ζ_k. It is described by a flow potential ϕ which is a solution of Laplace's equation proportional to $\cos(kx) f\{r\}(\partial\zeta_k/\partial t)$; the function $f\{r\}$ involves Bessel functions. Hence the constant of proportionality relating the fluid's mean kinetic energy per unit length to $(\partial\zeta_k/\partial t)^2$ may be found, and the equation of motion relating $\partial^2\zeta_k/\partial t^2$ to ζ_k follows immediately. According to Rayleigh, s_k, which is zero where $k = k_c$, reaches a maximum where $k = 0{\cdot}697 k_c$ or where $\lambda = 9{\cdot}02a$, in reasonable agreement with Savart's observations. The 2% discrepancy, in the wrong direction to be due to viscosity, is attributable to experimental error.



Rayleigh-Plateau