

UNIVERSO PRIMITIVO

Mestrado em Física Astronomia 2022-2023

Exercise Sheet 1

1. In a FRLW universe, fundamental observers experience no external forces and have fixed coordinates in the comoving coordinate system. The proper distance between two of such observers scales as $\mathbf{r}(t) = a(t) \mathbf{x}$, where $a(t)$ is the scale factor and \mathbf{x} is their comoving separation.
 - 1.1. Derivative this expression to obtain the Hubble law, $\mathbf{v}(t) = H(t) \mathbf{r}(t)$, where $H = \dot{a}/a$.
 - 1.2. Derive a similar expression for a pair of non-fundamental observers that have a relative peculiar velocity, $\mathbf{v}_p = \dot{\mathbf{x}}$, in the comoving coordinate system.

2. Consider a homogeneous and isotropic perfect fluid with an energy-stress tensor: $T_{\nu}^{\mu} = (\rho + p) U^{\mu} U_{\nu} - p g_{\nu}^{\mu}$.
 - 2.1. Apply the conservation law $T_{\nu;\mu}^{\mu} = 0$ to the $\nu = 0$ component to obtain the energy conservation equation $\dot{\rho} = -3H(\rho + p)$, where $H = \dot{a}/a$ is the Hubble constant.
 - 2.2. Use this equation to prove that $dE = -pdV$, where $dE = d(\rho a^3 L^3)$ is the energy inside a volume element, $dV = d(a^3 L^3)$, where L^3 is an arbitrary comoving volume.
 - 2.3. Integrate the energy conservation equation in 2.1 to prove that $\rho(t) = \rho_i \left(\frac{a(t)}{a_i}\right)^{-3(1+w)}$ where ρ_i, a_i are integration constants and w is the equation of state (EoS) parameter for a given fluid component.
 - 2.4. Use the expression in 2.3 to derive the time dependence of the scale factor for the following components: radiation ($w = 1/3$); collisionless matter ($w = 0$); and cosmological constant ($w = -1$) assuming the conditions (1), (2) and (3) at the bottom of slide 12 of chapter 2 of the course notes, respectively.

3. Consider the FLRW dynamic equations discussed in class.
 - 3.1. Use the Friedmann equation and the acceleration equations to derive the energy conservation equation in 2.1.
 - 3.2. Use the definition of the cosmological density parameters to re-write Friedmann equation in the following form (the subscript 'm' refers to all forms of matter, i.e. baryon and dark matter):

$$\begin{aligned}
 H^2(t) &= \frac{8\pi G}{3} (\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \\
 &= H_0^2 \left[\Omega_{r0} \left(\frac{a_0}{a}\right)^4 + \Omega_{m0} \left(\frac{a_0}{a}\right)^3 + \Omega_{k0} \left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda 0} \right]
 \end{aligned}$$

- 3.3. Consider the concordance model as in Baumann's lectures, page 25: $\Omega_{r0} = 9.4 \times 10^{-5}$, $\Omega_{m0} = 0.32$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0.68$. Derive approximate values for the redshift at radiation-matter equality and matter-dark energy equality epochs.
4. Use the Friedmann equation in 3.2 to compute the Age of the universe for a:
 - 4.1. Critical density universe ($\Omega_{r0} = 0$, $\Omega_{m0} = 1$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0$), with $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$
 - 4.2. Flat, Λ -Universe with $\Omega_{r0} \simeq 0$, $\Omega_{m0} = 0.32$, $\Omega_{k0} = 0$, $\Omega_{\Lambda 0} = 0.68$, $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$
[Hint: integrate the Friedmann equation with respect to the scale factor]