Aula 10-03-2021

9 de março de 2021 22:55

1) a) $|\psi\rangle = N e^{-\alpha r}$

$\langle\psi|\psi\rangle = 1 \Leftrightarrow \int_0^{2\pi} d\varphi \int_0^{\pi} \operatorname{sen}\theta \, d\theta \int_0^{\infty} \psi^* \psi \, r^2 \, dr = 1$

$\boxed{P(\operatorname{sen}\theta) = -\cos\theta \quad P(1) = \theta}$

$\Leftrightarrow [\varphi]_0^{2\pi} [-\cos]_0^{\pi} \int_0^{\infty} \psi^* \psi \, r^2 \, dr = 1 \Leftrightarrow$

$\Leftrightarrow 2\pi \times 2 \int_0^{\infty} \psi^* \psi \, r^2 \, dr = 1 \Leftrightarrow 4\pi \int_0^{\infty} \psi^* \psi \, r^2 \, dr = 1$

$\Leftrightarrow 4\pi \int_0^{\infty} \underbrace{N e^{-\alpha r} N}_{N^2} e^{-\alpha r} r^2 \, dr = 1 \Leftrightarrow$

$\Leftrightarrow 4\pi N^2 \int_0^{\infty} \underbrace{r^2 e^{-2\alpha r}}_{\substack{n=2 \\ a=2\alpha}} dr = 1 \Leftrightarrow 4\pi N^2 \left[\frac{2!}{(2\alpha)^3}\right] = 1$

$\Leftrightarrow 4\pi N^2 \times 2 = (2\alpha)^3 \Leftrightarrow N^2 = \frac{2^3 \times \alpha^3}{2^3 \pi} \Leftrightarrow N^2 = \frac{\alpha^3}{\pi}$

$N = \sqrt{\frac{\alpha^3}{\pi}} \Rightarrow |\psi\rangle = \sqrt{\frac{\alpha^3}{\pi}} e^{-\alpha r}$

b) $T|\psi\rangle = -\frac{1}{2} \nabla^2 |\psi\rangle = -\frac{1}{2} r^{-2} \frac{d}{dr}\left[r^2 \frac{d}{dr} N e^{-\alpha r}\right]$

$$b)\quad T|\psi\rangle = -\frac{1}{2}\nabla|\psi\rangle = -\frac{1}{2}r^{-2}\frac{d}{dr}\left[r^2\frac{d}{dr}Ne^{-\alpha r}\right]$$

$$(e^{x})' = x'e^{x} \Rightarrow (e^{-\alpha r})' = (-\alpha r)'e^{-\alpha r} = (-\alpha)e^{-\alpha r}$$

$$\underbrace{-\frac{N}{2}r^{-2}\frac{d}{dr}\left[r^2(-\alpha)e^{-\alpha r}\right]}_{+} = \frac{N}{2}r^{-2}\left[2r\alpha e^{-\alpha r} + r^2\alpha(-\alpha)e^{-\alpha r}\right]$$

$$= \frac{N}{2}\left[\frac{2\cancel{r}\alpha e^{-\alpha r}}{r^{\cancel{2}}} - \frac{\cancel{r^2}\alpha^2 e^{-\alpha r}}{\cancel{r^2}}\right] =$$

$$= \frac{N}{2}\left[\frac{2\alpha e^{-\alpha r}}{r} - \alpha^2 e^{-\alpha r}\right]$$

$$\langle\psi|-\frac{1}{2}\nabla^2|\psi\rangle = \langle T\rangle =$$

$$= \underbrace{\int_0^{2\pi}d\varphi\int_0^{\pi}\operatorname{sen}\theta\, d\theta}_{4\pi}\int_0^{\infty}\overbrace{Ne^{-\alpha r}\,\frac{N}{2}}^{N^2/2}\left[\frac{2\alpha e^{-\alpha r}}{r} - \alpha^2 e^{-\alpha r}\right]r^2 dr$$

$$= \frac{4\pi N^2}{2}\int_0^{\infty}e^{-\alpha r}\left[\frac{2\alpha e^{-\alpha r}}{r} - \alpha^2 e^{-\alpha r}\right]r^2 dr =$$

$$= 2\pi N^2\int^{\infty}\frac{2\alpha r^{\cancel{2}}e^{-2\alpha r}}{\cancel{r}} - \alpha^2 r^2 e^{-2\alpha r}\,dr \Rightarrow$$

$$= 2\pi N \int_0 \frac{\ldots}{\cancel{\alpha}} - \alpha r \ldots$$

$$\Leftrightarrow 2\pi N^2 \left[2\alpha \underbrace{\int_0^{\infty} r e^{-2\alpha r} dr}_{\substack{n=1 \\ \alpha = 2\alpha}} - \alpha^2 \underbrace{\int_0^{\infty} r^2 e^{-2\alpha r} dr}_{\substack{n=2 \\ \alpha = 2\alpha}} \right] \Leftrightarrow$$

$$\Leftrightarrow 2\pi N^2 \left[\frac{2\alpha}{(2\alpha)^2} - \frac{\alpha^2 \, 2}{(2\alpha)^3} \right] \Leftrightarrow 2\pi N^2 \left[\frac{\cancel{2\alpha}}{2^{\cancel{2}} \alpha^{\cancel{2}}} - \frac{\cancel{\alpha^2} \, \cancel{2}}{2^{\cancel{3}} \cancel{\alpha^{3}}} \right]$$

$$\Leftrightarrow 2\pi N^2 \left[\frac{1}{2\alpha} - \frac{1}{4\alpha} \right] = 2\pi N^2 \left[\frac{2}{4\alpha} - \frac{1}{4\alpha} \right] =$$

$$= \frac{4\pi N^2}{4\alpha} - \frac{2\pi N^2}{4\alpha} = \frac{2\pi N^2}{4\alpha} \qquad N^2 = \frac{\alpha^3}{\pi}$$

$$\frac{2\pi \alpha^{\cancel{3} 2}}{4\cancel{\alpha}\cancel{\pi}} = \frac{\alpha^2}{2} = \langle T \rangle$$

---

c) $\langle V \rangle = \langle \psi | -\frac{1}{r} | \psi \rangle .$

$$= \underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta}_{4\pi} \int_0^{\infty} \overbrace{N e^{-\alpha r} \left(-\frac{1}{\cancel{r}}\right) N}^{N^2} e^{-\alpha r} r^{\cancel{2}} dr =$$

$$= -4\pi N^2 \int_0^\infty \underbrace{r e^{-2\alpha r}}_{\substack{n=1 \\ \alpha = 2\alpha}} dr = \frac{-4\pi N^2}{(2\alpha)^2} = -\frac{\cancel{4}\pi N^2}{\cancel{4}\alpha^2} = -\frac{\pi N^2}{\alpha^2}$$

$$N^2 = \frac{\alpha^3}{\pi}$$

$$= -\frac{\cancel{\pi}\alpha^{\cancel{3}}}{\cancel{\alpha^2}\cancel{\pi}} = -\alpha$$

---

d) $\langle E \rangle = \langle T \rangle + \langle V \rangle = \frac{\alpha^2}{2} - \alpha$

e) $\frac{dE}{d\alpha} = 0 \Leftrightarrow \frac{\cancel{2}\alpha}{\cancel{2}} - 1 = 0 \Leftrightarrow \alpha = 1$

logo

$\langle E \rangle = \frac{1^2}{2} - 1 = -1/2 \; H_a$