Probes of Geometry

The Astrophysical approach

Which FRW model better corresponds to the reality?

To answer this question, we need to **determine the values of its parameters H_0, Ω_i, w_i**

Then we will be able to:

determine the global geometry, dynamics, future behaviour (expansion, big bang, big crunch)

and also to:

determine the evolution of the universe - thermal history, transition epochs, structure formation

find out what energy forms exist in the universe (dark matter, dark energy, neutrinos)

Direct measurements

The best way to find the cosmological parameters is to estimate them from measurements of "cosmological functions", i.e., quantities like distances or power spectra that depend on the cosmological model.

Before presenting that approach in detail (based on model-dependent **indirect measurements** of cosmological properties), we will discuss an alternative approach, which is based on making **direct measurements** of some astrophysical properties that may provide good approximations to the values of the cosmological parameters.

Examples of these direct measurements are:

Determination of baryonic matter density

Big Bang Nucleosynthesis

Cosmic budget → measuring all the baryonic mass in the Universe should provide a good approximation to Ω_b

Determination of total mass and Mass-to-Light ratio (M/L)

Galaxy mass (from rotation curves)

Cluster mass (from various methods: kinematic, X-ray gas, strong gravitational lensing, weak gravitational lensing)

Determination of radiation density

CMB Temperature → directly gives the energy distribution of primordial CMB photons

Determination of the age of the universe

Nuclear chrono-cosmology (decay of radioactive elements)
Age of oldest stars (globular clusters) → lower-bound to the age of the Universe

Cooling of White Dwarfs

Determination of the Hubble constant (independently of the values of the density parameters)

Redshift drift → The redshift is a ratio between the scale factor at two different times. In a second observation of the same object, this ratio will be changed (because of acceleration) → **the redshift of a comoving object changes with time** → Measuring the redshift at different times gives information on H(z).

Calibration of the distance ladder
Gravitational lensing time-delays in double images of variable sources

Let us now discuss the direct measurements of the density parameters.

We will not discuss here the direct measurements of the Hubble parameter (which have recently become an active field again due to the so-called Hubble tension).

Radiation

$$\Omega_r = \Omega_\gamma + \Omega_\nu \text{ (relativistic)}$$

The main contribution to the cosmological radiation are the **CMB photons**.

The **energy density of the CMB photons** is found by summing up the energy of all photons. The CMB has a blackbody spectrum and so the energy distribution of the photons is well-known and is determined by the temperature.

The energy density is then the integral of hν with a window function (the Bose-Einstein distribution):

$$\rho_{CMB} = \frac{2}{(2\pi)^3} \int d^3\nu \, h\nu \, \frac{1}{e^{h\nu/kT} - 1} \qquad c=1$$

2 d.o.f. — Bose-Einstein distribution

(here h is the Planck constant)

$$\rho_{CMB} = \frac{1}{c^2} \frac{\pi^2}{15} \frac{(k_B T_{CMB})^4}{(\hbar c)^3} \simeq 4.5 \times 10^{-34} \, g/cm^3$$

using T_CMB = 2.725 K

Dividing by the critical density, $\rho_c = 3\,H_0^2 / 8\pi G = 1.88 \times 10^{-29}\,h^2\,g\,cm^{-3}$,

the dimensionless radiation density is $\Omega_\gamma = 2.4 \times 10^{-5}\,h^{-2}$

The **massless neutrinos** also give an important contribution to the radiation of the Universe. The **energy density of massless neutrinos** is computed in the same way, but using the Fermi-Dirac distribution instead and a different number of degrees-of-freedom:

Add massless neutrinos

Fermi-Dirac

Fermion factor

$$\rho_\nu = \frac{6}{(2\pi)^3} \int d^3E \; E \, \frac{1}{e^{E/kT_\nu}+1} = 6 \, \frac{7}{8} \, \frac{\pi^2}{30} \, T_\nu^4 = 3 \cdot \frac{7}{8} \left(\frac{T_\nu}{T_\gamma}\right)^4 \rho_{CMB}$$

6 d.o.f.

$\nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau$

From the thermal history of the Universe, we know that neutrinos decouple before the CMB, when the temperature was higher, such that:

$$\frac{T_\nu}{T_{CMB}} = \left(\frac{4}{11}\right)^{1/3}$$

So, their density is $\rho_\nu = 0.68\, \rho_\gamma$

$$\rho_{r\,\mathrm{total}} = \left(1 + 3 \cdot \frac{7}{8} \left(\frac{4}{11}\right)^{4/3}\right) \rho_{CMB} = 1.68\, \rho_{CMB}$$

and in terms of the dimensionless density parameter: $\Omega_r = \Omega_\gamma + \Omega_\nu$ **~ 0.00004 h^{-2}** (a negligible contribution to the density of the Universe today).

Note however, that **neutrinos are massive**, and the massless neutrinos scenario is only a good approximation when the temperature of the Universe is $T \gg M_\nu$.
Later in the Universe, neutrinos become non-relativistic fermionic particles and the **density of massive neutrinos** is computed as:

$$\rho_\nu = M_\nu n_\nu = M_\nu \frac{6}{(2\pi)^3} \int d^3 E \frac{1}{e^{E/kT} + 1}$$

using again:

$$\frac{T_\nu}{T_{CMB}} = \left(\frac{4}{11}\right)^{1/3}$$

The result is:

$$\Omega_\nu = M_\nu \frac{1}{94\ \mathrm{eV}} h^{-2}$$

The density depends on the neutrino mass (here M_v is the sum of the masses of the 3 neutrinos) and is no longer fully determined by the temperature.

For example, a neutrino mass of 0.1 eV would give a small but non-negligible contribution to the total energy density of $\Omega \sim 0.001$

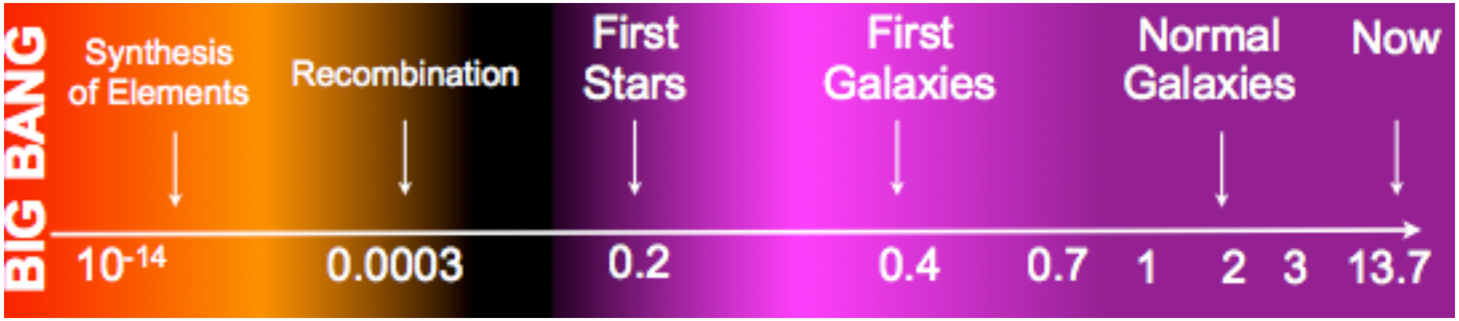
Neutrinos then contribute both to the radiation density and to the matter density.

An additional cosmological parameter N_eff (effective number of relativistic species) was introduced to model what fraction of neutrino density is considered relativistic and contributes to the radiation density, and what fraction is non-relativistic and contributes to the matter density affecting structure formation on small scales.

Baryonic matter

Ω_b

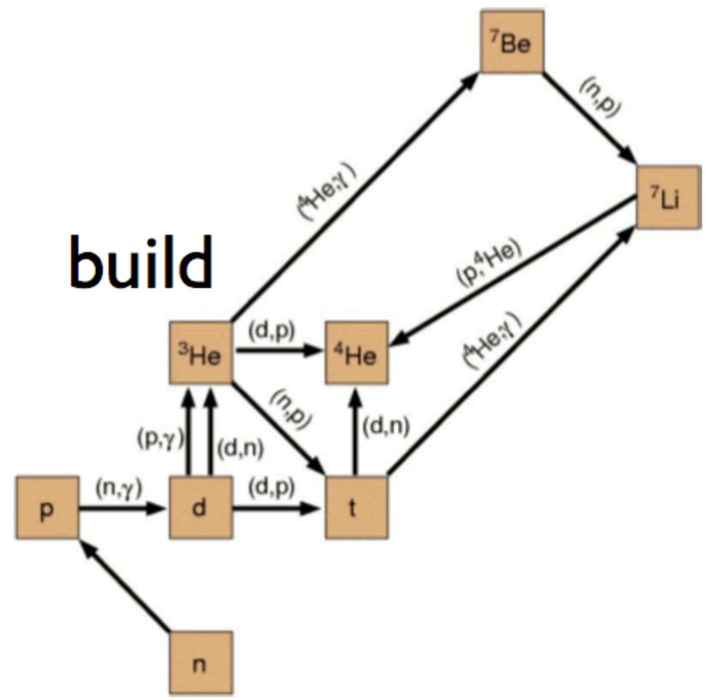
Its total density is determined by nucleosynthesis and also by cosmological probes (such as CMB anisotropies)



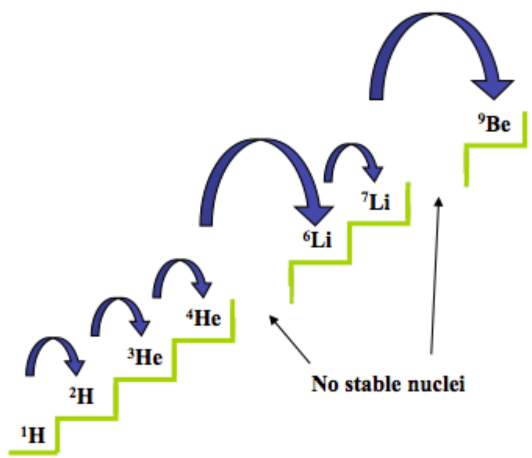
1. "Direct" measurement of Ω_b: nucleosynthesis

(nuclear fusion in early Big Bang)

The formation of elements during the first 2-5 minutes of the Universe

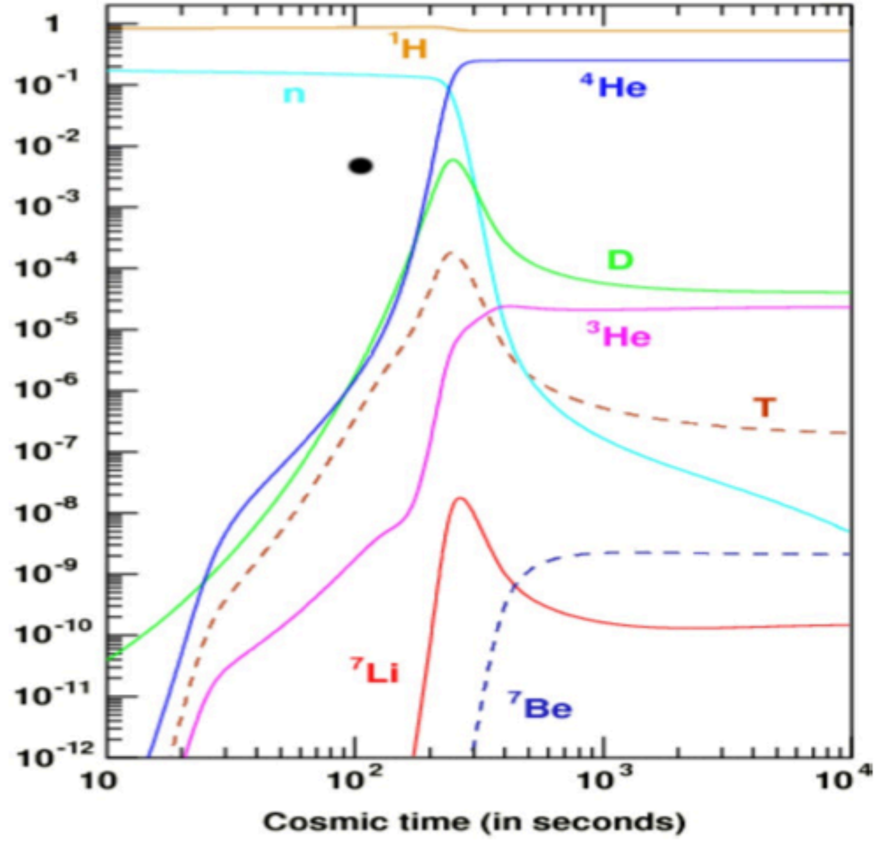


The lack of stable elements with masses 5 and 8 make it more difficult for nucleosynthesis to progress beyond Lithium and even Helium

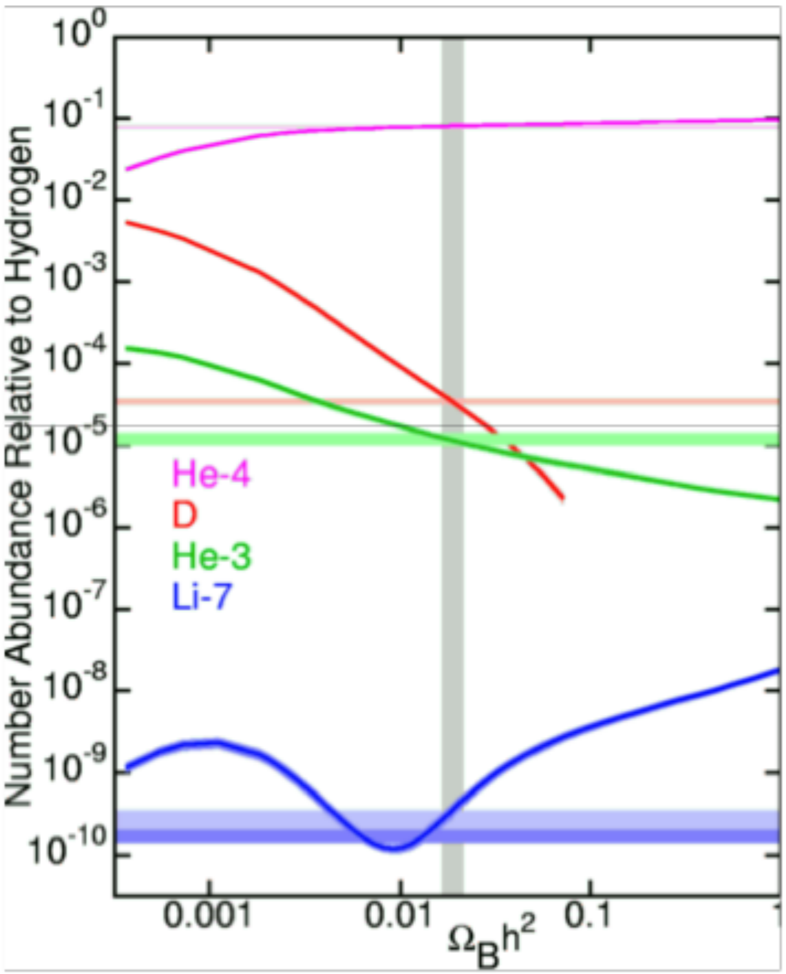


The **evolution of the abundances with time** (shown as M_element/M_H)

Formation of these elements is finished before 1000 s of cosmic time.



The interesting point is that **the reaction rate for element formation depends on the total amount of baryons present in the Universe** (before nucleosynthesis they are mainly in the form of protons and neutrons)



higher Ω_b → more He4 forms (the most stable species)

higher Ω_b → less D ou He3 form (because He4 is formed instead)

This provides a powerful way to estimate Ω_b : we just need to be able to measure the total amount of one of these species.

But this is difficult because **the abundances of the species do not remain constant**.

After star formation, stars destroy some elements and create others:

Deuterium → destroyed in stars from fusion

He 4 → produced from fusion

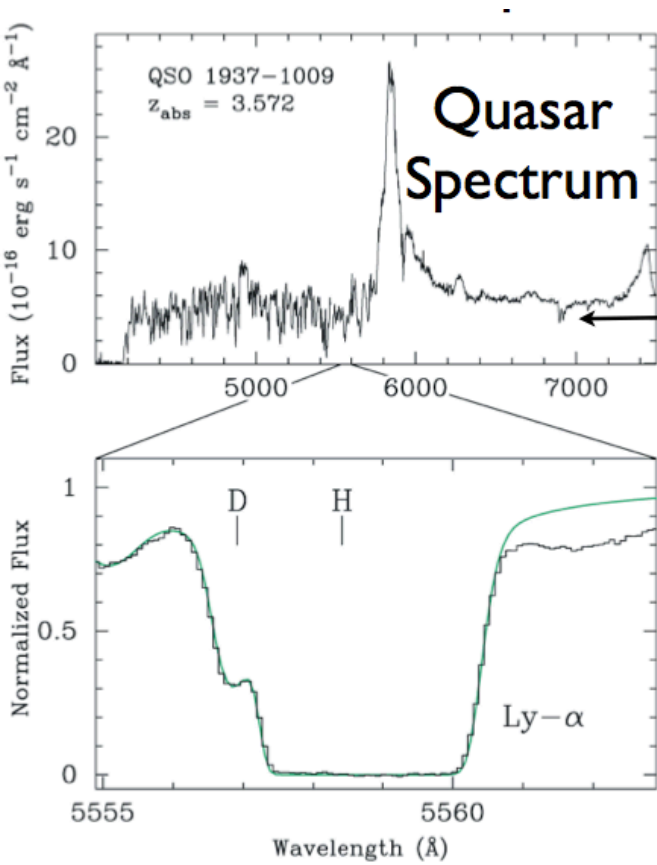
Li 7 → destroyed in stars from fusion and
also created in the interstellar medium from impact of cosmic-rays (spallation)

He 3 → produced by burning deuterium and
also destroyed to produce He 4

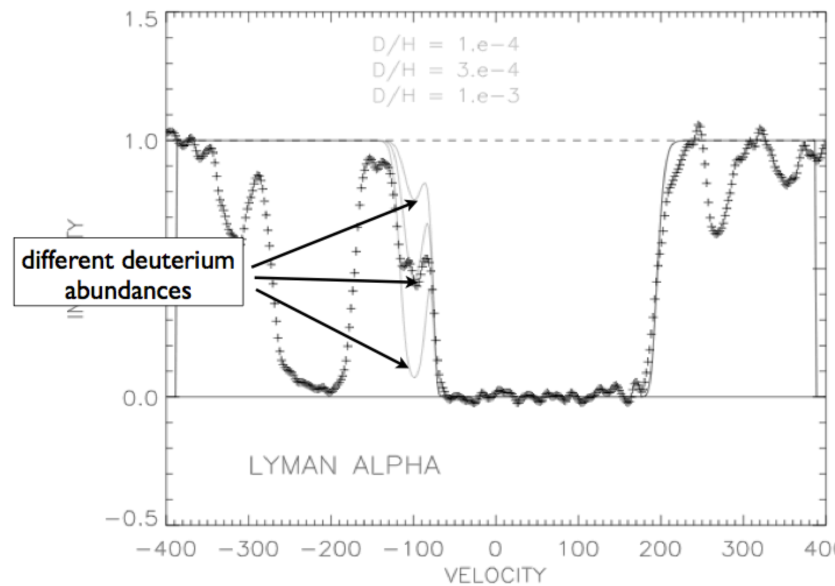
Measuring the abundances

Deuterium

Observe gas clouds in the early universe (where stars have not yet formed), looking for absorption features of rare elements (deuterium) on the spectrum of background bright sources (quasars)



to control, check if there are heavier elements



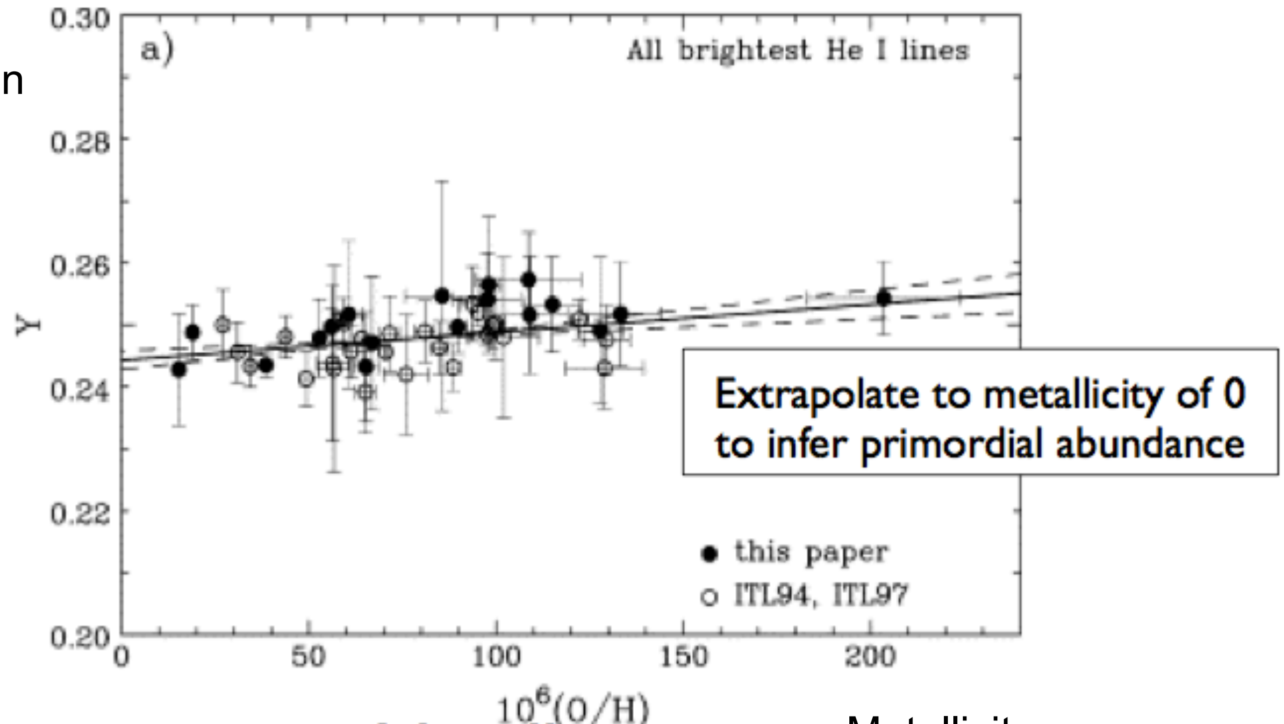
Result: [D/H] ~3 x 10^-5

Helium 4

Observe recombination lines from HII regions in low metallicity galaxies (oldest galaxies)

Measure abundance ratios of many elements He, O, N, H (metallicity)

Mass fraction of baryons in He 4

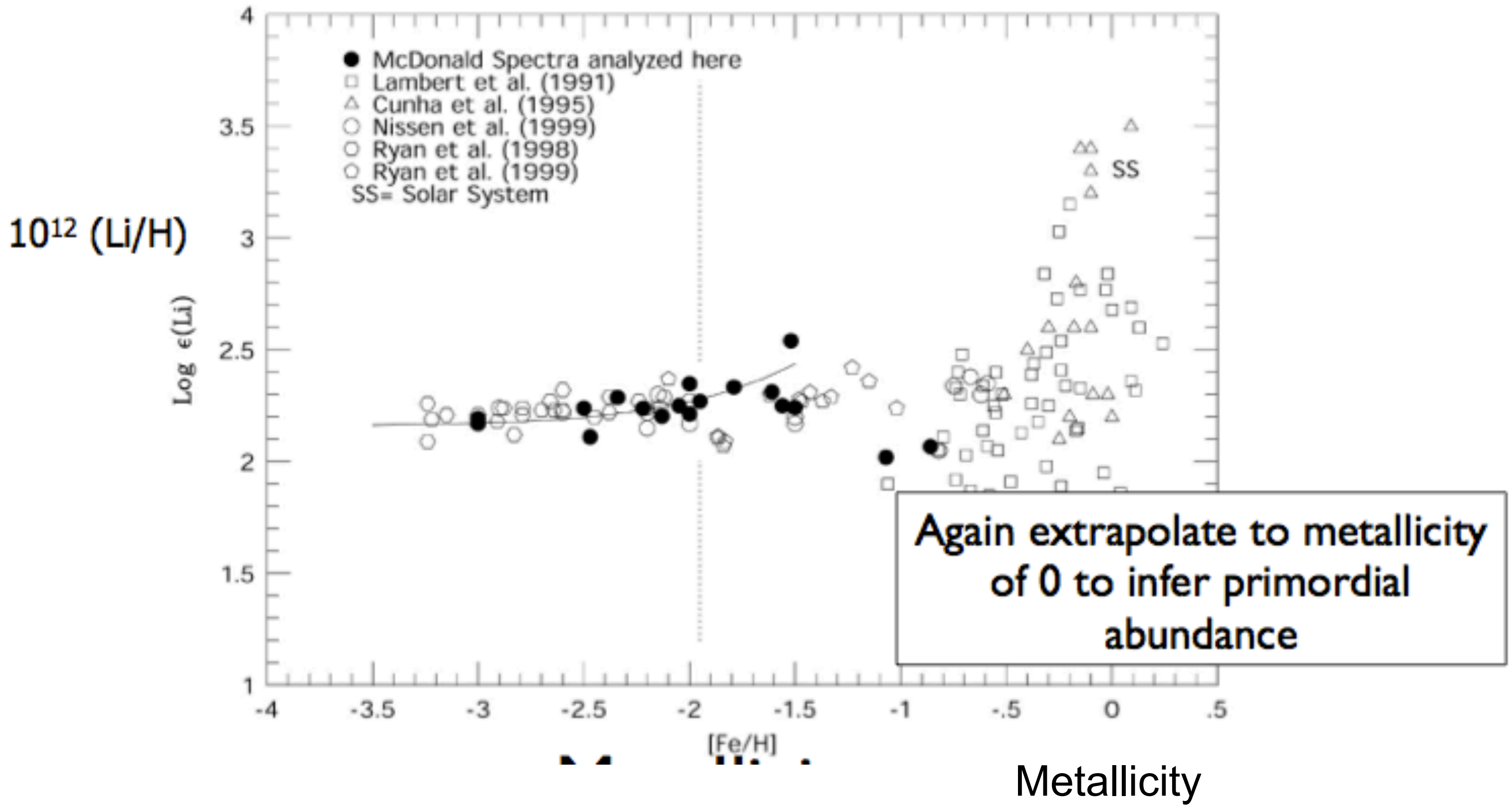


Metallicity

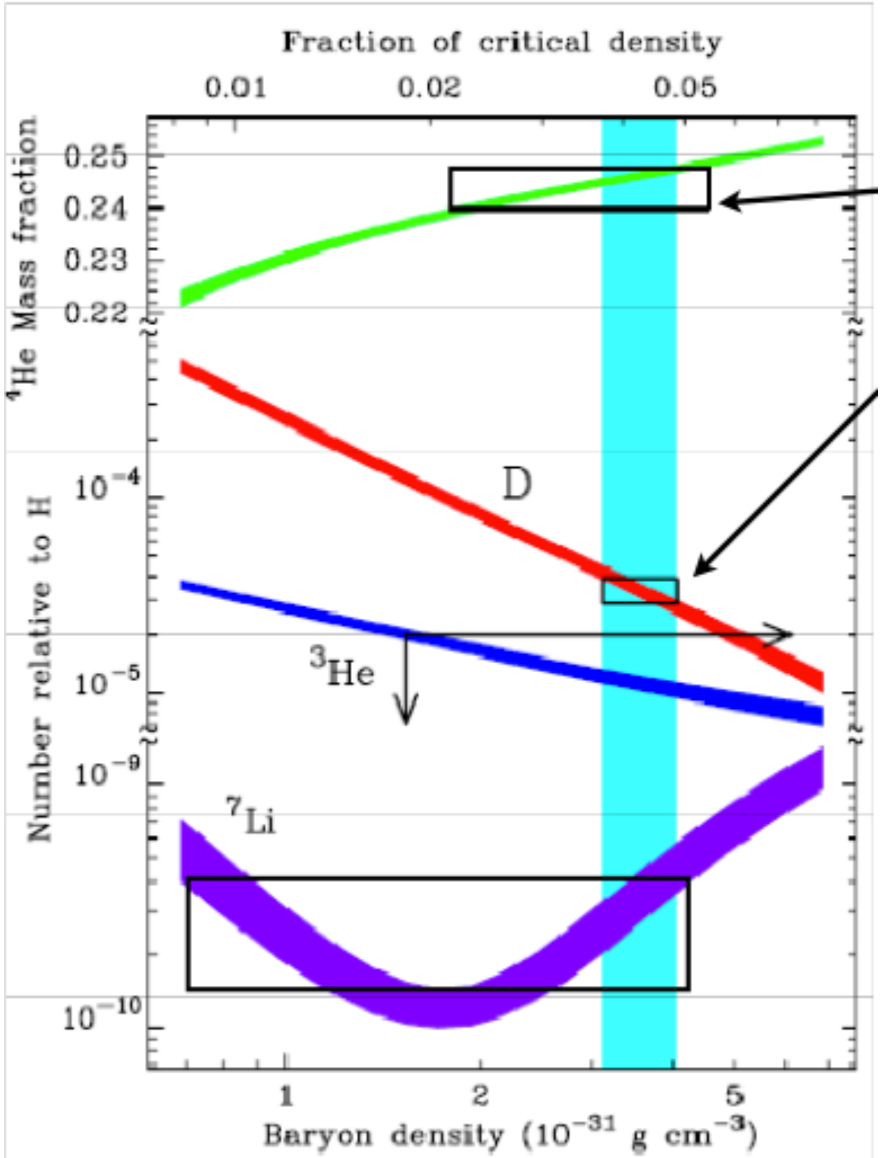
Lithium 7

Observe absorption in the atmospheres of cool, metal poor population II halo stars

Need to model the atmosphere of stars



Results



(the most useful result comes from Deuterium measurements)

Best Fit Baryon Density

$$\Omega_B h^2 = 0.019 \pm 0.0024$$

$$\Omega_B = 0.037 \pm 0.009$$

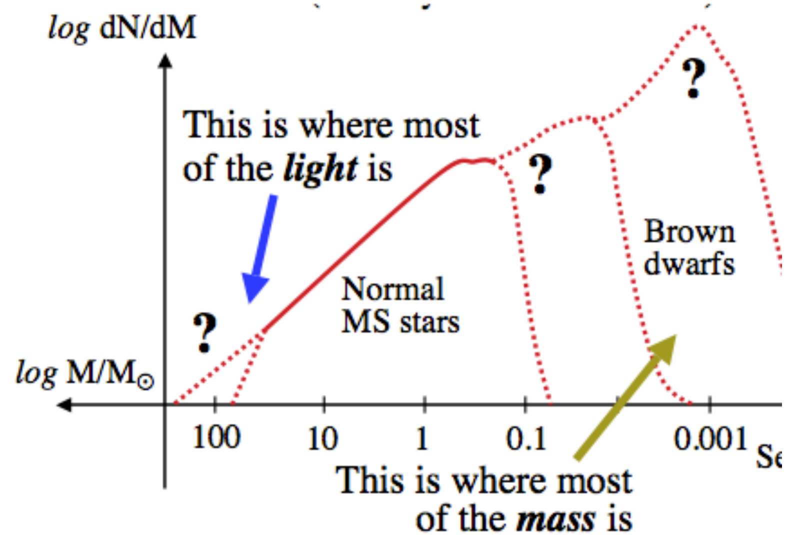
Ω_b ~ 0.04

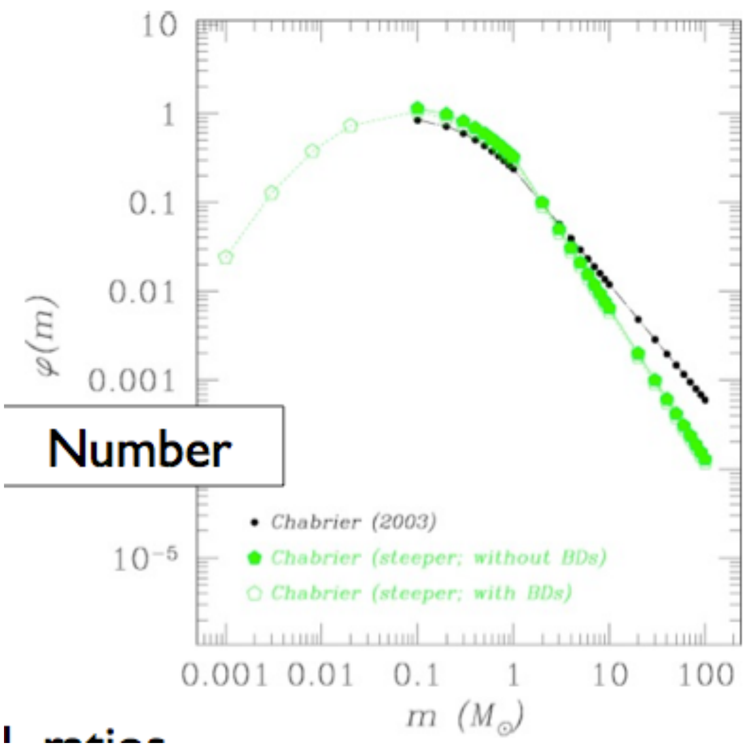
2. Cosmic baryon budget

We found out that $\Omega_b \sim 0.04$.

It would be interesting to “count all the baryons” and try to find out where is the baryonic matter → the cosmic baryon budget

Baryonic mass density in stars
(in galaxies)





Estimate mass from light

L ~ M^3
M/L ~ M^-2

low mass stars → high M/L ratios
high mass stars → low M/L ratios

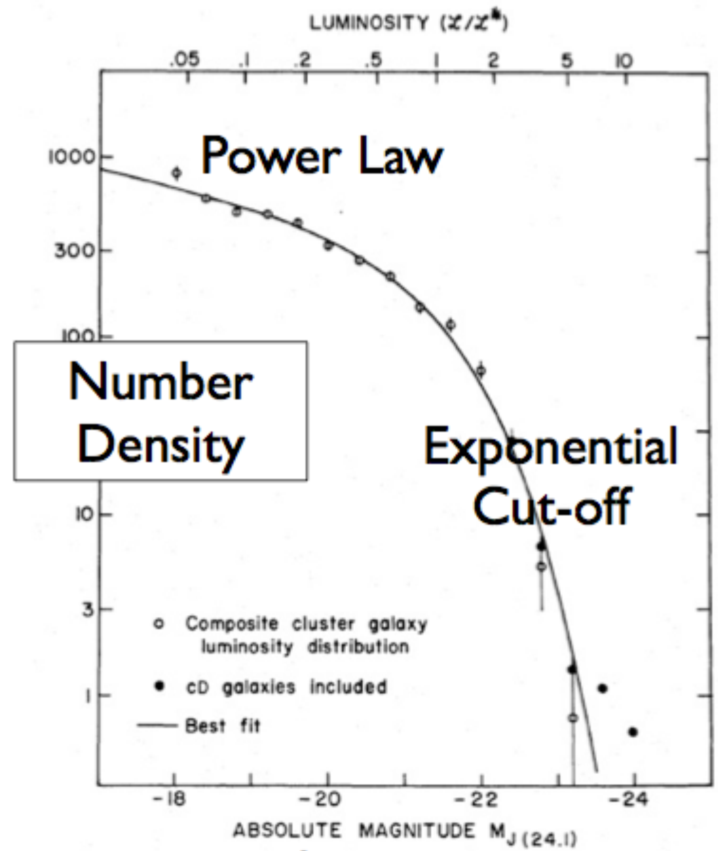
Integrating over the Initial Mass Function, we can compute an average M/L ratio.

complication: the M/L ratio of a population of stars (eg. stars from the same galaxy) depends also on the age

but age can be estimated from color: $\log_{10} M/L \sim -0.4 + 1.1\,(g-r)$

red galaxies → old
blue galaxies → young

Need to sum all luminosities (which are proxies for mass) and using the corrected mass functions, over the various populations (i.e., over many galaxies), using the luminosity function



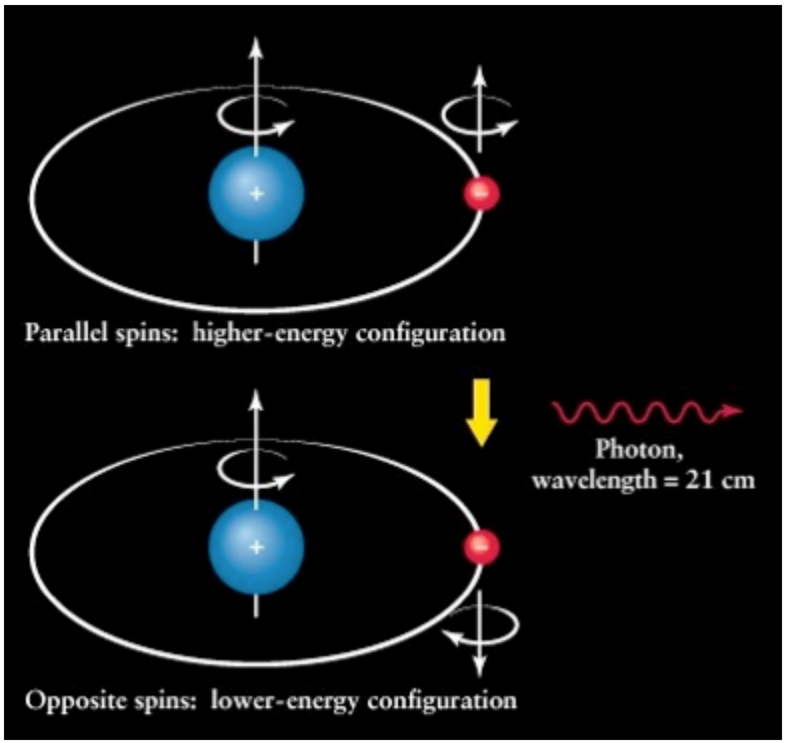
Results:

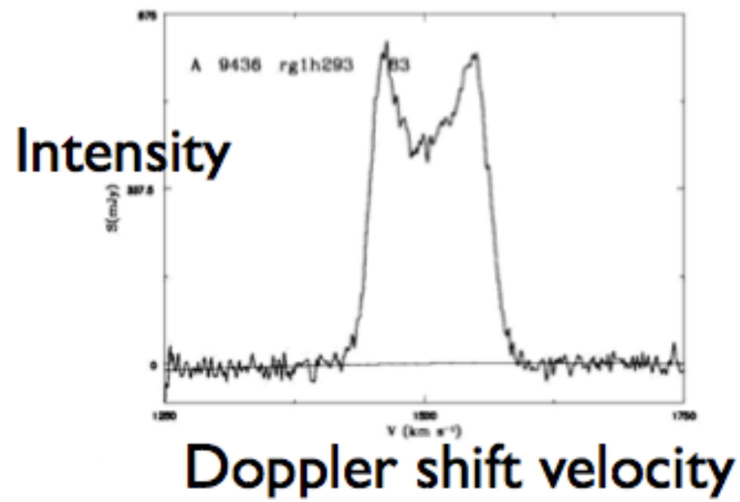
$$M_{stars} = 3 \times 10^8\ M_{sol}/Mpc^3$$

$$\Omega_{stars} = 0.002$$

Baryonic mass density in neutral atomic hydrogen
HI (in galaxies)

Hyperfine structure 21 cm





Spectra of a spiral galaxy at 21cm

Results: $\Omega_{HI\ gas} = 0.0003$

Baryonic mass density in molecular hydrogen H2
(in galaxies)

Difficult to search for H2 since it has no observable transitions

Assume CO emission is a good tracer of H2 (CO emission caused by H2 molecules colliding with CO)

Examine ratio of atomic hydrogen to molecular hydrogen in galaxies and then use this to convert from atomic hydrogen mass density

Results: $\Omega_{H2} = 0.0003$

So the total from galaxies is (very low, only 6.5% of the total baryonic density):

$$\Omega_{stars+gas} = 0.0026$$

Baryonic mass density in galaxy clusters (intra-cluster medium ICM)

Measured from X-ray photons coming from bremstrahlung radiation →

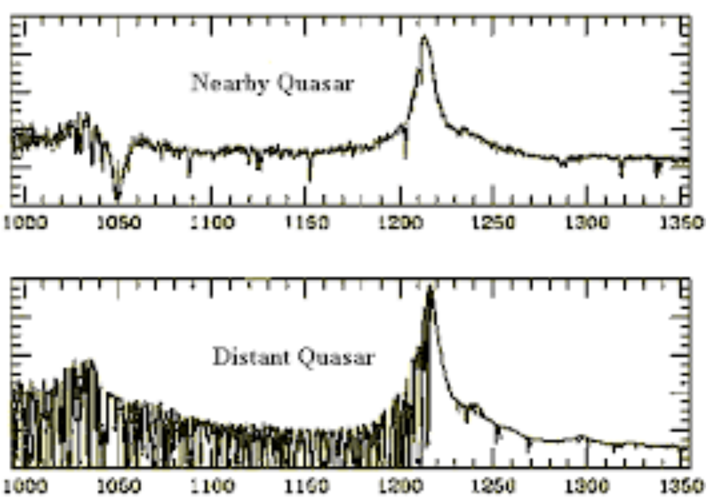
proportional to $\rho_{gas}^2 \times T_{gas}^{1/2}$

Measuring T and light → mass (isothermal)

Results: $\Omega_{ICM} \sim 0.001$

Baryonic mass density in between galaxies (inter-galactic medium IGM)

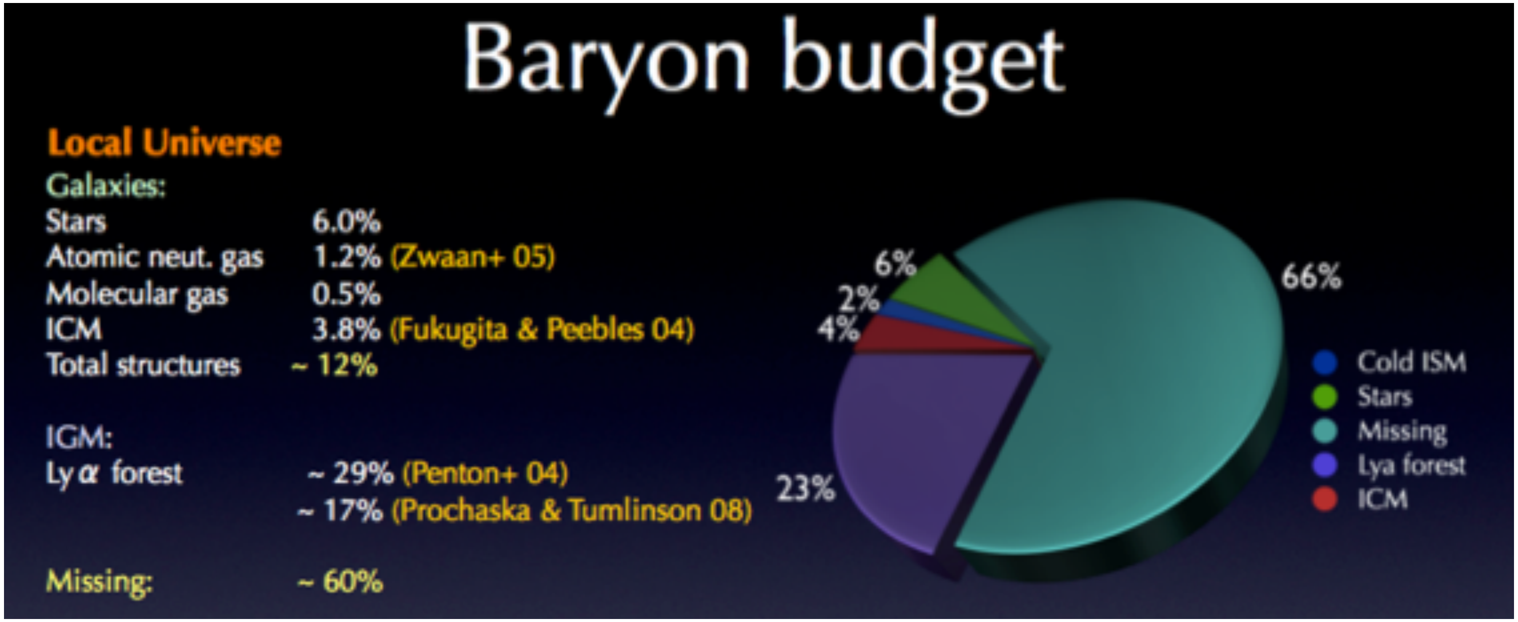
Measurements of Lyman_α forest (absorption by IGM clouds of photons emitted by background quasars)



Results: $\Omega_{IGM} \sim 0.008$

Putting all the results together, we get

Ω_b from galaxies+IGM+clusters ~ 0.014

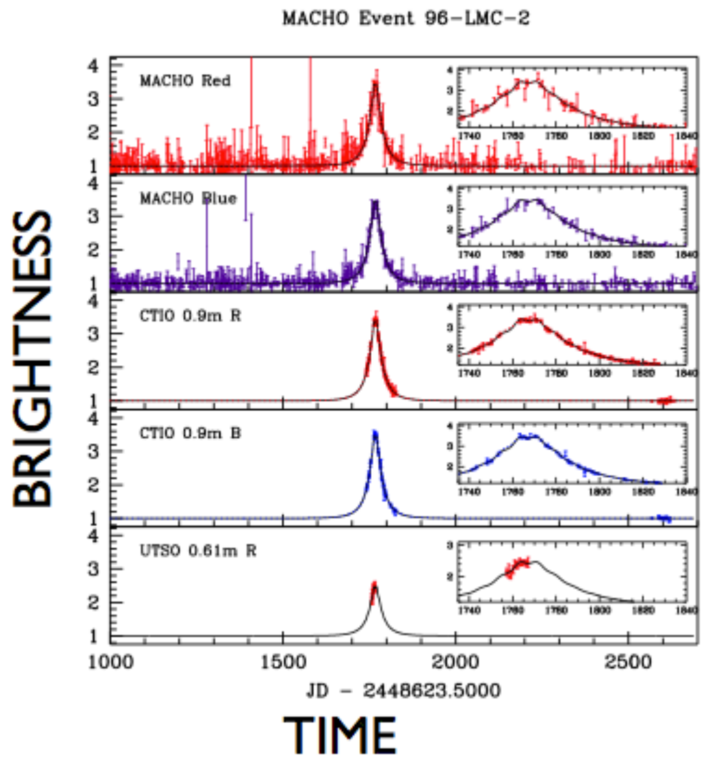


This is known as the problem of the missing baryons

Missing baryons

Baryonic mass density in MACHOs

First it was thought the solution might be a large amount of low-brightness objects: the Massive Compact Halo Objects,
i.e., a large amount of black holes, white dwarfs, neutron stars, large planets

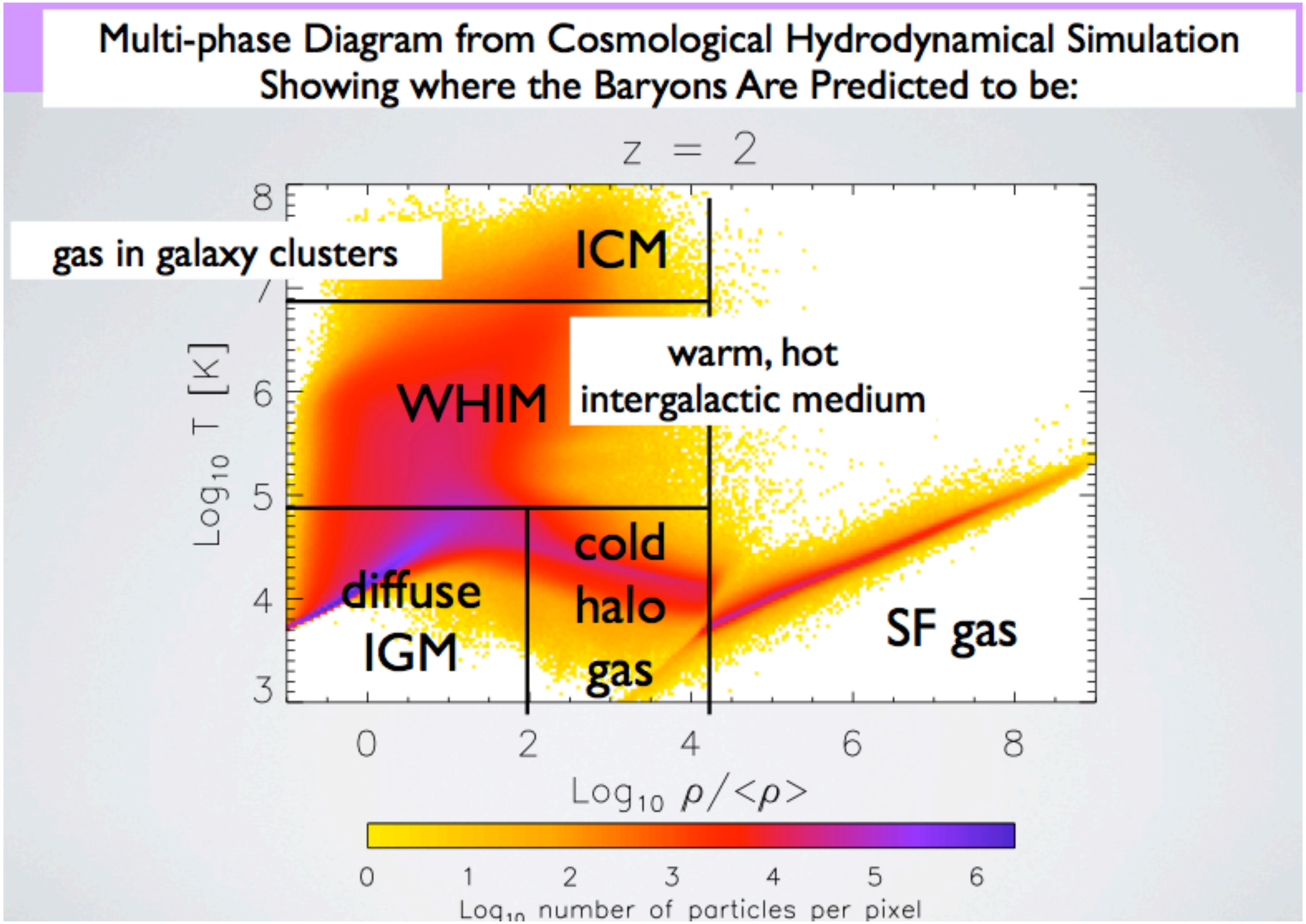


In the 1990s:

Extensive search for microlensing events

Results: Ω_MACHOs ~ negligible

Baryonic mass density in the cosmic web between clusters or field galaxies (warm/hot intergalactic medium WHIM)



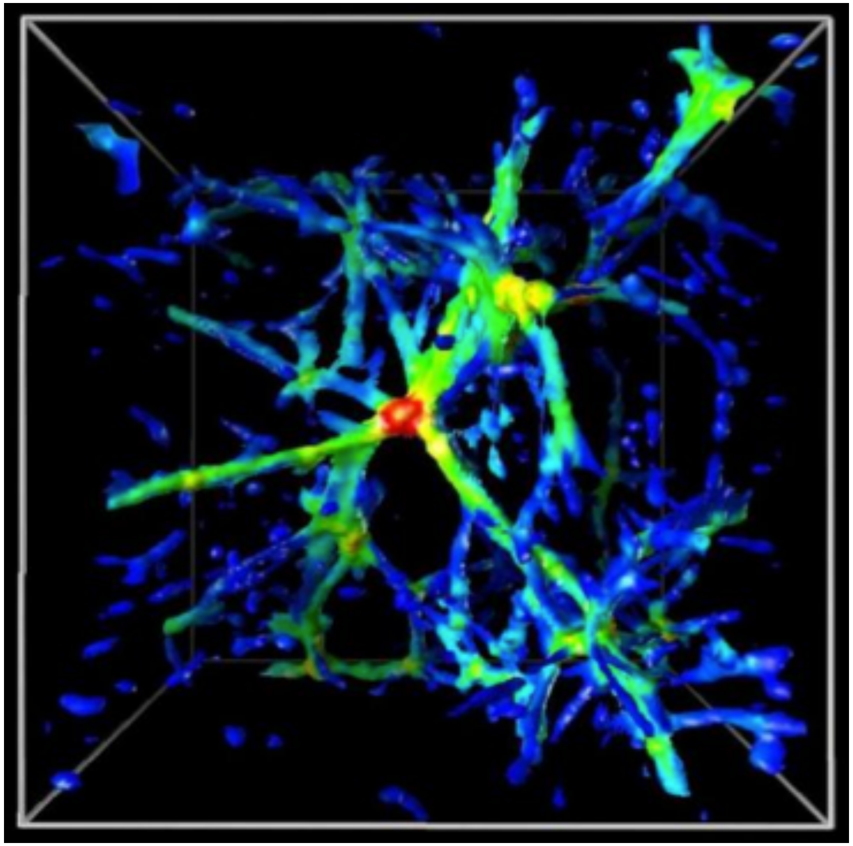
ICM intracluster medium : hot gas inside clusters

IGM intergalactic medium: diffuse gas between galaxies (Ly alpha forest)

WHIM warm/hot IGM: cosmic web between clusters

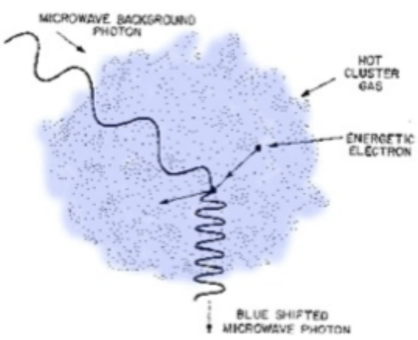
In the 2010s:

Detecting the WHIM



Thermal Sunyaev-Zeldovich effect

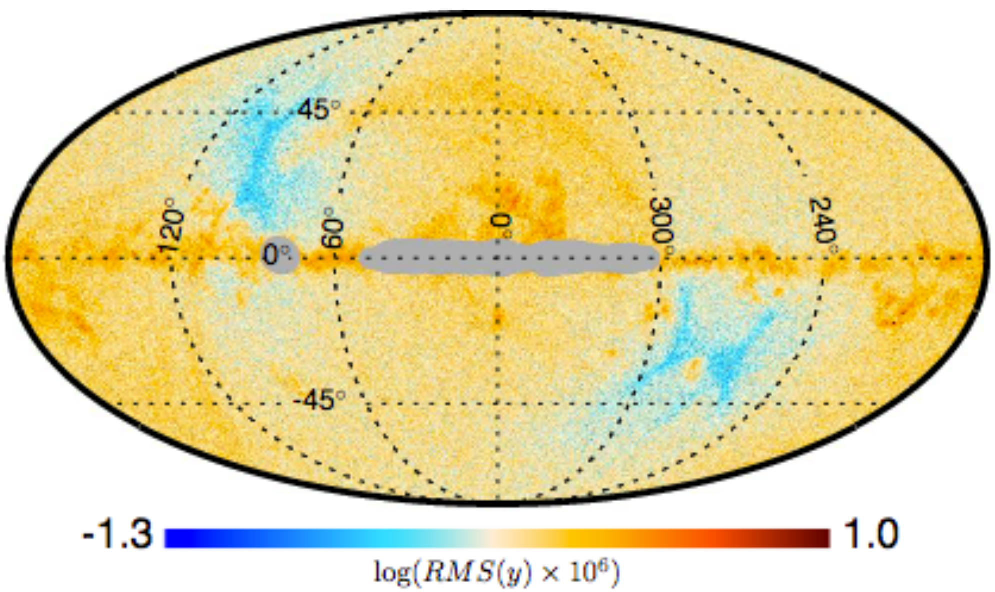
When CMB photons pass by hot ionized gas (like in a cluster), the photons can gain energy by scattering off of the hot electrons → T_CMB increases in the direction of a cluster

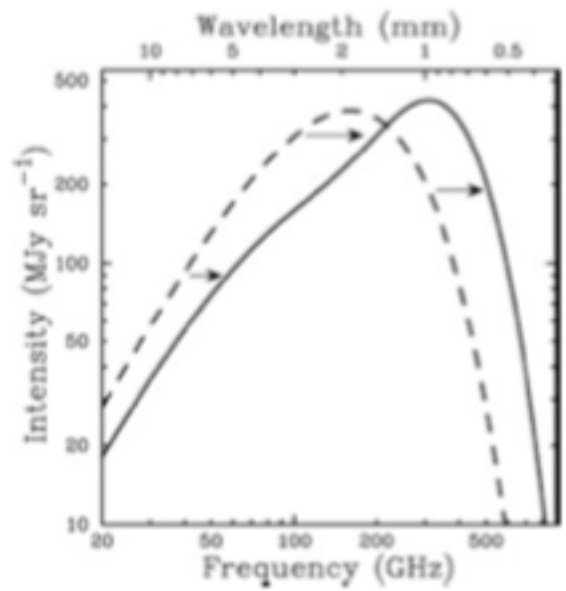


The amplitude of the effect (y) depends on the density of hot electrons → SZ is a measurement of the baryonic density

$$y = \frac{k_B \sigma_T}{m_e c^2} \frac{1}{A} \int n_e T_e dV = \frac{k_B \sigma_T}{m_e c^2} \frac{1}{A} \frac{0.88 f_B}{m_p} \sum_i M_i T_i$$

for clusters → y ~10^-6





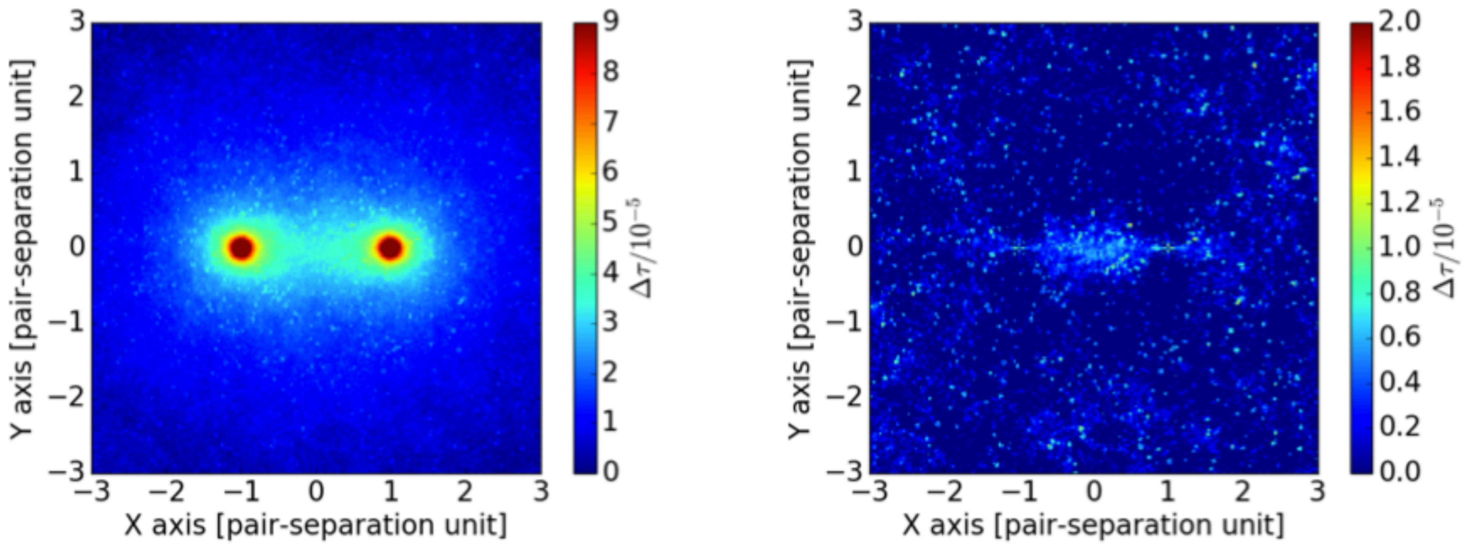
"A Search for Warm/Hot Gas Filaments Between Pairs of SDSS Luminous Red Galaxies", H. Tanimura, G. Hinshaw, I. McCarthy et al, MNRAS 483, 1, Feb. 2019 (arXiv:1709.05024)

Used the tSZ map from Planck 2015 and the luminous red galaxies LRG catalog from SDSS-DR12 (luminous galaxies at cluster centers) → found **260 000 LRG** pairs.

Stacking the signal from all pairs in one image (to increase SNR) and subtracting the cluster tSZ signal (with a model for cluster amplitude y), they found the residual signal coming from WHIM:

$$y = (1.31 \pm 0.25) \times 10^{-8}$$

Stacked pairs: before and after subtracting the cluster signal



First direct detection of a LSS filam

It has **δ ~ 5** → filaments are the largest and weakest-clustered structures

Absorption lines

The ionized WHIM should emit thermal bremstrahlung radiation.

But compared with the ICM, the WHIM gas has much lower temperature and density → impossible to detect its X-ray emission.

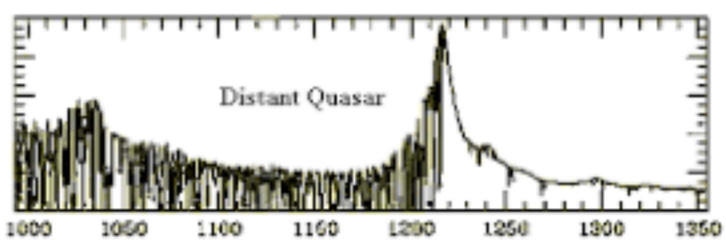
Use absorption techniques from its effect on background bright X-ray sources.

In the distant Universe, sources appear so faint that it is usually easier to detect them through absorption that through direct emission.

But still it may be needed a burst, like from a blazar (AGN with radio jet pointed toward us), to have enough signal for detection.

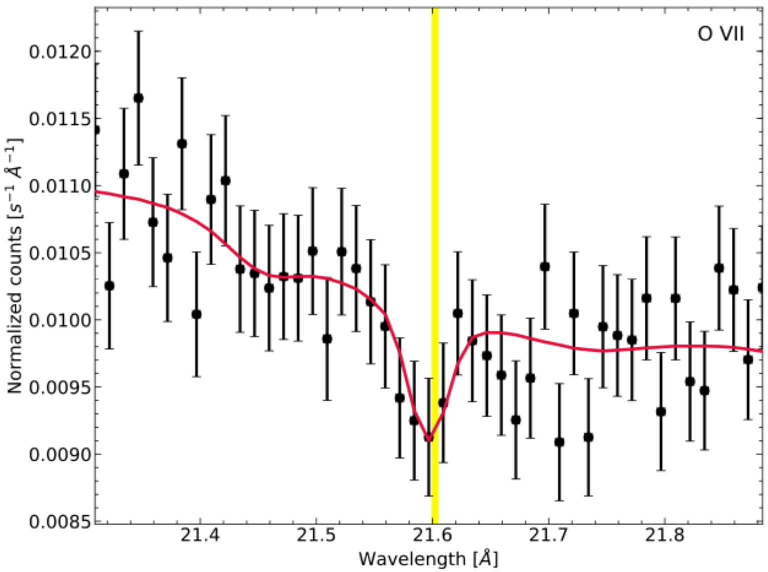
Unless some way of increasing the signal is found.
Again, through **stacking**:

Using all the absorptions at different redshifts of a single quasar → blueshift and stack them to increase SNR



“Detection of the Missing Baryons toward the sightline of H 1821-643”, O. Kovacs, A. Bogdan, R. Smith et al, ApJ 872, 1, Feb. 2019 (arXiv:1812.04625)

Adding 17 OVII absorption lines, the absorption from WHIM was seen in the spectra.



$$\Omega_b(\mathrm{O\,VII}) = \frac{\mu m_p H_0}{\rho_c c}\left[\left(\frac{\mathrm{O}}{\mathrm{H}}\right) f_{\mathrm{O\,VII}}\, Z/Z_\odot\right]^{-1} \cdot \frac{\sum_i N_i(\mathrm{O\,VII})}{\Delta X}$$

Results: Ω_WHIM = 0.017 (+/- 0.005)

It seems that the baryons on large-scales dominate the baryonic content of the Universe.

The current count is thus: Ω_b = 0.014 + 0.017

Mystery solved?

Dark matter

Ω_{dm}

The value of Ω_dm can be determined in various ways:

- direct mass measurements

- probes of structure formation: CMB anisotropies, weak lensing, galaxy clustering

- probes of geometry: Supernovas, BAO

There are 2 general types of dark matter:

- Cold dark matter (CDM): heavy particles (eg. WIMPs - weakly interacting massive particles)

- Hot dark matter (HDM): low mass particles (eg. neutrinos) - can erase small-scale perturbations

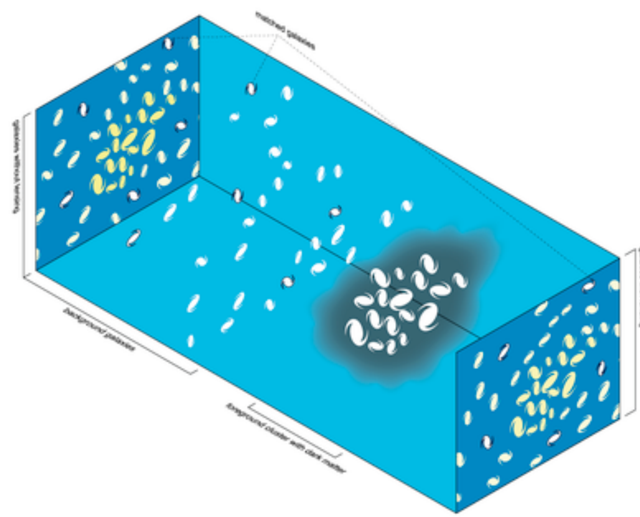
I. Evidence for dark matter

There is evidence for the existence of dark matter on various scales

Large-scale structure (LSS)

- From the small amplitude of CMB anisotropies → not enough time for baryonic matter to form the observed collapsed structures

- From the detection of correlations between galaxy ellipticities → well explained by the coherent deflection induced by “invisible” gravitational potentials



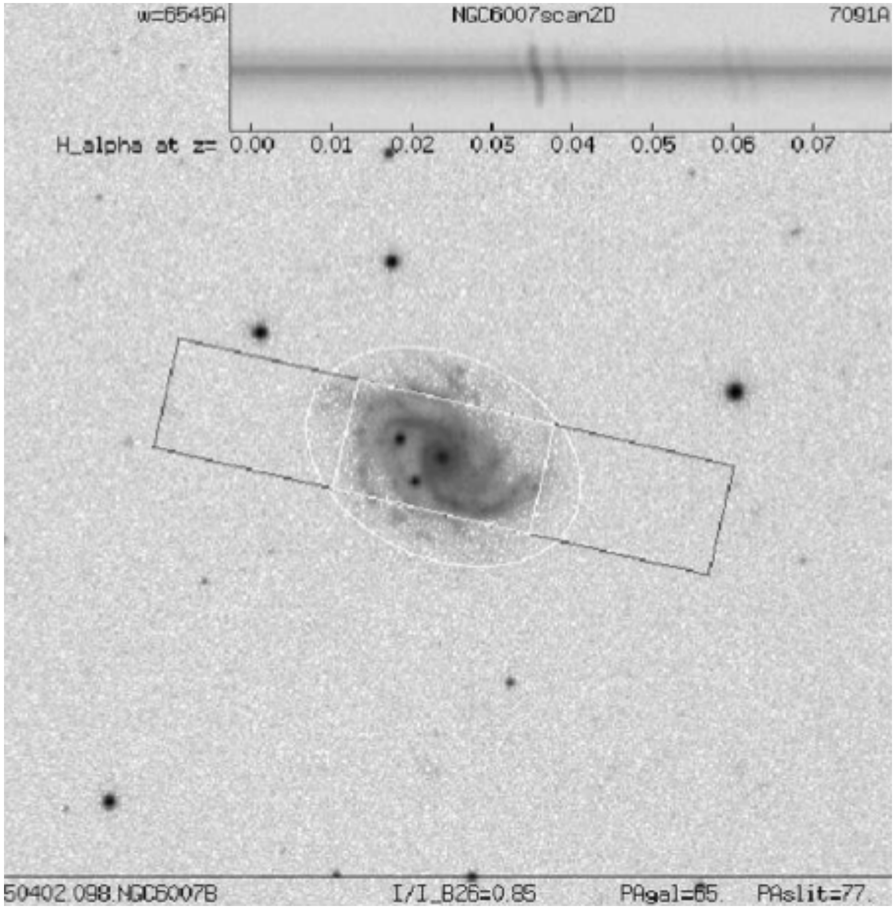
Clusters

- From their dynamics → need more mass

Galaxies

- From their rotation curves → need more mass

Dark matter in Galaxies (rotation curves)



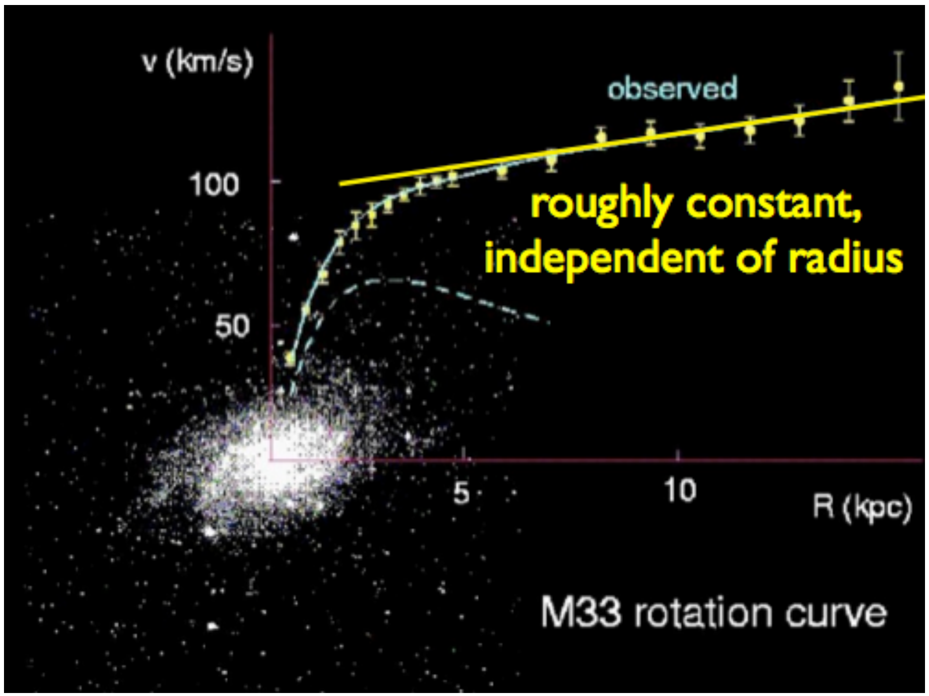
The rotation rate of a spiral galaxy can be measured by letting light pass through a slit along the axis of the galaxy and taking a spectrum

If the galaxy is not edge-on, we need to apply an inclination angle correction

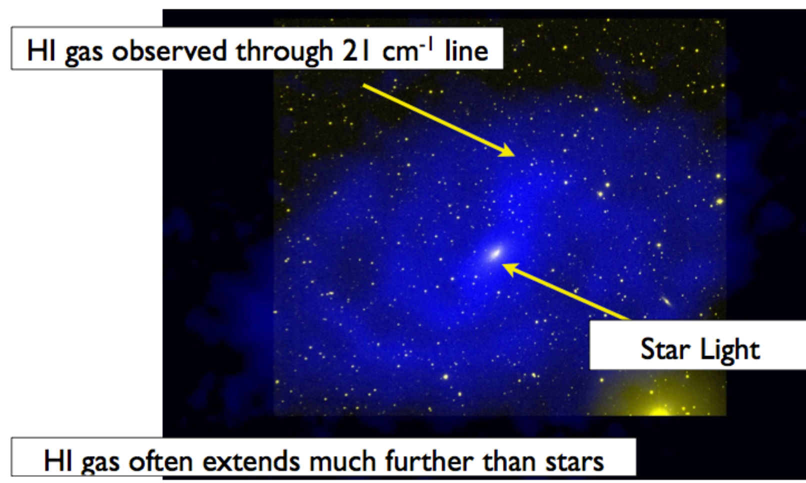
If the mass of the galaxy is mostly at the inner part → v_rot decreases with distance to the center (r)

$$\frac{GM(\leq r)}{r^2} = \frac{v_{\rm rot}^2(r)}{r}$$

force per unit mass   acceleration



However, the observed **v_rot is approximately constant** (beyond a certain radius) and it continues flat to very large distances



if v_rot flat → M increases with r

$$M(\leq r) = \frac{v_{\rm rot}^2 r}{G}$$

In principle, this does not need to be a problem, the distribution of mass in the galaxy could naturally be such that it increased with radius (no need to be concentrated in the center).

But the problem is that the **light in a galaxy decreases exponentially with radius** → for large radius, the total light inside the radius tends to a constant

$$I(r) = I_0 \exp(-r/h)$$

$$I_0 \int_0^{r_0} 2\pi r \exp(-r/h)\, dr$$
$$\propto h^2 - h(r+h)\exp(-r/h)$$

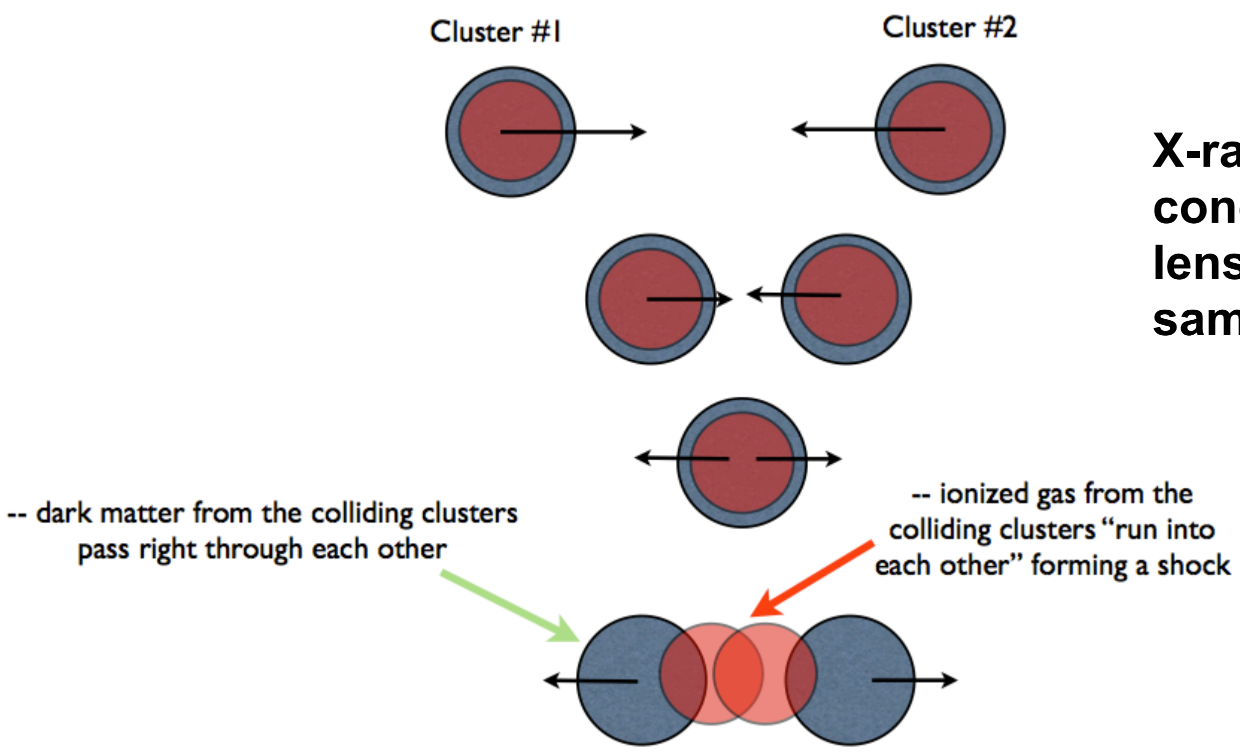
This means that the light is restricted to the inner part, up to a typical scale r = h

→ matter emitting light (baryonic matter) is also mostly in that part → the mass that increases with r cannot be luminous → **dark matter is needed**

Dark matter in Clusters (clusters collisions)

The observation that the velocities of individual galaxies in clusters could only be explained if total mass of the cluster was much greater than that seen in galaxies was already suggested in 1933 (Fritz Zwicky), based on observations of the nearby Coma cluster.

The observations of the colliding Bullet Cluster (2006) are well understood if there is dark matter in clusters.



X-ray emission and mass concentration (from weak lensing) are not at the same position

ICM (X-ray emission) and mass concentration (from weak lensing) are not at the same position → ICM gas is not the dominant mass contribution.

This is dark matter.

Notice that the galaxies of the clusters also passed right through.

II. "Direct" measurement of Ω_{dm}

A useful way to quantify the amount of dark matter in a structure is the mass-to-light ratio (M/L). It compares the total mass with the mass expected based on the luminosity.

The stars set the scale : $(M/L)_{stars} \sim 1$

Since stars have almost no dark matter and we saw that in stars $\Omega_b \sim 0.002$
→ M/L =1 means $\Omega_m = 0.002$ (and $\Omega_{dm} \sim 0$)

Dark matter density in galaxies

Total mass measured from rotation curves:

$$M(\leq r) = \frac{v_{\rm rot}^2 r}{G}$$

is $(M/L)_{gal} = 20$ → $\Omega_{m\_gal} = 0.04$ → $\Omega_{dm\_gal} \sim 0.04$

Dark matter density in clusters

The total mass of a cluster can be determined in 3 different ways.

Each method makes some assumptions about the state of equilibrium of the cluster

1. Dynamics of the cluster galaxies → virial theorem
2. X-rays emission → hydrostatic equilibrium
3. Gravitational lensing → cluster symmetries

1. Galaxy motions

For systems that have collapsed gravitationally and are relaxed, the virial theorem is

E_kin = -1/2 E_pot

Galaxy are observed in spectroscopy → Doppler shifts are measured along the line-of-sight → the measured dispersion in the average velocity along the l-o-s is

$$\left\langle v_{\parallel}^2 \right\rangle$$

dispersion of the average velocity $\left\langle v^2 \right\rangle = \left\langle v_x^2 \right\rangle + \left\langle v_y^2 \right\rangle + \left\langle v_z^2 \right\rangle = 3 \left\langle v_{\parallel}^2 \right\rangle$

$$E_{\mathrm{kin}} = \frac{1}{2} \sum_i m_i \mathbf{v}_i^2 = \frac{3}{2} M \left\langle v_{\parallel}^2 \right\rangle$$

$$E_{\mathrm{pot}} = \frac{GM^2}{R_{\mathrm{cl}}} \quad \rightarrow \quad M = \frac{3}{G} \left\langle v_{\parallel}^2 \right\rangle R_{\mathrm{cl}}$$

Typical values: v ~1000 Km/s; R_cl ~1 Mpc → **M ~10^15 Msun**

Knowing that the total mass of the galaxies in a cluster is ~10^13 Msun

→ **M/L = 160**

→ Ω_{m_cl} = 160 x 0.002 = 0.32 → **Ω_{dm_cl} ~0.28**

The other 2 methods give similar results:

2. X-ray profiles

Ionized gas in clusters - assumed to be in hydrostatic equilibrium

+ ideal gas p = nKT (n= ρ /m_p)

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{GM(r)}{r^2},$$

→ $$M(r) = -\frac{rkT}{Gm_\mathrm{p}}\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r}$$

This is the total mass needed to keep the hot gas (with pressure p, temperature T and density ρ) in equilibrium.

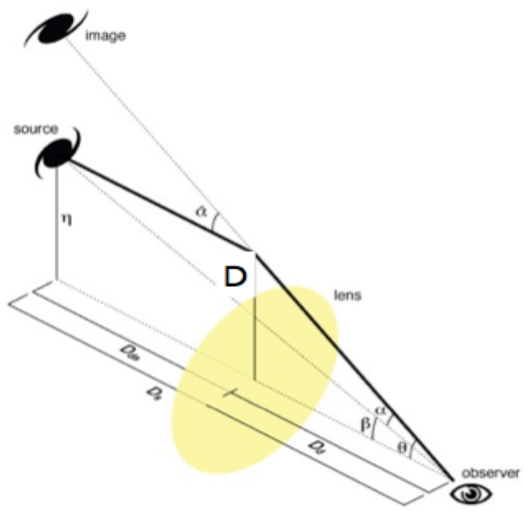
Also need to assume a density profile for the cluster (to be able compute dρ /dr), i.e., assume a model:

$$n(x) = \frac{n_0}{(1+x^2)^{3\beta/2}}, \quad x \equiv \frac{r}{r_\mathrm{c}}$$

→ $$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r} = \frac{\mathrm{d}\ln n}{\mathrm{d}\ln r} = -3\beta\frac{r^2}{1+r^2}$$ → $$M(r) = \frac{3\beta rkT}{Gm_\mathrm{p}}\frac{r^2}{1+r^2}$$

Typical values: kT ~10 KeV; r ~1 Mpc; β = 2/3 → **M ~10^15 Msun**

3. Gravitational lensing



Measuring the positions of multiple images and giant arcs, we can constrain the mass distribution of the lens.

Need to model the lens. Also need to know the distance to the lens and to the source

Simple approximation: modeling the cluster as a sphere of mass M concentrated in the center, it produces a deflection of α for a light ray passing at a distance D from the center →

$$\tilde{\alpha} = \frac{4GM}{Dc^2} \qquad D = D_d D_{ds}/D_s$$

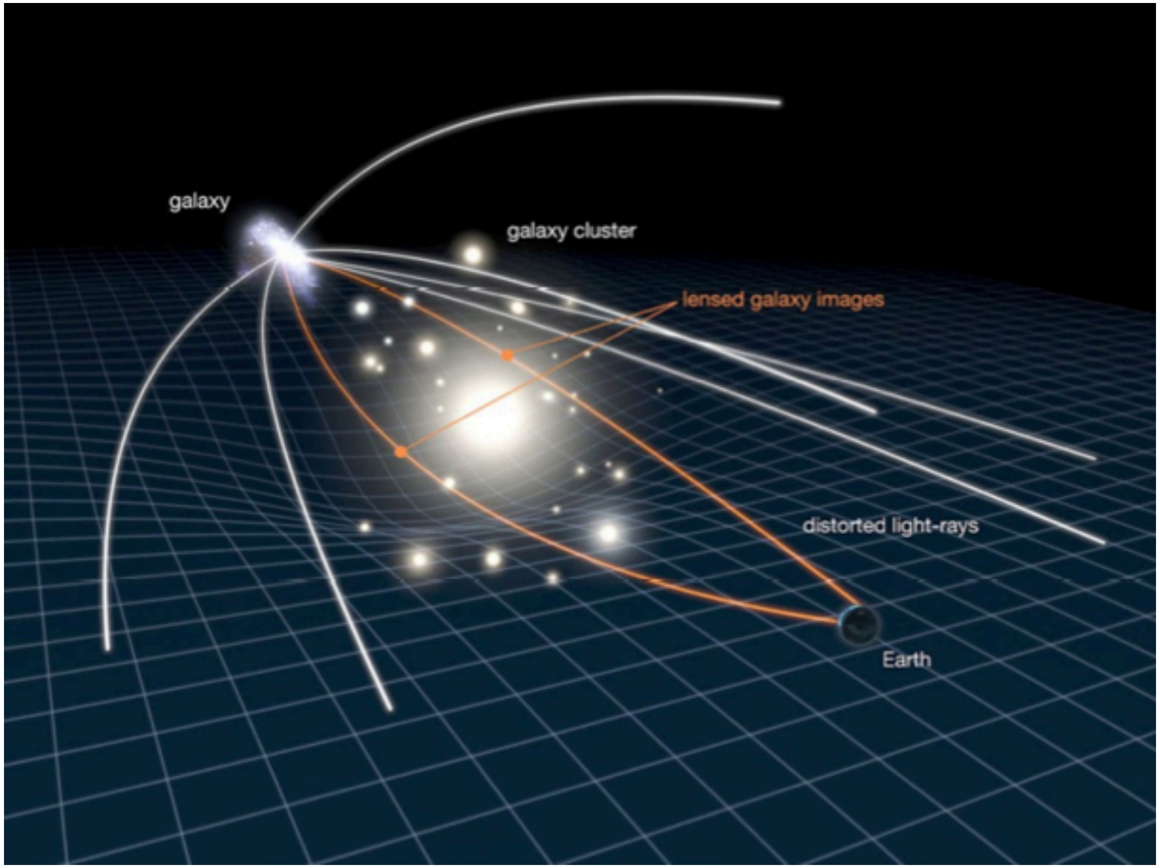
Measure the deflection α, measure the distances → get the mass **M ~10^15 Msun**

Gravitational Lensing

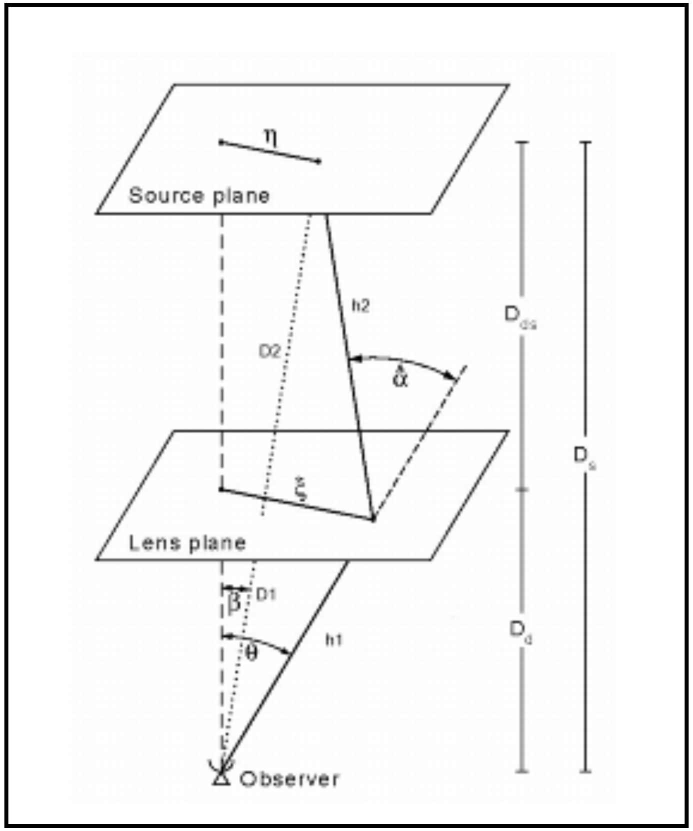
(Let us take this opportunity to introduce the important effect of **gravitational lensing**)

1. Deflection of light

The basis of gravitational lensing is the effect of deflection of light caused by gravity.



In general, we define a **source - lens - observer system**



source position in the source plane

deflection angle

impact parameter in the lens plane

image position in the image plane

optical axis

Light from a point emitted at an angular position β is observed at a different angular position θ.

It is deflected by a deflection vector α induced by gravity.

The lens equation, relating source and lens planes can be found from the diagram above, by using simple trigonometry (vector addition on the source plane):

$$D_s\vec{\theta} = D_s\vec{\beta} + 2D_{ds}\frac{\hat{\vec{\alpha}}}{2}$$

α is determined by the properties of the lens : it contains the physical (gravitational field) information we want to find out.

Measuring the change between θ and β we can find α , if we know the distances (there is a degeneracy with the distance).

How does the deflection angle relate to the lens gravitational potential?

Let us consider **light propagation from source to observer** in the Universe described by the Robertson-Walker metric with a small inhomogeneity representing the lensing potential:

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right)\left[dx_1^2 + dx_2^2 + dx_3^2\right]$$

The deflection may be derived using the principle of Fermat: **light follows a path of extremal time.**

Light follows null geodesics, and setting $ds^2 = 0$ we can immediately write the speed of light when travelling in the gravitational field of the lens. It is:

$$c' = \frac{|d\vec{x}|}{dt} = c\sqrt{\frac{1+\frac{2\Phi}{c^2}}{1-\frac{2\Phi}{c^2}}} \approx c\left(1+\frac{2\Phi}{c^2}\right)$$

We can think of the gravitational field as a “change of medium” since it effectively changes the speed of light propagation. This medium is thus associated to an effective index of refraction, given by:

$$n = c/c' = \frac{1}{1+\frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$

In terms of properties of light propagation, the perturbed metric is like a medium where the speed of light is $v < c$ → it bends the light, with respect to the homogeneous spacetime where $v = c$.

Now, let x(l) be a light path crossing the medium.

The light travel time is then proportional to:
(since the refraction index is basically dt/dx)

$$\int_A^B n[\vec{x}(l)]\mathrm{d}l$$

and we want to find the path of extremal (minimum) time, i.e.,

$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

This is a standard variational problem, that as we know will lead to the Euler-Lagrange equations.

The extremal light path verifies:

$$\delta \int_A^B n(\vec{x})dx = 0 = \delta \int_{\lambda_A}^{\lambda_B} n(\vec{x}(\lambda))\frac{dx}{d\lambda}d\lambda = \delta \int_{\lambda_A}^{\lambda_B} n(\vec{x}(\lambda))|\dot{\vec{x}}|d\lambda = \delta \int_{\lambda_A}^{\lambda_B} L(x, \dot{x}; \lambda)d\lambda,$$

where λ is an arbitrary affine parameter, labeling the positions along the path,

and we found out that $n[\vec{x}(\lambda)]\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right|$ role of a Lagrangian.

Having found the Lagrangian we can now describe the light path using the Euler-Lagrange equations:

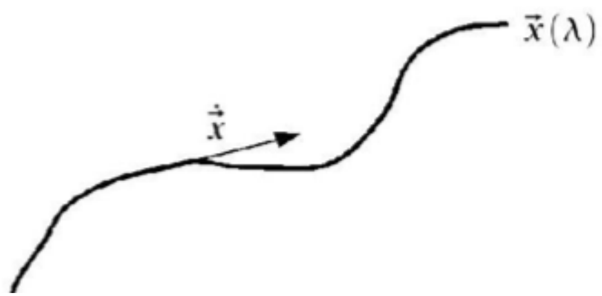
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0$$

From our Lagrangian, we compute:

$$\frac{\partial L}{\partial \dot{\vec{x}}} = n\frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$$

$$\frac{\partial L}{\partial \vec{x}} = |\dot{\vec{x}}|\frac{\partial n}{\partial \vec{x}} = (\vec{\nabla}n)|\dot{\vec{x}}|$$

This means that the Euler-Lagrange equation is an equation for the evolution of $\dot{\vec{x}}$, which is a vector tangent to the light path.



$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \qquad \frac{d}{d\lambda}(n(\vec{x})\vec{u}_x) - \vec{\nabla}n = n\dot{\vec{u}}_x + (\vec{\nabla}n.\vec{u}_x)\vec{u}_x - \vec{\nabla}n.$$

(u is the normalised vector tangent to the path) and so the Euler-Lagrange equation is: $n\dot{\vec{u}}_x + (\vec{\nabla} n.\vec{u}_x)\vec{u}_x - \vec{\nabla} n = 0$

$$\Leftrightarrow \quad \dot{\vec{u}}_x = \frac{1}{n(\vec{x})}\left(\vec{\nabla} n - (\vec{\nabla} n.\vec{u}_x)\vec{u}_x\right)$$

this is the gradient of n perpendicular to the light path

$$\Leftrightarrow \quad \dot{\vec{u}}_x = \frac{1}{n(\vec{x})}\vec{\nabla}_\perp n(\vec{x}) = (1+\frac{2\Phi}{c^2})(-\frac{2}{c^2}\vec{\nabla}_\perp\Phi) \approx -\frac{2}{c^2}\vec{\nabla}_\perp\Phi$$

and therefore, the gradient of the potential.

Now, the derivative of the tangent vector is by definition the deflection. So **we found that the deflection is the gradient of the lens potential in the plane orthogonal to the tangent to the path (i.e. on the lens plane).**

Notice the minus sign, meaning the gradient of the potential points away from the lens centre and the deflection angle points toward the lens (light is pulled towards the lens).

The potential changes from point to point along the light path, so the **total deflection** is the integral over the "pull" of the gravitational potential perpendicular to the light path:

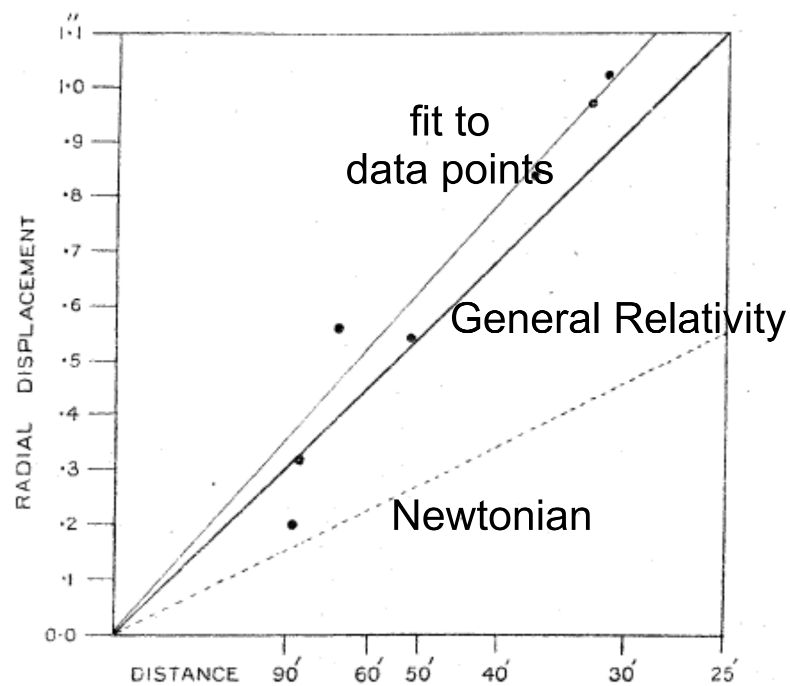
$$\vec{\alpha} = -\frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \dot{\vec{u}}_x \, d\lambda = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \, d\lambda,$$

Notice that:

- the integral should be made over the actual $\Phi/c^2 \ll 1$ priori unknown before computing the deflection → so it is a recursive problem). However, given the smallness of the potential the deflection angle is usually small and in practice it is usual to integrate over the unperturbed light path. (This is exactly like the Born approximation used in scattering theory).

- since the speed of light is effectively slowed down in the gravitational field, the travel time to cross a given length is larger than it would be in the absence of a lens. This is called the Shapiro delay.

- the value of the deflection angle computed in GR (that was we saw contains a factor of 2) is twice the value predicted by Newtonian gravity, or by considering the equivalence principle (gravity - acceleration) in special relativity. The well-known Eddington eclipse expedition of 1919 measured the deflection angle produced at the edge of the Sun disk with the purpose of comparing the measurement with the two predictions. It was the first test of GR.



Point source

Having found the relation between deflection angle and gravitational potential, we can compute the deflection of the light emitted by a point source when passing by a lens.

Let us consider first a point mass lens, with the usual potential

$$\Phi = -\frac{GM}{r}$$

Light from the source travel along the z-axis towards the observer and crosses the lens plane (x,y)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{b^2 + z^2},\; b = \sqrt{x^2 + y^2}$$

the light path crosses the lens plane at a distance b from the point mass. b is called the impact parameter

The potential on the lens plane (the orthogonal plane) is

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix}$$

and the resulting deflection vector is:

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{(b^2 + z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2 (b^2 + z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \end{aligned}$$

with

$$\begin{pmatrix} x \\ y \end{pmatrix} = b \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

From the x and y components of the deflection angle vector, we compute its norm, which is the well-known result:

$$|\hat{\vec{\alpha}}| = \frac{4GM}{c^2 b}$$

Note that the impact parameter is strongly constrained. The source emits in all directions, and various light paths reach the lens plane. But only one is deflected towards the observer. From the lens equation (from the source-lens-observer diagram), we can see it is the one that passes at $b = D_d D_{ds} / D_s$

D_d = distance from observer to lens (deflector)
D_{ds} = distance from lens to source
D_s = distance from observer to source

For this reason, **all lensing systems have a fundamental degeneracy between distances and lens properties.** We can only compute the mass of the lens if we know the distances involved in the system. Conversely, lensing can be used as a geometric probe of the Universe (i.e., it can be used to measure cosmological distance and use them to infer the density parameters) if the mass of the lens is known.

Let us consider now an extended lens

Since the deflection angle depends linearly on the mass M, the effect from a finite lens in a plane is just the sum of the deflection angles created from all points in the lens. If we discretize the lens as a set of N point lenses of masses M_i at positions ξ_i on the lens plane, then the deflection angle of a light ray crossing the plane at ξ will be:

$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

We can also consider a lens in 3D with mass density ρ. The z extension of the lens is always just a small segment of the full source-observer light path, and it can be considered that it is in a plane - the thin-screen approximation. In this approximation, the lensing matter distribution is completely described by its **surface mass density**:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\, \mathrm{d}z$$

and the total deflection is given by:

$$\hat{\vec{\alpha}}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi}')\Sigma(\vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}\, \mathrm{d}^2\xi'$$

2. Gravitational Lensing

Gravitational lensing, in a strict sense, refers to the case of **extended sources**, which give rise to differential effects.

Indeed, neighbouring points from the source suffer slightly different deflections in the lens plane: it is a differential effect that makes **the image of an extended source** (i.e. non-pointlike) **to become distorted**.

This is easily seen if we Taylor-expand the lens equation. Remember the lens equation is a mapping from image positions to source positions (it is usually written in that order, and not as a mapping from source to image). So a given point θ in the image plane corresponds to an original position β(θ) in the source plane, related by the deflection angle:

$$\vec{\beta}(\theta) = \vec{\theta} - \vec{\alpha}$$

(here the vectors have absorbed the distance factors present in the original lens equation) $\vec{\alpha}(\vec{\theta}) \equiv \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\hat{\vec{\alpha}}(\vec{\theta})$

The Taylor expansion of β(θ) to linear order is $\beta(\theta) = \beta(\theta_0) + A(\theta_0).(\theta - \theta_0)$

where A is the amplification matrix (the Jacobian) and describes the **lensing transformation** between source and image planes to first order:

$$A_{ij}(\theta) = \frac{\partial \beta_i}{\partial \theta_j} = \left(\delta_{ij} - \frac{\partial \alpha_i}{\partial \theta_j}\right)$$

it is a 2D matrix, since β (position in the source plane and θ (position in the lens plane) are 2D vectors.

Now, remember that a general matrix can be decomposed in 3 parts:

(traceless) symmetric + (traceless) antisymmetric + diagonal

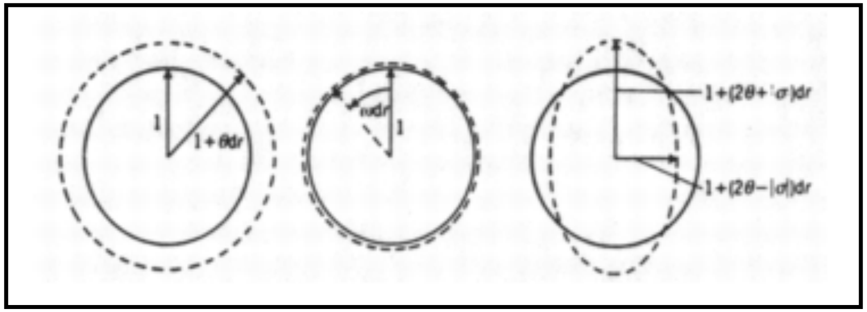
$$\begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Applying a diagonal matrix to an image will: expand it (or contract it) radially in an isotropic way → k is called **convergence.**

Applying an antisymmetric matrix to an image will: rotate it → ω is called **rotation.**

Applying an symmetric matrix to an image will: distort it in an anisotropic way, contracting in one dimension and expanding in the other → γ is called **shear.**

This means that any linear distortion of an image is a combination of convergence/expansion, rotation and shear



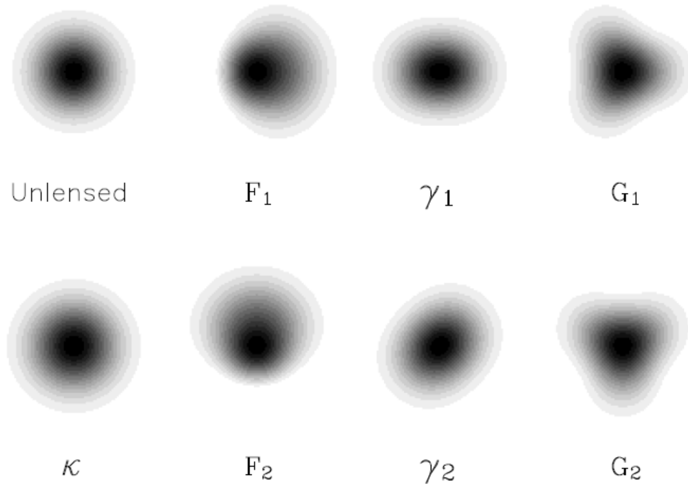
The amplification matrix is then written as

$$A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

Note that the lensing distortion does not includes rotation because the gravitational field is a gradient field (completely defined by a potential), and so its rotational is zero (it is a so-called E field) and the deflection vector field does not produces rotations.

The presence of rotations in a lensed image (due to so-called B-modes) is an indication of systematic effects, i.e., distortion effects with non-lensing origin.

For example, the distortions applied to a circular image result in:



isotropic distortion (k, convergence) → a circle expands/contracts (full rotational symmetry)

anisotropic distortion (γ, shear) → a circle transforms into a π-rotational symmetric shape (an ellipse)

second-order distortions (by continuing the Taylor expansion) (F, G, flexion) → a circle transforms into a 120º-rotational symmetric shape (a banana-shape F or a "Mercedes logo" G)

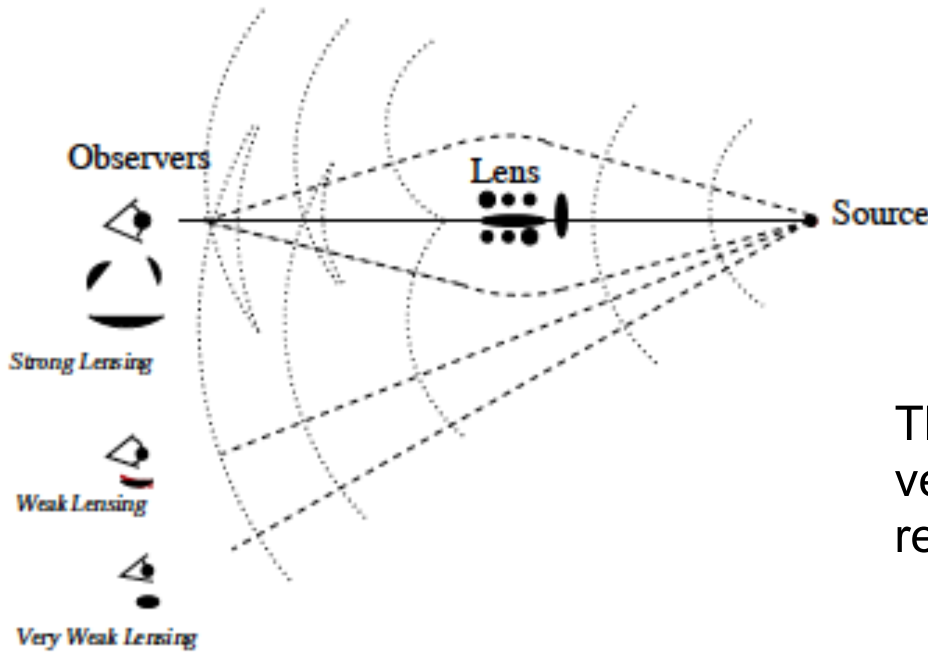
These are the fundamental distortions (also called the optical scalars) and contain the dependence on α → which contains the information on gravity

The determinant of the amplification matrix defines the magnification:

$$\mu = \frac{1}{det\,A} = \frac{1}{(1-\kappa)^2 - \gamma^2}$$

The magnification, and the amplitude of the optical scalars - which are fields in the 2D sky - define the **gravitational lensing regime** that occur in the positions of the sky.

There are two general regimes - **weak lensing** and **strong lensing** - that occur in regions of the image plane where the values of the k(θ) and γ (θ) fields are small (<<1) (weak lensing) or large (strong lensing).

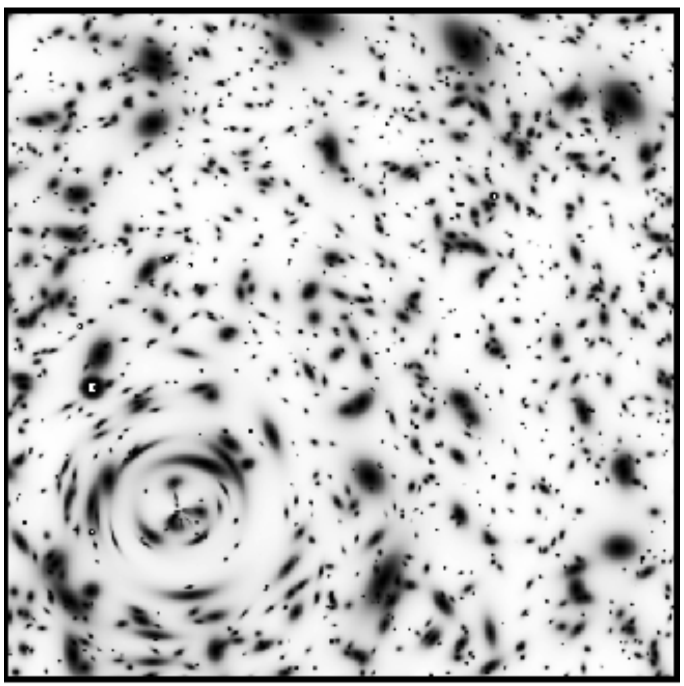


The observable effects are very different in the two regimes

Weak Lensing occurs further from the line of alignment of source-lens-observer, or with lenses of lower density contrast.

The effects are: small increase of ellipticity of the source galaxy (shear), alignment of images.

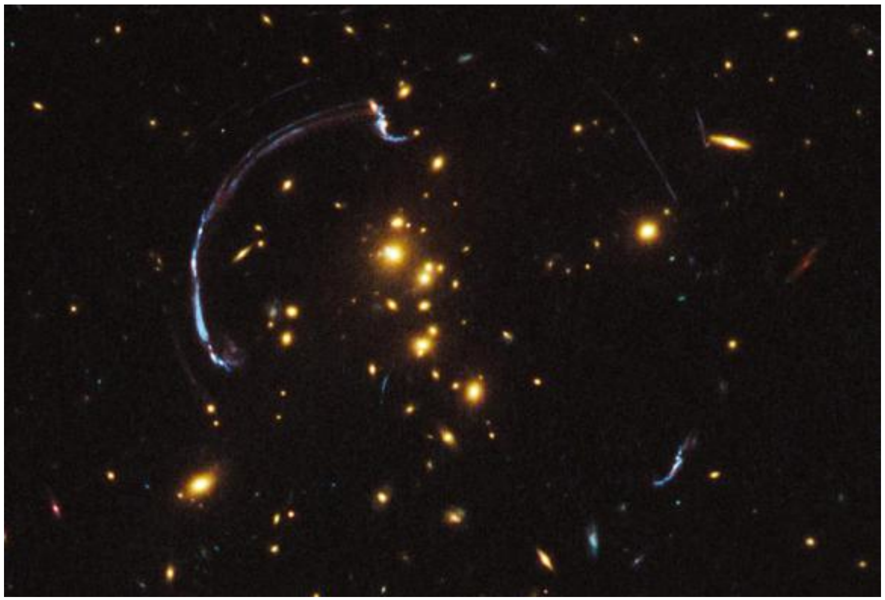
Weak lensing is a very useful probe in a cosmological system where the lens is the large-scale structure of dark matter distribution. In this case the shear is so small that it cannot be detected in individual galaxies. What can be detected is a correlation of those ellipticities because their orientations get some degree of alignment and cease being randomly oriented → this effect is used to probe the structure formation of the Universe.



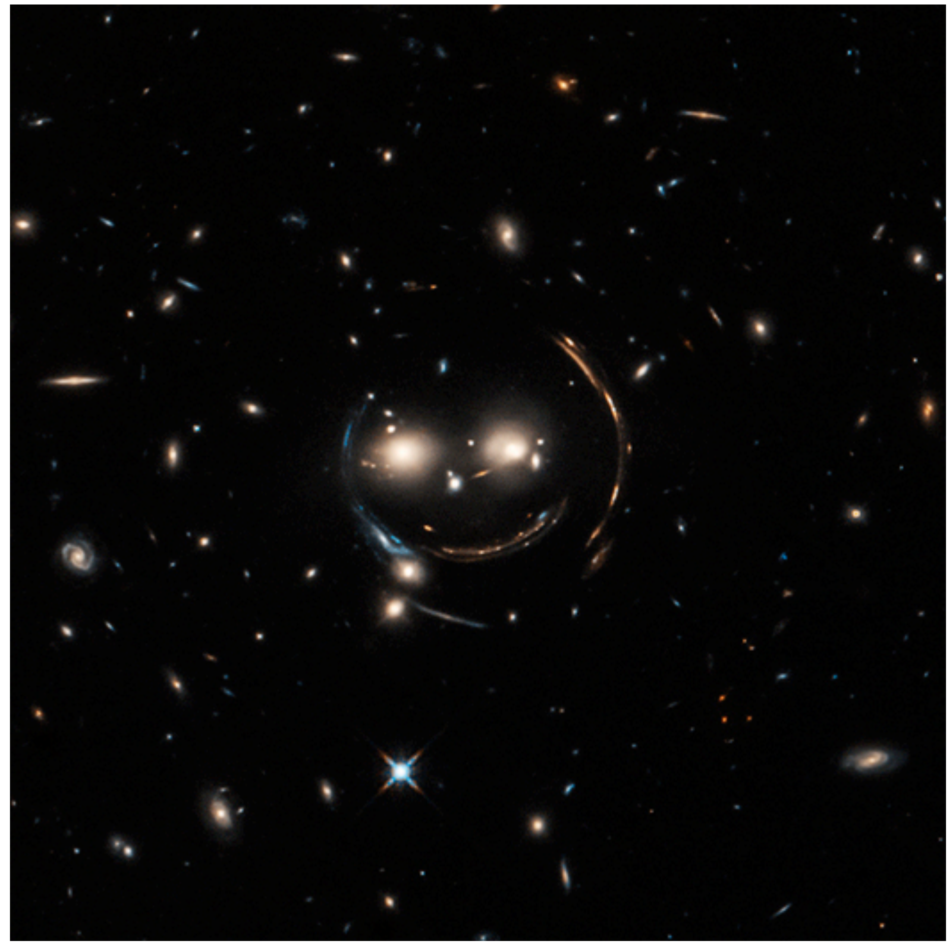
Increased ellipticities: Weak lensing of galaxies by a cluster (The Bullet Cluster)

Strong Lensing occurs near the line of alignment of source-lens-observer, with lenses of high density contrast.

The effects are: very strong distortions (giant arcs), multiple images, flux magnification. They occur near lines where det A = 0 (infinite magnification), which are called critical lines of the image plane (the observed sky), and map back to the source plane to lines known as caustic lines.

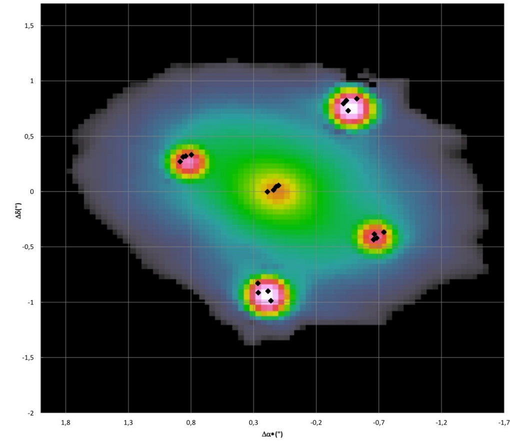


Giant Arcs: Strong lensing of galaxies by a cluster



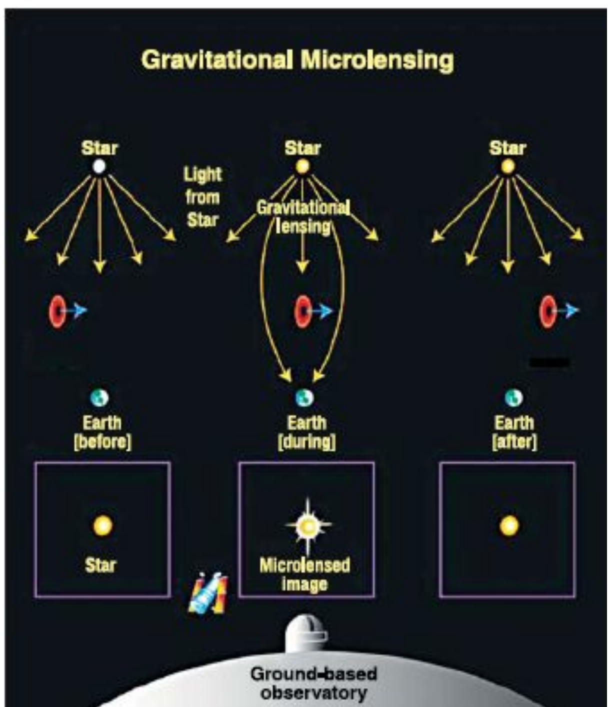
Giant Arcs: Strong lensing and Einstein ring of galaxies by a group that includes 2 massive ellipticals

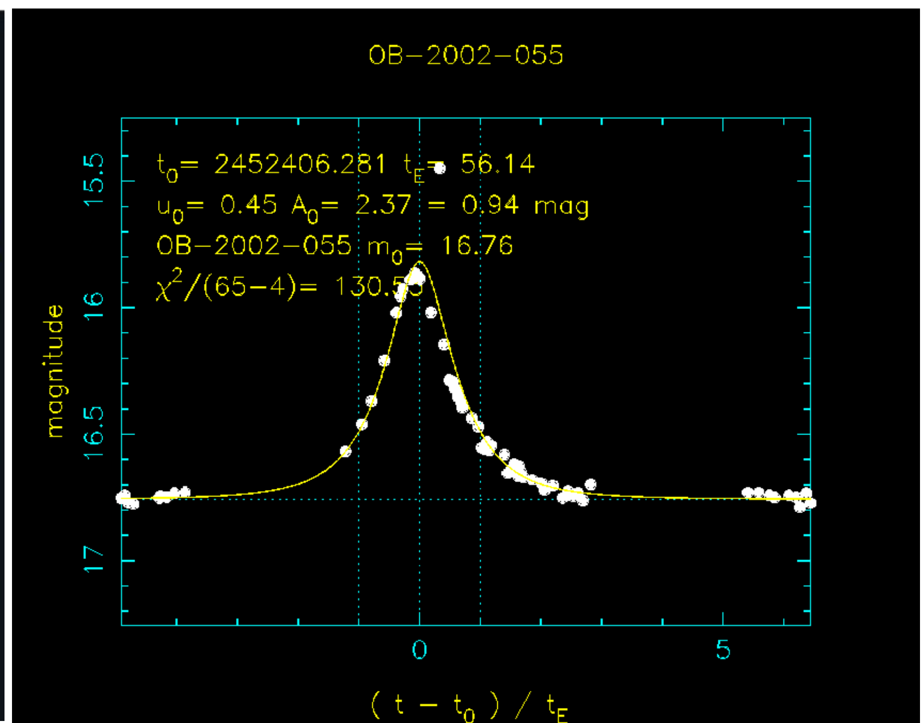
Einstein ring: Strong lensing of a galaxy by a galaxy, an infinite number of multiple images forming a circle



Einstein cross: Strong lensing of a quasar by a galaxy, forming a quadruple image of the quasar

When the scale of the strong lensing effects is small (ex: multiple images have small angular separation and are not resolved) this type of strong lensing is called **microlensing**.



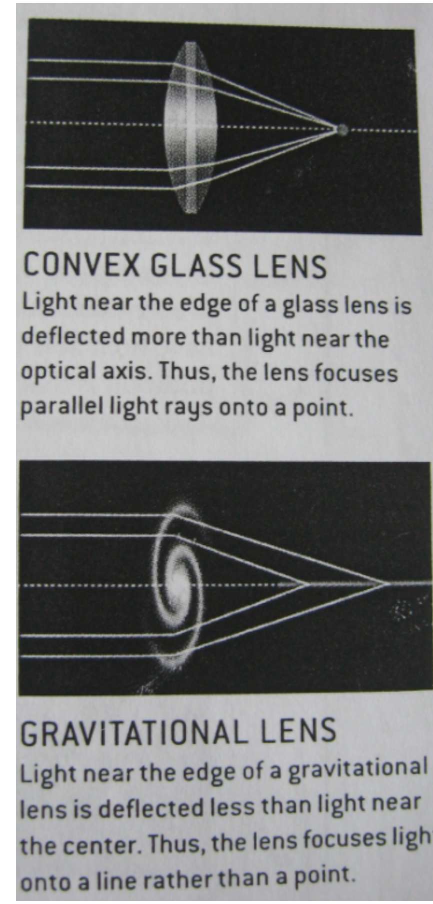


Produces also an increase of flux → used to detect exoplanets.

3. Conclusion

Gravitational Lensing has a number of fundamental properties:

- it depends on the projected 2d mass density distribution of the lens
- it is independent of the luminosity of the lens
- it does not have a focal point
- it is achromatic, there is no frequency shift from source to image
- it involves no emission or absorption of photons
- it conserves the surface brightness



This leads to a number of observable features:

- change of apparent **positions**
- magnification (increase of size), which combined to the conservation of brightness implies an increase of **flux** → natural telescope
- **distortion** of extended sources (ellipticities, tangential giant arcs, radial arclets)
- **multiple images**
- **time-delay** between multiple images

These are the lensing observables

These observables (positions, fluxes, distortions) can be used **to estimate the total mass and mass distribution of the lens**. For example:

- in (strong or weak) cluster lensing → mass distribution of the cluster
- in LSS weak lensing (cosmic shear) → dark matter power spectrum

In all systems, the general **recipe** is similar. We need to:

i) (theoretical) ***define a lens model and derive its gravitational potential.***

ii) (theoretical) ***derive the deflection and optical scalar fields from the gravitational potential***

From the definitions in the amplification matrix, it is clear that shear and convergence are derivatives of the deflection field, and second-order derivatives of the potential. In particular:

shear
$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \quad , \quad \gamma_2 = \psi_{,12} \, .$$

convergence
$$\kappa = \frac{1}{2}(\psi_{,11} + \psi_{,22}) .$$

where ψ is the gravitational potential projected on the lens plane (i.e. integrated along z) and dimensionless (with the distance factors included), i.e.,

$$\Psi = \frac{D_{\mathsf{L}}^2}{\xi_0^2} \cdot \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z) \mathrm{d}z$$

This is called the **lensing potential**. Note that indeed:

$$\begin{aligned}
\vec{\nabla}_x \Psi(\vec{x}) &= \xi_0 \vec{\nabla}_\perp \left(\frac{D_{\mathsf{LS}} D_{\mathsf{L}}}{\xi_0^2 D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(\vec{x}, z) \mathrm{d}z \right) \\
&= \frac{D_{\mathsf{LS}} D_{\mathsf{L}}}{\xi_0 D_{\mathsf{S}}} \frac{2}{c^2} \int \vec{\nabla}_\perp \Phi(\vec{x}, z) \mathrm{d}z \\
&= \vec{\alpha}(\vec{x})
\end{aligned}$$

Note also that the convergence is the Laplacian of the lensing potential.
This means, from Poisson equation, that the convergence is a (projected) mass.
In particular, it is the (dimensionless) surface density:

$$\kappa(\vec{x}) \equiv \frac{\Sigma(\vec{x})}{\Sigma_{\mathsf{cr}}} \quad \text{with} \quad \Sigma_{\mathsf{cr}} = \frac{c^2}{4\pi G} \frac{D_{\mathsf{S}}}{D_{\mathsf{L}} D_{\mathsf{LS}}}$$

iii) (theoretical) ***predict the observables from the optical scalars fields*** (shear, image positions, fluxes)

iv) (observational) ***measure the observables in astrophysical images***

v) (statistical***) estimate the lens model parameters by fitting the theoretical predictions to the data***

For example to estimate the mass of a galaxy cluster (to get the astrophysical measurement of Ω_m)**, we need to build a complex model that takes into account different components of mass distribution: dark matter halo, gas, galaxy distribution, and predict the distortions, positions and fluxes on the image plane of source background galaxies.**

Let us consider just the component of the matter distribution of the dark matter halo - a NFW density profile:

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$ (with 2 free parameters)

The 2D **surface mass density** can be computed from the 3D density profile and it is:

$$\Sigma(x) = \frac{2\rho_s r_s}{x^2 - 1} f(x) \ ,$$

with

$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \text{arctan}\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \text{arctanh}\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

and so the **convergence** is

$$\kappa(x) = \frac{\Sigma(\xi_0 x)}{\Sigma_{cr}} = 2\kappa_s \frac{f(x)}{x^2 - 1} \qquad \text{with} \qquad \kappa_s \equiv \rho_s r_s \Sigma_{\rm cr}^{-1}$$

from which we can obtain the **mass**,

$$m(x) = 2\int_0^x \kappa(x')x'dx' = 4k_s h(x)$$

with

$$h(x) = \ln\frac{x}{2} + \begin{cases} \frac{2}{\sqrt{x^2-1}} \text{arctan}\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ \frac{2}{\sqrt{1-x^2}} \text{arctanh}\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 1 & (x = 1) \end{cases}$$

We can also compute the **lensing potential**, which is,

$$\Psi(x) = 4\kappa_s g(x) \,,$$

where

$$g(x) = \frac{1}{2}\ln^2\frac{x}{2} + \begin{cases} 2\,\text{arctan}^2\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ -2\,\text{arctanh}^2\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

and the **deflection angle**, which is

$$\alpha(x) = \frac{4\kappa_s}{x} h(x)$$

From this, we can for example predict the image positions of source galaxies, fit to the observed positions and constrain the two parameters r_s and ρ_s needed to determine the value of the cluster mass.

Epilogue: Total density

$$\Omega = \Omega_m + \Omega_x = \Omega_{dm} + \Omega_b + \Omega_\nu \text{ (non-relativistic)} + \Omega_x = 1$$

Going back to the direct measurements of the densities in the Universe, we found that Ω_dm_clusters ~ 0.28

Sun: $M/L = 1$

$(M/L)_{galaxy} \sim 10\text{-}20\ M_{solar}/L_{solar}$

$(M/L)_{cluster} \sim 100\text{-}200\ M_{solar}/L_{solar}$

and that **dark matter is increasingly important as we go into larger scales**

Universe: $M/L = 1400\Omega_M h^2$

$\sim 200\ (\Omega_M/0.3)(h/0.7)^2$

This value Ω_dm ~0.28 is a good representation of the dark matter density in the Universe because clusters are very large quasi-linear structures that represent well the average densities of the whole Universe.

Indeed, **this value is confirmed by Ω_dm measurements of various cosmological probes** (i.e., by the more cosmological approach and model-dependent "indirect measurements"), including the well-known supernovae observations and CMB.

Also notice that the ratio between dark matter and baryonic matter is 7 (much lower than the M/L ratio of 160, which is the ratio between dark mater and luminous matter) → this shows that most of baryonic matter is not in the form of stars/galaxies that contribute to the luminous matter of galaxies and clusters but as we saw, it is in the form of hot ionized gas - in clusters and in the cosmic web -

So, since the 1980-90s, much before the modern SN and CMB probes, cluster observations already gave a hint that the total matter density in the Universe was less than 1 : $\Omega_{dm} + \Omega_b + \Omega_v \sim 0.3$ → this implies there is something else missing to reach the needed total of $\Omega = 1$ (from Friedmann eq.), and moreover it is the dominant contribution!

It was first thought that it could be a hint for an **open Universe, oCDM**.
Indeed, curvature can be moved to the right-side of Einstein equation and be considered as a contribution to the densities, Ω_K

Is it curvature? → the Universe would need to have negative curvature (which would also imply it is open) in order to have $\mathbf{\Omega_K >}$ 0 (the sign of Ω_K is opposite to the sign of K).
But no, later on CMB measurements found that most likely **Ω_K = 0** → flat Universe (even though some recent data also point to $\mathbf{\Omega_K <}$ 0 → this is part of the debate of the cosmological tensions - see later)

It has been a long story of missing components → missing baryons, missing matter, and now the missing 70% of the Universe

Following the discovery of the dimming of distant supernovae, there was evidence that the expansion of the Universe started accelerating in recent times. The driver for this acceleration had to be the missing density: an additional component that only became dominant recently and that has the property of accelerating the expansion.

Since we do not know what is this new source of energy, it was decided to call it **Dark Energy**, $\Omega_x = \Omega_{DE}$

Today there are many theoretical and phenomenological models of dark energy. The simplest one capable of producing the acceleration is the **the famous Einstein's cosmological constant, La**

The direct measurements are in agreement with modern results: **Planck final results (2018)**

Ω_cdm = 0.268 +/- 0.8%

Ω_b = 0.049 +/- 0.4%

→ Ω_m = 0.317 Ω_Λ = 1 - Ω_m = 0.683