# The Homogeneous Universe

## The cosmological fluid

## The Energy-momentum tensor

The energy-momentum tensor, also called stress-energy tensor, represents the total energy present in the Universe in various forms.
It contains the energy-momentum contributions from all sources of gravity.

In the homogeneous universe, each source of gravity (e.g. matter, radiation, dark energy, curvature) is treated as a fluid, and together they form the **cosmological fluid.** The general form of the 4D energy-momentum tensor of the cosmological fluid is:

$$T_{ab} = (\rho + p)\, u_a u_b + p\, g_{ab} + q_a u_b + q_b u_a + \Pi_{ab}$$

This expression shows what are the physical quantities that contribute to the energy-momentum. They are:

- $\rho$ - energy **density** ($T_{00}$)
- $q_a$ - **momentum** density and flux ($T_{0i}$)
- $\Pi$ - **anisotropic stress** ($T_{ij}$)
- $p$ - **pressure** ($T_{ii}$)

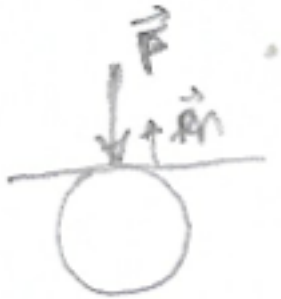
and $u^a$ is the 4-vector velocity of the fluid reference frame → $u^a = (-1,0)$ for comoving

$$
\begin{array}{cc}
 & \begin{array}{cccc} c^{-2}\cdot\begin{pmatrix}\text{energy}\\ \text{density}\end{pmatrix} & \text{momentum density} & & \end{array} \\
\begin{array}{c} \\ \text{momentum density} \\ \\ \\ \end{array} &
\begin{bmatrix}
T^{00} & T^{01} & T^{02} & T^{03} \\
T^{10} & T^{11} & T^{12} & T^{13} \\
T^{20} & T^{21} & T^{22} & T^{23} \\
T^{30} & T^{31} & T^{32} & T^{33}
\end{bmatrix}
\begin{array}{l} \\ \\ \text{shear stress} \\ \text{pressure} \end{array} \\
 & \begin{array}{cccc} & \text{momentum flux} & & \end{array}
\end{array}
$$

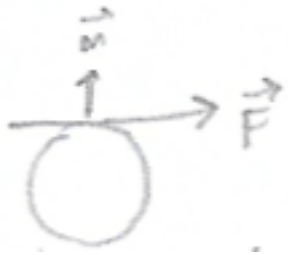
Note that all these quantities have dimensions of energy/volume.

The spatial part is the stress-tensor, containing isotropic and anisotropic pressure:

$$
\begin{bmatrix}
p_x & \pi_{ab} & \\
 & p_y & \\
\pi_{ab} & & p_z
\end{bmatrix}
$$




isotropic force :
produces expansion/contraction



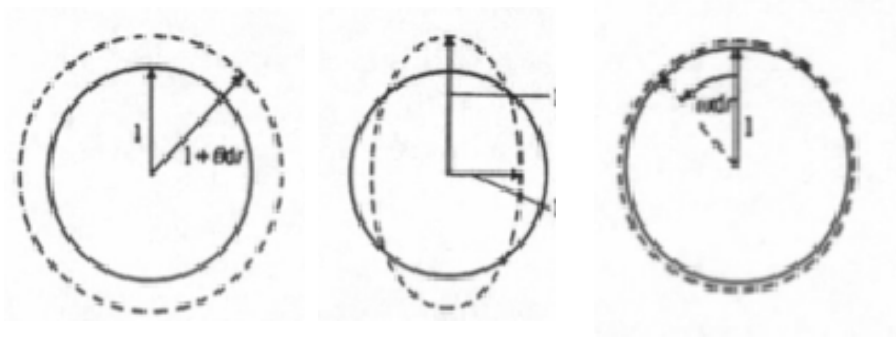

anisotropic force :
produces shear (ellipticity)

The stress tensor can be decomposed in three contributions:

$$\begin{bmatrix} p-\sigma_1 & 0 \\ 0 & p+\sigma_1 \end{bmatrix} + \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & -\sigma_1 \end{bmatrix} + \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}$$

(simplified for 2D spatial dimensions)

expansion shear rotation




However, in the homogeneous Universe, the metric is RW, and $T_{ab}$ is forced to have the same symmetries → off-diagonal terms are necessarily zero, and the energy-momentum tensor is that of a **perfect fluid**:

$$T_{ab} = (\rho+p)u_a u_b + p\, g_{ab}$$

$$T_{00} = (\rho+p)(-1)^2 + p(-1) = \rho+p-p = \rho \quad \longrightarrow T^0{}_0 = -\rho$$

$$T_{ii} = 0 + p = p \quad \longrightarrow T^i{}_i = p$$

$$T_{ab} = \begin{bmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix}$$

**Density and pressure are thus the only relevant quantities. How can they be computed (for each constituent of the cosmological fluid)?**

The elements of the energy-momentum tensor may be obtained from the **least action principle**, by varying the action with respect to the metric.

The **action** is

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}$$

It is defined in the 4D volume, and th integration measure must include the determinant g of the metric.

In cartesian coordinates, t,x,y,z :

$$g_{ab} = \begin{bmatrix} -1 & & & \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{bmatrix} \Rightarrow g = -a^6 , \qquad \sqrt{-g} = a^3$$

In GR, the **Lagrangian** density of the **homogeneous and empty Universe** is given by the Ricci scalar:

$$\mathcal{L} = \frac{R}{16\pi G}$$

So we need to compute the derivative of the action with respect to the metric $\frac{\delta S}{\delta g^{ab}}$

$$\delta S = \frac{1}{16\pi G}\,\delta\left(\sqrt{-g}\,R\right)\,\delta g^{ab} = 0$$

We need to compute,

$$\delta\left(\sqrt{-g}\,R\right) = \delta\left(\sqrt{-g}\,g^{ab}R_{ab}\right) = \delta\left(\sqrt{-g}\right)g^{ab}R_{ab} + \sqrt{-g}\,g^{ab}\,\delta R_{ab} + \sqrt{-g}\,R_{ab}\,\delta g^{ab}$$

The derivative of the determinant is

$$\delta g = g\,g_{ab}\,\delta g^{ab}$$

$$\Rightarrow \delta\sqrt{-g} = \frac{1}{2}(-g)^{-1/2}\,\delta g = -\frac{1}{2}\frac{1}{\sqrt{-g}}\,g\,g_{ab}\,\delta g^{ab} = -\frac{1}{2}\sqrt{-g}\,g_{ab}\,\delta g^{ab}$$

Inserting above, we write,

$$\delta\left(\sqrt{-g}\, g^{ab} R_{ab}\right) = -\frac{1}{2}\sqrt{-g}\, g_{ab} R\, \delta g^{ab} + \sqrt{-g}\, g^{ab}\, \delta R_{ab} + \sqrt{-g}\, \delta g^{ab} R_{ab}$$

$$= \underbrace{\sqrt{-g}\left(R_{ab} - \frac{1}{2} g_{ab} R\right)}_{\text{This is } T_{ab} \text{ (from Einstein eq.)}} \delta g^{ab} + \sqrt{-g}\, g^{ab} \underbrace{\delta R_{ab}}_{\text{the derivation of the Ricci tensor is zero}}$$

$$\delta R_{ab} = 0$$

and so:

$$\delta S = 0 \quad (*)$$

$$\delta S = \frac{1}{16\pi G}\sqrt{-g}\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = 0$$

$$\Rightarrow \quad R_{ab} - \frac{1}{2} g_{ab} R = 0 \quad \Rightarrow \quad T_{ab} = 0$$

This is the result for the energy-momentum tensor of the empty Universe (which is zero, as expected)

Now, if we consider a **matter-energy component**, described by a Lagrangian L, the action becomes

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + L \right)$$

and the variational principle leads to:

$$\delta S = 0 \iff \delta \left( \frac{\sqrt{-g}\, R}{16\pi G} + \sqrt{-g}\, L \right) = 0 \iff \sqrt{-g}\, \frac{8\pi G\, T_{ab}}{16\pi G} + \delta\left(\sqrt{-g}\, L\right) = 0$$

$$\iff \boxed{T_{ab} = -\frac{2}{\sqrt{-g}}\, \delta\left(\sqrt{-g}\, L\right)}$$

This shows that if we consider a (energy/particle) field in the Universe (for example a dark matter particle or a dark energy field) described by a potential V and a field $\phi$, we can compute its energy-momentum tensor elements as function of the field's V and $\phi$ → the theoretical approach

If a Lagrangian cannot be computed for the new field, we can alternatively give directly a prescription for the density and pressure → the phenomenological approach

# The Einstein equations

Having the Einstein and the energy-momentum tensors, we can write the Einstein equations

$$G_{00} = \frac{f'^2 + 2ff'' - 1}{a^2 f^2} - \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{c^4}\rho$$

$$G_{11} = \frac{f'^2 - 1}{a^2 f^2} - \frac{1}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{2\ddot{a}}{a}\right] = \frac{8\pi G}{c^4} p$$

$$G_{22} \text{ or } G_{33} \rightarrow \frac{1}{a^2}\frac{f''}{f} - \frac{1}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{2\ddot{a}}{a}\right] = \frac{8\pi G}{c^4} p$$

Note that for example for K > 0

$$f'_K(\chi) = \cos(K^{1/2}\chi)$$

$$f''_K(\chi) = -K^{1/2}\sin(K^{1/2}\chi)$$

and so $(f'^2-1)/f^2 = -K$ and $f''/f = -K$

Inserting this in $G_{00}$ we get $G_{00} = \frac{3K}{a^2} + \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2$

and the 00 element of the Einstein equations is:

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}}$$

**Friedmann equation**

Adding the elements 00 and ii we get a second equation:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)}$$

**Second Friedmann equation**

The second Friedmann equation turns out to be the **Raychaudhuri equation** for the case of a perfect fluid in the RW space-time.

To see this, consider the general **decomposition** of a (3D timelike) vector field:

$$u_{i;j} = \frac{1}{2}(u_{i;j} + u_{j;i}) - \frac{1}{3}h_{ij}u_{k;k} + \frac{1}{2}(u_{i;j} - u_{j;i}) + \frac{1}{3}h_{ij}u_{k;k}$$

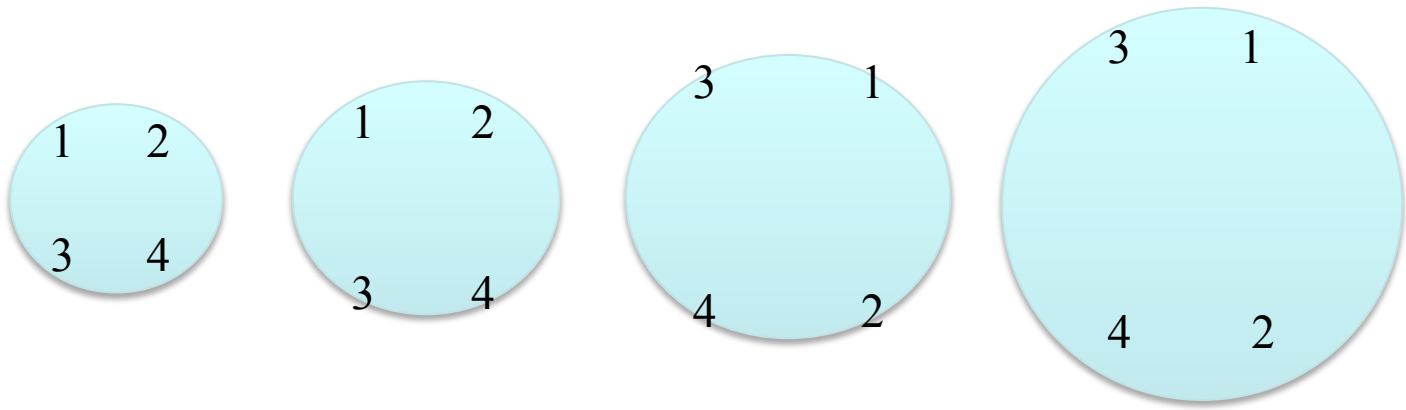
symmetric and traceless: shear $\sigma$

anti-symmetic: rotation $\omega$

trace (diagonal): expansion $\theta$ (or convergence)

$h_{ij}$ is the metric projected in the 3D space defined by the vector field

An important application is the case of the vector field of a set of comoving galaxies in the cosmic flow.




$t_1$

$t_2$ : expansion

$t_3$ : expansion and rotation

$t_4$ : expansion and shear

We are interested in **the time evolution of this vector field**.

In particular, working from the expression for the decomposition, the evolution of the expansion/convergence θ = div(u) is found to be given by:

$$\ddot{\theta} + \frac{1}{3}\theta^2 + \sigma^2 - \omega^2 = + R_{ab} v^a v^b$$

**This equation for the evolution of the trace of the separation vector of a congruence of timelike geodesics is the Raychadhuri equation**

From the Einstein equation we get:

$$R_{ab} u^a u^b = -4\pi G(\rho + 3p)$$

and using comoving coordinates:

$$\theta = v_{i;i} = \mathrm{div}(\dot{a}\chi) = \mathrm{div}\left(\frac{\dot{a}}{a} r\right) = \frac{\dot{a}}{a}\mathrm{div}(r) = \frac{\dot{a}}{a} 3 \qquad (r = a\chi)$$

note the factor 3, that comes from the divergence in 3D, like in the ij terms of the Einstein equations

and so

$$\dot{\theta} = \frac{d}{dt}\left(3\frac{\dot{a}}{a}\right) = 3\,\frac{\ddot{a}a - \dot{a}\dot{a}}{a^2} = 3\frac{\ddot{a}}{a} - 3\left(\frac{\dot{a}}{a}\right)^2$$

Inserting in the Raychaudhuri equation we get:

$$\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) - (\sigma^2 - \omega^2)$$

We see that **density** and **pressure** are sources of attractive gravity → **contributing to attraction, or decelerated expansion**

**Shear** is also a source of attractive gravity

**Rotation** is a source of repulsive gravity → **contributing to repulsion, or accelerated expansion** (like a centrifugal force)

So a rotating cosmological fluid could be an alternative to dark energy. But from where would it get its rotation → also from some mysterious extra energy? from internal coherent rotation of all dark matter particles at a fundamental level?

However, for a perfect fluid in a homogeneous and spherical symmetric universe, shear and rotation are zero, and we see that the Raychadhuri equation is indeed identical to the second Friedmann equation.

The two Friedmann equations are **constraint equations**, connecting the field (i.e. the metric) quantities, a and K, to the source quantities ρ and p.

(Note that a, ρ and p evolve in time, while K is constant)

The evolution of the sources are determined by **energy conservation equations**, which in principle are independent from the Einstein equations.

Energy conservation equations are equivalent to conservation of the energy-momentum tensor:

$$T^{ab}{}_{;b} = 0$$

From here, in principle we can get 4 energy conservation equations.
For the RW metric and perfect fluid, there is only one:

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0}$$

**Continuity equation**

In the RW case, the continuity equation is already contained in the Einstein equations and can be found by combining the time-derivative of Friedmann equation with the second Friedmann equation:

Derivative $\rightarrow$ $$\left(\frac{\ddot{a}}{a}\right) 2\frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^3 2 = \frac{8\pi G}{3}\dot{\rho} + 2K\dot{a}\frac{1}{a^3}$$

Insert $$\left[\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho+3p)\right. \Rightarrow \overbrace{-\frac{8}{3}\pi G(\rho+3p)\frac{\dot{a}}{a}} - \left(\frac{\dot{a}}{a}\right)^3 2 = \overbrace{\frac{8\pi G}{3}\dot{\rho}} + 2K\dot{a}\frac{1}{a^3}$$

Now, what is $$\left(\frac{\dot{a}}{a}\right)^3 2 + 2K\dot{a}\frac{1}{a^3}$$

Insert again eq. Friedmann $\rightarrow$ $$\left(\frac{\dot{a}}{a}\right)^2 2\frac{\dot{a}}{a} + 2K\dot{a}\frac{1}{a^3}$$

$$\left(\frac{8\pi G}{3}\rho\right) 2\frac{\dot{a}}{a} + \cancel{\frac{K}{a^2} 2\frac{\dot{a}}{a}} - \cancel{2K\dot{a}\frac{1}{a^3}}$$

So,

$\Rightarrow$ Derivative of Fried. eq. is,

$$-\frac{8\pi}{3}G(\rho+3p)\frac{\dot{a}}{a} - \frac{8\pi G}{3}\dot{\rho} = \frac{8\pi G}{3}\rho\,\frac{2\dot{a}}{a}$$

[d Fried + Fried + Raych = Conv.]

$$-\rho\frac{\dot{a}}{a} - 3p\frac{\dot{a}}{a} - \dot{\rho} - \rho\, 2\frac{\dot{a}}{a} = 0$$

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho+p) = 0}$$

Conservation eq.

or Continuity eq.

## The cosmological fluid

We saw that the cosmological fluid is described by density and pressure.
In addition, there are in general several different **species** in the cosmological fluid, and so we need to consider the densities and pressures of all of the species.

In general, for each species, the properties are not independent. For a perfect fluid, density and pressure are related through an **equation-of-state** w(t) (like in thermodynamics, when p,V,T, etc may be related under certain conditions).

$$w(t) = p(t) / [\rho(t)\, c^2]$$

Note: in the inhomogeneous universe, perturbations in denstity may also be related to perturbations in pressure. That relation determines the speed of sound in the fluid.

Note: there may also be constraints, relating the density and pressure of different species.

**We have one equation of energy conservation, involving density and pressure. For some species, the pressure is known, or may be determined independently. For those case, the continuity equation may then be solved to find ρ(t).**

## Matter

Matter (of any type: baryonic or dark matter) is defined by

$$p=0$$

In this case, the continuity equation is easily solved:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \text{or} \quad \frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \quad \text{or} \quad \frac{1}{\rho}\,d\rho = -3\,\frac{1}{a}\,da$$

$$\frac{d}{dt}\ln\rho = -3\,\frac{d}{dt}\ln a \quad \Rightarrow \quad \rho \propto a^{-3}$$

This means the density dilutes linearly with the expansion of the volume $a^3$.

Note that the factor 3 comes from having 3 spatial dimensions. In the Einstein equations this appears from terms like $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j$, and $\delta^i_i = 3$

In the Newtonian derivation, it appeared more directly from the volume in the first law of thermodynamics: $d/dt\,(a^3) = 3a^2\dot{a}$

Note that we found the functional form of $\rho(t)$ but not its amplitude.
In reality the solution is

$$\ln \rho + C_1 = -3 \ln a + C_2,$$ where $C_1$ and $C_2$ are integration constants.

We can choose the constants in any form we wish. A usual is to choose the "initial" conditions at $t_0$ = today. The solution is then:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3}$$

With $a_0 = 1$, we are left with 1 free parameter: **the matter density today $\rho_{m,0}$**

## Radiation

Radiation is the flux of relativistic particles present in the Universe, mostly photons from the CMB, but also neutrinos from the cosmic neutrino background. They have radiation pressure that will be a source of gravity, in addition to their energy density.

Let us compute this pressure.

Fluid made of radiation. Particles move with $c$ $\Rightarrow$ density fluctuations (sound or pressure waves) should propagate with $c_s = c$. So $p$ should be $p = \rho$?

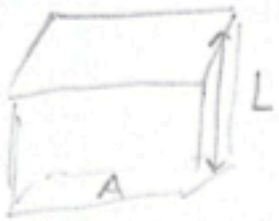
Let us try to compute it:

Consider 1 photon arrives at an area $A$


$$p = \frac{F}{A}$$

A photon distribution arrives. At which rate?

Put a minimum to get a rate:




$V = AL$

photons bounce

$$\rho_{1\gamma} = \frac{E}{V} = \frac{E}{AL}$$

$$P_{1\gamma} = \frac{F}{A} \quad \text{or} \quad = \frac{\text{moment transfer}}{\text{time} \times A}$$

A photon arrives every $t = \frac{2L}{c}$ (bouncing)

and transfers a moment $2p$

So $\text{Pressure}_\gamma = \frac{2p}{\frac{2L}{c} A} = \frac{pc}{V} = \frac{E}{V} \Rightarrow \underline{P_{ress\,\gamma} = \rho_\gamma}$

So, is w = 1 ?

But note that this result is only valid for 1D — when all photons come from the same direction.

Each photon arrives from a direction $\theta$

$\Rightarrow$ momentum transfered is just $2p\cos\theta$ and not $2p$

and $t = \dfrac{2L}{c\cos\theta}$ (travels $\dfrac{2L}{\cos\theta}$ inside the box)

$$\Rightarrow \text{Pressure}_\gamma = \frac{2p\cos\theta}{\frac{2L}{c\cos\theta}A} = \frac{pc}{LA}\cos^2\theta = \underbrace{P_\gamma \cos^2\theta}$$

The photons come isotropically from all directions $\Rightarrow$ on average they come from the mean direction

Mean of $\cos^2\theta$ in 3D is:

$$\langle\cos^2\theta\rangle = \frac{\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\, d\theta\, \text{sen}\,\theta\, d\varphi}{\int_0^{2\pi}\int_0^{\pi} d\theta\, \text{sen}\,\theta\, d\varphi} = \frac{2\pi\int_0^{\pi}\cos^2\theta\, \text{sen}\,\theta\, d\theta}{2\pi\int_0^{\pi}\text{sen}\,\theta\, d\theta}$$

$$\begin{pmatrix}\cos^2\theta = x \\ dx = -\text{sen}\,\theta\, d\theta\end{pmatrix} \rightarrow \langle\cos^2\theta\rangle = \frac{-\int_1^{-1} x^2\, dx}{-\int_1^{-1} 1\, dx} = \frac{\int_{-1}^{1} x^2\, dx}{\int_{-1}^{1} dx} = \frac{x^3/3\,\Big|_{-1}^{1}}{x\,\Big|_{-1}^{1}} = \frac{1/3 \times 2}{2} = 1/3$$

So on average each photon only has $P_\gamma = \frac{1}{3}\rho_\gamma$

And for the full fluid of N photons $\rightarrow$ $P = \frac{1}{3}\rho$ $\quad w = 1/3$, while in 1D $w = 1$

The result is then $w = 1/3$

We know the pressure of the radiation species, now we can find the density evolution:

$p = 1/3\,\rho$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\frac{4}{3}\rho \Rightarrow \dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \Rightarrow \frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \Rightarrow d\ln\rho = -4\,d\ln a$$

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-4}$$

The radiation energy density dilutes faster than the matter density, and this is because it is affected by both expansion and redshift.

Again at $a_0 = 1$, we have found another cosmological parameter: **the radiation density today $\rho_{r,0}$**.

Note that from the Stefan-Boltzmann law: $\rho_r \sim T^4$

(this is the temperature of the radiation fluid, which is the temperature of the Universe), and given the density evolution → this implies that $T \sim 1/a$

So T is also a unique indicator of the instants in the Universe evolution, just like the redshift : they provide **model-independent** indicators of the Universe events.

## “Attractive” vs “Repulsive”

From the comparison of the two cases (matter and radiation), we see that for a species with **constant (non-evolving) pressure**, the solution for the density evolution is:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(w+1) \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \rho \propto a^{-3(1+w)}$$

On the other hand, from the second Friedmann equation we see that

$$\ddot{a} > 0 \text{ if } \rho + 3p < 0 \Rightarrow 3p < -\rho \Rightarrow p < -\frac{1}{3}\rho \Rightarrow w < -1/3$$

So, the higher is w of a species, the faster is the dilution of its energy density, and the stronger is its contribution to gravity → a faster deceleration of the Universe (or a contraction, if the Universe had not started from an expanding beginning)

Conversely, the lower is w, the slower is the dilution of its energy density (could even remain constant, or increase), and the weaker is its contribution to gravity (can even be repulsive if w < -1/3) → a slower deceleration of the Universe, or even an acceleration.

## “Model” vs “Cosmology”

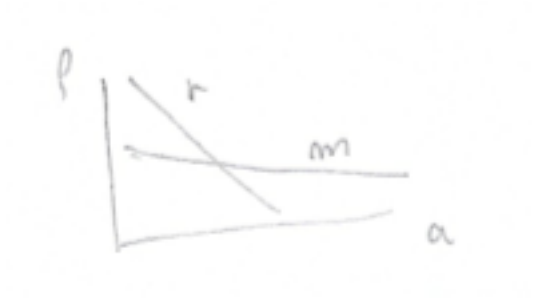
The choice of which species to include in the cosmological fluid (e.g., only matter, matter + radiation + dark energy) + their $T_{ab}$ properties (e.g., only density, anisotropic stress, type of w(t)) + the functional form of fundamental quantities of the Universe (e.g. ρ(t) ) derived from the equations of the theory → defines the **model.**

**Examples:** CDM, ΛCDM, Milne, Einstein-de Sitter, etc.

However, the models have free parameters and are only completely defined once the values of the parameters are known. The parameters values are constrained with observations → the set of parameter values for a given model defines the so-called **cosmology** (also sometimes confusingly called the model).

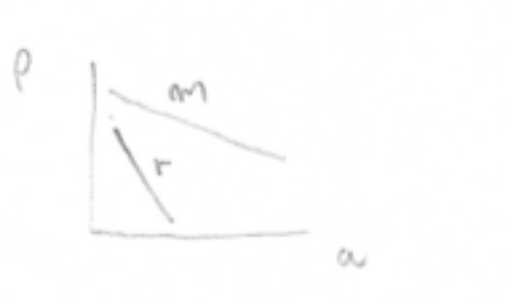
**Examples:** Concordance model (ΛCDM with $\rho_{m,0}$ = 30%, $\rho_{\Lambda CDM,0}$ = 70%, h = 0.70 )
Planck cosmology (ΛCDM with $\rho_{m,0}$ = 32%, $\rho_{\Lambda CDM,0}$ = 68%, h = 0.67 )

Another example: with different density parameter values, the **"matter + radiation" model** may have completely different properties:




$\rho_{r,0} << \rho_{m,0}$

Epoch of radiation and epoch of matter, CMB is an important feature, baryonic matter clusters slowly, DM needed




$\rho_{r,0} <<<< \rho_{m,0}$

No epoch of radiation, matter dominates at all times, CMB is not an important feature, baryonic matter may cluster fast, DM may not be needed

**The main purpose of the cosmological surveys is to constrain the parameters of the cosmological functions from astrophysical observations.**

Only with precise and accurate estimates of the cosmological parameters can the cosmological model be fully established.

## Curvature

Inserting the density evolution of matter and radiation in Friedmann equarion, this becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\frac{\rho_{m,0}}{a^3} + \frac{\rho_{r,0}}{a^4}\right] - \frac{\mathrm{K}}{a^2}$$

We see that the curvature term has the same structure of the others. So, even though the curvature is a parameter of the metric, it also has a gravitational effect like an effective density. By analogy, a curvature density may be defined, and its evolution is then,

$\rho_K(a) = \rho_{K,0}\, a^{-2}$,

introducing the parameter **curvature density today** $\rho_{K,0} = -3K/8\Pi G$

Note that to keep the structure of the equation, the curvature density is defined as the negative of the curvature, and so the negative curvature is the one that contributes to a positive density.

Now we can reason the other way around, and find the curvature pressure (or equation-of-state) associated to a density $a^{-2}$ evolution. It is -3 (1+w) = -2, i.e.,

w = - 1/3, in the limit between attractive and repulsive regimes

## Cosmological constant

The Einstein equation has the freedom to contain an extra degree of freedom that we did not yet consider, known as the cosmological constant $\Lambda$:

$$G_{ab} + \Lambda g_{ab} = 8\Pi G T_{ab}$$

This way, the Friedmann equation has a new term:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}$$

This allow to define an effective energy density with evolution: $\rho_\Lambda (a) = \rho_{\Lambda,0}\, a^0$, that keeps a constant amplitude during all the evolution of the Universe, given by the **cosmological constant density parameter** $\Lambda / 8\Pi G$

The equation-of-state associated to a constant is -3 (1+w) = 0, i.e.,

$w = -1$, well into the repulsive regime

This represents an extreme case of negative pressure (an outward tension), where $p = -\rho$. It is an unknown fluid, with exotic properties.

## Dark energy: phenomenological approach

Any cosmological species with w < -1/3, capable of producing acceleration may be considered dark energy, and this includes the cosmological constant.

Since the first observations that the Universe is accelerating, there has been great activity in building dark energy models.

An interesting approach is the **phenomenological approach**, where the functional form of the evolution of the density, or the pressure or the equation-of-state is imposed and parameterized.

This method produces models with more free parameters than the standard $\Lambda$CDM, which need then to be fitted by observations. There is no theory of the dark energy species to provide the functional forms.

### CPL

A popular method is to **parameterize the equation-of-state** based on a Taylor expansion around a pivotal redshift. This is known as the **Chevalier-Polarski-Linder** dark energy:

$$w(a) = w(a=1) + \frac{\partial w}{\partial a}\Big|_{a=1}(1-a) + \delta\,(1-a)^2$$

This species introduces three new cosmological parameters, the **equation-of-state today** $w_0$, the derivative of the **equation-of-state today** $w_a$, and the **dark energy density today** $\rho_{DE,0}$

$$w_0 = w(a=1)$$

$$w_a = \frac{dw}{da}(a=1)$$

If $w_a$ is set to zero and only $w_0$ is used (with $w_0 \neq -1$, otherwise it would just be ΛCDM) then this is called the **wCDM** model.

The density evolution for these models is computed as usual from the continuity equation,

$$d\ln\rho = -3\,(1+w(a))\,d\ln a$$

$$\Rightarrow \rho(a) = \rho_0\, a^{-3}\, e^{-3\int_0^a \frac{w(a)}{a}\,da}$$

For CPL, with $w(a)=w_0 + w_a\,(1-a)$, the density evolution is,

$$\rho(a) = \rho_0\, a^{-3(1+w_0+w_a)}\, e^{3w_a a}$$

Since current data favors the $\Lambda$CDM model, most dark energy parameterizations are built to have a **$\Lambda$CDM limit**, and the data best-fit to the DE parameters are usually values close to this limit.

In the case of CPL, this means **$w_0$ slightly larger than -1 and $w_a$ close to zero**. In this "cosmology", the CPL dark energy density stays constant in the early universe (as a cosmological constant), and as the scale factor approaches 1, the exponential term starts to dominate and the DE density increases, being able to produce a faster acceleration than $\Lambda$CDM.

Phenomenological models can thus be tuned to include the desired behaviors (in this case, a faster acceleration).

## UDM

Another example is the **Unified Dark matter - dark energy model**

This dark energy species proposes to behave both as matter and as dark energy, in different periods of the Universe.

To produce this behavior, the **parameterization is made on the density** and not on the equation of state. This way, we can directly choose the desired feature. One example is:

$$\rho = \begin{cases} \rho_t \left(\frac{a_t}{a}\right)^3 & a < a_t \\ \rho_\Lambda + (\rho_t - \rho_\Lambda)\left(\frac{a_t}{a}\right)^{3(} & a > a_t \end{cases}$$

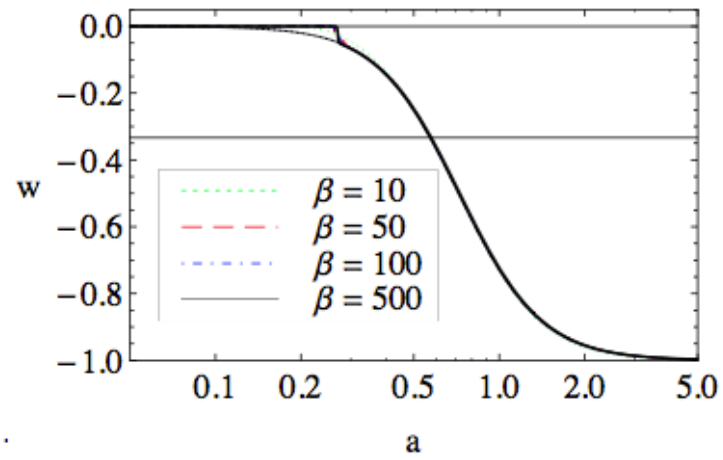
Before a transition scale factor $a = a_t$ the species behaves as dark matter, with the density decreasing with $a^3$. After the transition a constant term $\rho_\Lambda$ arises that will eventually dominate and in the late universe the density will tend to that constant value.

Note: one advantage of giving a **prescription for the density** instead of the equation-of-state, is that the continuity equation is a differential equation for the density, but does not involve derivatives of p or w.
This way, **having ρ, we can differentiate it and directly get p (and w)** without introducing an additional parameter for p (or w).
The other way around, **having p (or w), we need to integrate to get ρ**, which introduces an additional parameter.

This species introduces two new cosmological parameters, the **dark energy density today** $\rho_{UDM,0}$ , the **dark energy density at transition** $\rho_{UDM,t}$ (or alternatively the transition scale factor).

To ensure a fast but smooth transition between the two regimes, a Heaviside-type function can be used, which introduces a third parameter β.

The behavior of the equation-of-state, computed from the continuity equation is the expected one, starting at w=0 (matter) and reaching a=1 with w < -1/3 (dark energy)



## Dark energy: theoretical approach

Most dark energy models are not built phenomenologically, but are built as a **physical model**, defining its Lagrangian and deriving its energy-momentum tensor.

### Quintessence

Quintessence was one of the first physical DE models proposed, and it is based on a **scalar field** $\phi$.

$$L_\phi = \underbrace{+\frac{1}{2} g^{ab} \partial_a \phi \, \partial_b \phi}_{\text{kinetic}} \underbrace{- V(\phi)}_{\text{potential}}$$

Note: (For homogeneous $\phi = \phi(t)$ this is

$$L_\phi = +\frac{1}{2} \partial_0 \phi \, \partial^0 \phi - V(\phi) \quad )$$

or

$$\boxed{L_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi)}$$

As we saw, the energy-momentum tensor is computed from

$$T_{ab} = -\delta(\sqrt{-g}\,L_\phi)\frac{2}{\sqrt{-g}}$$

The goal is to obtain ρ and p as function of ϕ and V. This approach does not introduce additional free parameters for the density and pressure, but density and pressure parameters will be related to the model's underlying parameters: e.g., **amplitude of the scalar field, amplitude of the potential, slope of the potential**, etc. So, in this case, the observations will constrain the parameters of the physical model.

Now, computing $T_{ab}$ yields,

$$\delta(\sqrt{-g}\,L_\phi) = \delta(\sqrt{-g})\,L + \sqrt{-g}\,\delta L = -\frac{1}{2}\sqrt{-g}\,g_{ab}\,\delta g^{ab}\,L + \sqrt{-g}\left(+\frac{1}{2}\right)\underbrace{\delta g^{cb}\,\partial_a\phi\,\partial_b\phi}_{\text{No dependence of } \phi \text{ and } V \text{ on } g_{ab}} + 0$$

$$= -\frac{1}{2}\sqrt{-g}\,\delta g^{ab}\left(g_{ab}L - \partial_a\phi\,\partial_b\phi\right)$$

So we get $\Rightarrow$ $\boxed{T_{ab} = -\partial_a\phi\,\partial_b\phi + L\,g_{ab}}$

we may compute $T_{00} = -\partial_0\phi\,\partial_0\phi + \left(\frac{1}{2}\dot{\phi}^2 - V\right) g_{00} = -g_{00}\,\partial^0\phi\,\partial_0\phi + \frac{1}{2}\dot{\phi}^2 \cdot g_{00} - V g_{00}$ or

$$\text{or } T_{00} = \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + V \quad \text{or} \quad T_{00} = \frac{1}{2}\dot{\phi}^2 + V$$

Note: $T_{00} = g_{00} T^0{}_0 = -T^0{}_0$, como $T^0_0 = -\rho$

$\Rightarrow T_{00} = +\rho$

$$\Rightarrow \underbrace{\rho = \frac{1}{2}\dot{\phi}^2 + V}$$

$$\Rightarrow T_{11} = \overset{\partial_1\phi}{0} + L\, g_{11} = \left(\frac{1}{2}\dot{\phi}^2 - V\right) g_{11}$$

$$T_{11} = g_{11} T^1{}_1 = g_{11} p = p$$

$$\Rightarrow \underbrace{p = \frac{1}{2}\dot{\phi}^2 - V}$$

i.e., the cosmological scalar field has

(like the inflationary field)

$$\begin{cases} \rho = \frac{1}{2}\dot{\phi}^2 + V \\ p = \frac{1}{2}\dot{\phi}^2 - V \end{cases}$$

We have thus found ρ and p as function of ϕ and V.

Like for an inflationary field, the case of **slow-rolling** , i.e., $\dot{\phi}^2 << V$

leads to ρ ~V and p ~ - V → w ~ -1 and a dark energy behavior.

Now, we still need to find out the evolution ρ(a) and p(a). For this we need to know the evolution of the scalar field ϕ (a) and V (ϕ). This can be found from the Euler-Lagrange equation,

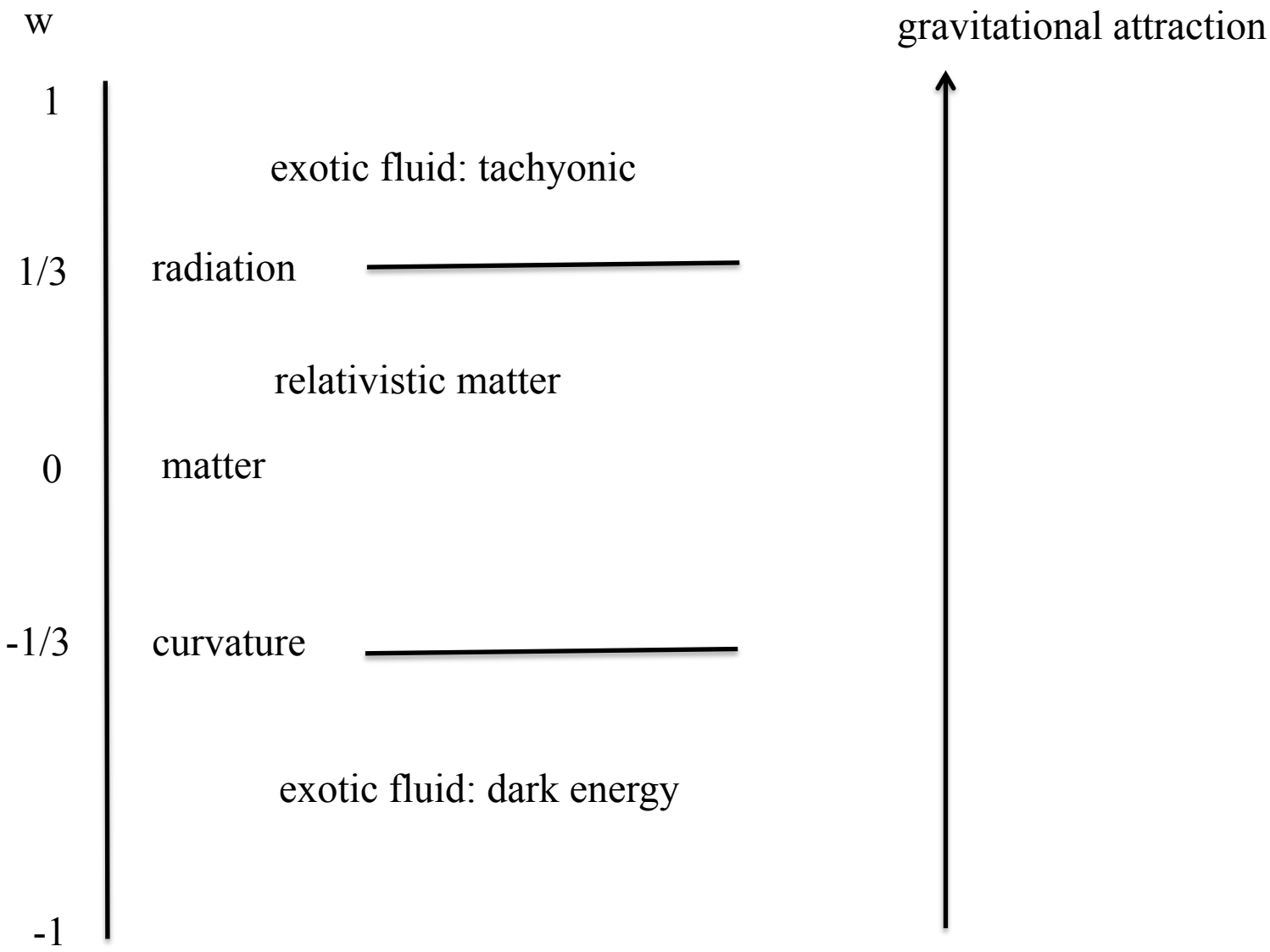
$$\frac{d}{dt}\left(\frac{\partial \sqrt{-g}\,L}{\partial(\partial^{\mu}\phi)}\right) - \frac{\partial(\sqrt{-g}\,L)}{\partial\phi} = 0$$

that leads to the equation of motion of the field: **the Klein-Gordon equation**:

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V' = 0}$$

In order to solve for ϕ (a), we nay need to fix V (ϕ), introducing additional free parameters. This choice defines a particular quintessence model.

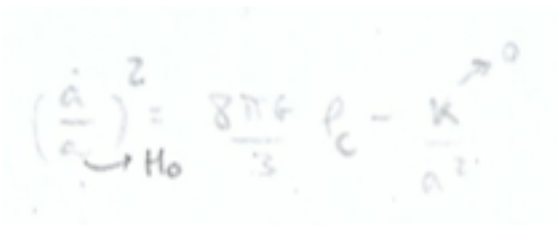
## Summary


w
gravitational attraction
1
exotic fluid: tachyonic
1/3
radiation
relativistic matter
0
matter
-1/3
curvature
exotic fluid: dark energy
-1

## Dimensionless density parameters

It is usual to normalize the density parameters $\rho_0$ by the **critical density.**

**The critical density** $\rho_c$ is the $\rho_0$ density today of a flat universe with a cosmological fluid containing only matter. So, from Friedmann equation:



$$\rightarrow \quad \rho_c = 3H_0^2 / 8\Pi G$$

Since in this case there is only one density parameter (one species), its value is completely determined by the Hubble constant, and there is no need to determine it independently from observations → **the Friedmann equation provides a constraint** → if there are N species there are only N-1 free density parameters.

It follows that

$\rho_c = 1.88 \times 10^{-26}\ h^2\ \mathrm{Kg\ m^{-3}}$ (where $H_0$ was left undetermined)

Note this is a very small value.

For example, if the volume between the Earth and the Moon,
$V = 4/3\ \Pi\ (384400\ \mathrm{km})^3$, would be filled with matter with this mean density, this would correspond to a mass of 2.2 Kg (assuming h=0.7).

So the mass in the form of dust in the "empty" solar system is larger than the mass in the same volume of the "empty" universe.

Now, normalizing the density parameters by the critical density, we define the dimensionless **Ω density parameters:**

$$\Omega_m = \rho_{m,0} / \rho_c\ ,\quad \Omega_r = \rho_{r,0} / \rho_c\ ,\quad \Omega_K = \rho_{K,0} / \rho_c = -K / H_0^2\ ,\quad \Omega_\Lambda = \rho_\Lambda / \rho_c = \Lambda / 3H_0^2$$

Note that with this definition, the Ω parameters are only defined today. There is no analogous definition of a $\Omega_m$ (a) function.

Note that because of the dependence of the critical density on the Hubble parameter, the values of Ω implicitly depend on the value of h. It is also usual to define h-independent parameters, called the **physical densities**:

$$\omega_m = \Omega_m h^2, \quad \omega_r = \Omega_r h^2, \text{ etc}$$

We can now write the Friedmann equation for the case of a cosmological fluid with matter, radiation, curvature and cosmological constant:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_{m,0}}{a^3} + \frac{8\pi G}{3}\frac{\rho_{r,0}}{a^4} - \frac{K}{a^2} + \frac{\Lambda}{3}$$

Inserting the critical density, we find:

$$H^2(a) = \frac{3H_0^2}{8\pi G}\left[\frac{8\pi G}{3}\frac{\Omega_m}{a^3} + \frac{8\pi G}{3}\frac{\Omega_r}{a^4} - \frac{K}{a^2}\frac{1}{\rho_c} + \frac{\Lambda}{3}\frac{1}{\rho_c}\right]$$

$$\boxed{H^2(z) = H_0^2\left[\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]}$$

Usually this part, factoring out $H_0$ is labeled E(z)

Note that at z=0, the Friedmann equation reduces explicitly to the density constraint condition:

$$\Omega_r + \Omega_m + \Omega_K + \Omega_\Lambda = 1$$