The Inhomogeneous Universe

Parameterization of the density contrast field

Space and Time description of Gaussian random fields

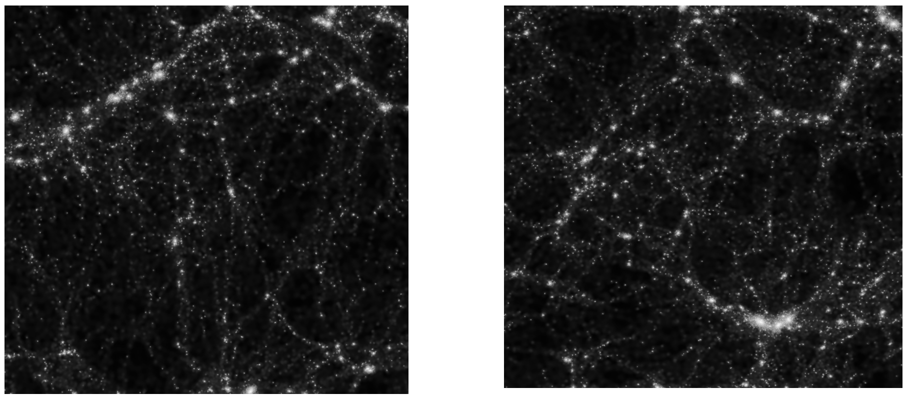
Two-point functions (power spectra and 2-pt correlation functions) of a given Gaussian cosmological inhomogeneous field contain the complete **spatial information** of the field.
(example of cosmological inhomogeneous fields are: **density contrast**, **peculiar velocity**, **gravitational lensing shear**, **CMB temperature anisotropy**, metric perturbations such as **Φ and Ψ gravitational potentials**).

From a 2-pt function, we can make a realization of the field, obtaining a map, i.e., the field as function of spatial coordinates (2D or 3D depending if we consider 3D or projected 2-pt functions), at a given time.

Remember: the density power spectrum does not give the values of δ(x) in specific coordinates, but it has all the information needed to produce **realizations** of δ(x) → **cosmological theory does not predict the exact maps of the universe**

These 2 universes have the same cosmological parameters values → the same power spectrum.

However, the values of δ(x) in points with the same coordinates are different.



The **time evolution** of the field is obtained from the time evolution of the two-point correlation functions or power spectrum.

The time evolution of the **power spectrum of the density contrast** can be computed from the cosmological theory, and this evolution is what is known as **structure formation.**

It is computed from a system of differential equations of motion for the various modes (scales) δ_k (t) and for the various cosmological species (δ_cdm, δ_baryons, δ_radiation).

The equations of motion are given by the Perturbed Einstein equations:

from a perturbed metric (scalar, vector, tensor perturbations and gauge transformation) + energy-momentum tensor of inhomogeneous fluids → Einstein equations for the inhomogeneous Universe (‘Friedmann-like’, ‘Raychaudhuri-like’ and other new equations) + energy conservation equations (continuity-like or alternatively the perturbed Boltzmann equation, which is needed when considering energy distributions at particle level instead of coherent fluid, like in the case of relativistic species).

It is also possible to derive the equations of motion in the

Euclidean approximation (valid for non-relativistic species and for sub-Hubble scales, i.e. scales smaller than the Hubble radius)

In this case, the power spectrum can be computed from classical fluid equations (Poisson, Euler, continuity) and there is no need to use Einstein equations.

The equations of motion allow us to compute the evolution of the power spectrum for each cosmological species.

For this, besides the equations of motion, we also need **initial conditions for the power spectrum**. These introduce new **cosmological parameters**.

There are also **initial conditions between species,** which introduce additional constraints between the various power sepctra (ex: **adiabatic perturbations, isocurvature perturbations**).

Remember:

- The equations of motion compute δ_k (t) for all scales k and all cosmological species, i.e., they compute the cosmological variances, which is the function
$P(k) = \delta^2(k)$ (i.e., the power spectrum).

They do not compute a unique solution $\delta(x)$.

- The power spectrum is not enough to describe the inhomogenous Universe if the perturbations are **non-Gaussian** → in that case we also need to consider **higher-order n-pt functions**.

- Note that the evolution of δ occurs while the universe is expanding "in the background".
The evolution of the homogeneous universe is also called **background evolution**.

Initial conditions

For the **initial conditions** of the density perturbations we need a set of N values for each species:

the amplitude (i.e. the variance) of each scale at a fixed time → in principle we will need **N new cosmological parameters for each cosmological species.**

In the homogeneous universe the initial condition is a free parameter (an Ω value) and usually is set at today's value and not at an initial value.

In the inhomogeneous universe it is possible to derive some theoretical constraints on the early Universe (instead of using late-time conditions) from **inflation.**

Inflation considers that the Universe is filled with a primordial scalar quantum field (called the **inflaton**). The quantum fluctuations that naturally exist in this field, evolve in the inflationary expansion, resulting in an inhomogeneous gravitational field, i.e., after inflation **perturbations in space-time curvature** appear.

An interesting property of inflation is that the inflationary evolution for the curvature perturbation has an **attractor solution** → the result is independent of particular realizations of the quantum fluctuations.

This is a key aspect of inflation → **It allows the computation of the post-inflationary metric perturbations independently of the original initial conditions** → **no fine-tuning** (the result depends only on the inflationary model used).

A given inflationary model thus computes the metric perturbations (created by the inflaton field).

In particular, it computes the perturbed gravitational random field at all scales, i.e., **inflation provides the post-inflationary power spectrum of gravity P_Φ**

(we will introduce later the metric perturbations and the two fields Φ and Ψ)

(In addition, inflation also computes the post-inflationary power spectrum of tensor metric perturbations, not relevant for the matter power spectrum).

The post-inflationary power spectrum is the initial condition for the subsequent process of structure formation → it is known as the **primordial power spectrum of the gravitational field**.

Note that in practice this power spectrum is obtained up to a constant → in reality inflation computes only the **relative amplitudes between all scales**, but it does not compute their absolute values.

This means that the result is a function of scale k, with free amplitude.

The result of inflation is a **scale-invariant dimensionless power spectrum** of the gravitational potential.

$$k^3 \; P^0_\Phi \quad = \text{constant}$$

This means that the amplitudes of the metric perturbations are the same for all scales (which means that the primordial perturbations are **white noise**)

→ the '**gravitational potential**' at any scale starts the structure formation process with the same amplitude. A priori there is no scale that will be more favorable to collapse and form structure (note **there is also no homogeneity scale for the potential**).

This is also known as the Harrison-Zeldovich power spectrum - a flat spectrum

Note that this is the result expected when the expansion is exponential (like during the inflation): $a(t) \sim e^{Ht}$, with H constant during the short inflationary period.

We can see this by thinking of a discretized expansion, i.e.

on each **e-folding** (equal time intervals where the Universe expands by an order of magnitude) the Universe "remains a certain time with a certain size".

That size (the order of magnitude of the e-folding) **defines a logarithmic scale**.

The time that the Universe stays on each scale is the same.

Since the times are the same, there is the same probability of forming inhomogeneities on all these logarithmic scales → **leading to the same amplitude of Φ on all logarithmic scales** → the same power per logarithmic bin → **constant dimensionless power spectrum**.

It is usual to write the result in the form:

$$k^3 \, P^0_\Phi \propto k^{n_s - 1}$$

allowing for a small deviation from the exactly scale-invariant power spectrum (the case $n_s = 1$).

This means that the dimensionless primordial power spectrum of gravity is a power law (in scale k) with index n_s - 1.

This introduces a new cosmological parameter - the **power law index n_s** - which parametrizes the relative amplitude between all scales.

This parameter is related to inflation **slow-roll parameters**:

$n_s = 1 - 2\varepsilon + 2\eta \rightarrow n_s$ is close to 1 (and smaller than 1).

Now, **the fact that all scales have the same initial gravitational conditions to collapse, does not mean that all matter perturbations start with equal amplitudes.**

We still need to find out what is the **primordial power spectrum of the density contrast field.**

Since the gravitational potential is a metric perturbation, we need the Einstein equations to relate metric perturbations to matter perturbations.

The gravitational potential is a term in the metric (00) that makes the inhomogeneous metric deviate from a perfect Robertson-Walker metric.

The (first-order) Friedmann equation in the inhomogeneous metric relates this term of the metric to the matter density perturbation. It is a **Poisson-like equation** (as we will see later).

In the sub-Hubble (Euclidean approximation) the original Poisson equation is valid, and we can write:

$$\nabla^2\phi = \frac{3H_0^2}{2a}\Omega_m\delta$$

We want to relate the power spectrum of the gravitational field to the power spectrum of the density.

For this **we need to take the Fourier transform of the Poisson equation**.

The right-hand side only contains spatially constant quantities and the density contrast, so its transform is just the transform of the density contrast:

$$\frac{3H_0^2}{2a}\Omega_m \delta_k$$

The left-hand side contains the Laplacian of the gravitational potential.

Its Fourier transform is written as:

$$\int \nabla^2(\Phi(r)e^{-ik.r})d^3r$$

Using the product rule, the transform of the Laplacian of the potential may be written as:

$$\int \nabla^2(\Phi(r))e^{-ik.r}d^3r = \int \nabla^2(\Phi(r)e^{-ik.r})d^3r - \int \Phi(r)\nabla^2(e^{-ik.r})d^3r$$

Now, in the **second term** of the right-hand-side we can take the second-order derivative of the plane wave and we are left with

the Fourier transform of the potential multiplied by k^2.

In the **first term** of the right-hand-side we can replace the volume integral of the Laplacian by the integral of the flux (the gradient of the potential) across the surface enclosing the volume, using the **theorem of Gauss** (also known as the divergence theorem).

So, the Fourier transform of the Laplacian of the gravitational potential is:

$$\int \nabla^2(\Phi(r))e^{-ik.r}d^3r = \int \vec{\nabla}\Phi(r).\vec{e}_r e^{-ik.r}r^2d^2\Omega - k^2\int \Phi(r)e^{-ik.r}d^3r$$

Since the domain of a d^3r integral is infinity, the surface integral is made on a sphere with radius r → ∞ , and since $r^2 \nabla\Phi$ → 0, the first term is zero.

So, Poisson's equation in Fourier space is simply:

$$\nabla^2\phi = \frac{3H_0^2}{2a}\Omega_m\delta \quad \rightarrow \quad -|k|^2\,\phi_k = \frac{3H_0^2}{2a}\Omega_m\,\delta_k$$

This is a very useful result: to Fourier transform the spatial derivative of a quantity we just need to multiply it by $-(-ik)^n$.

So, grad(F) → ik F_k and lap(F) → - k^2 F_k

Now, taking the square of both sides of the Poisson equation, we find a **"Poisson equation for the power spectrum"**: **this is the relation between the gravitational potential power spectrum and the matter power spectrum**:

$$P_\delta^0 \sim k^4 P_\Phi \sim k^4 k^{-3} k^{n_s-1} = A_s(k_0)\,k^{n_s}$$

We have thus found that the **primordial matter power spectrum** (the initial conditions for the matter fluctuations after inflation) is also a power-law, like the gravitational potential power spectrum, but with a different slope.

In particular, it is not scale-invariant:
small-scales (large k) start with a larger clustering amplitude than large-scales.

The primordial power spectrum is parameterized by only 2 free parameters:

- a slope n_s (parameterizing the relative amplitudes between the various scales).

- an amplitude A_s (given at a chosen scale; any scale may be used, but usually $k_0 = 0.02$ h/Mpc is chosen)

The fact that inflation is able to predict a functional form for the power spectrum is responsible for reducing the initial condition free parameters from N to only 2.

Cosmological parameters

We have introduced 2 new fundamental cosmological parameters $\mathbf{n_s}$ and $\mathbf{A_s}$ to add to the list of parameters that describe the cosmological model

(which included already H_0, Ω_{cdm} , Ω_b, Ω_{rad}, Ω_Λ, Ω_K)

Alternatively, as we saw earlier, the amplitude may be parametrized by the amplitude of the matter power spectrum at z=0.

In that case, the scale $k=2\pi/8$ h/Mpc is used.

and this is called the $\boldsymbol{\sigma_8}$ parameter.

The relation between σ_8 and A_s depends on the evolution of the power spectrum from early times to z=0 → it is not a simple scaling, it depends on all cosmological parameters.

These two are the most important new parameters, but there are many other cosmological parameters (or functions of redshift that can be parametrized) that are needed to model all aspects of the cosmological model, such as:

- Describe extra species in the homogeneous universe:

neutrinos - Ω_ν, N_{eff}
dark energy - $w_{DE}(z)$ (and many other parameters depending on the specific dark energy model)

- Describe pressure perturbations - speed of sound $c_s(z)$

- Describe other mechanisms of the perturbed universe:

reioinization redshift (formation of the first stars) - optical depth τ_{re}
halo profiles in non-linear collapse - ρ_c, concentration c
power spectrum of tensor perturbations - n_t, A_t or r
modified spectrum of initial conditions - running of the spectral index $n_s(k)$

- Describe specific cosmological probes:

$$n_s(k) = n_s(k_0) + \frac{1}{2}\frac{dn_s}{d\ln k}(k_0)\ln\left(\frac{k}{k_0}\right) + O(k^2)$$

redshift of sources for weak lensing - $n(z)$
mass-to-light bias for galaxy clustering - $b(k,z)$

content of the Universe

total energy density

$\mathbf{\Omega_{tot}}$ **(=1?)**

matter density

$\mathbf{\Omega_m}$

baryon density

$\mathbf{\Omega_b}$

neutrino density

$\mathbf{\Omega_n}$ **(=0?)**

Neutrino species

$\mathbf{f_n}$

dark energy eq^n of state

w(a) (=-1?)

or $\mathbf{w_0, w_1}$

evolution to present day

Hubble parameter

h

Optical depth to CMB

$\boldsymbol{\tau}$

perturbations after inflation

scalar spectral index

$\mathbf{n_s}$ **(=1?)**

normalisation

$\boldsymbol{\sigma_8}$

running

$\mathbf{a = dn_s/dk}$ **(=0?)**

tensor spectral index

$\mathbf{n_t}$ **(=0?)**

tensor/scalar ratio

r (=0?)

Overview of structure formation

Clustering and Collapse

Structure formation starts then from the primordial power spectrum, and is driven by gravity described by General Relativity.

We can consider two regimes of structure formation:

Linear evolution (clustering)

While the density contrast is still small, **$\delta \ll 1$**, the harmonic modes evolve independently of each other, with a system of uncoupled equations.

Scales in this regime are called **linear**.

Linear scales follow the expansion (they are expanding with the comoving background) but they expand with a slower rate (their density decreases with a slower rate than a^{-3}) → they are slowly forming over-dense structures → they cluster, and so their spatial distribution is correlated.

Non-linear evolution (collapse)

When the density contrast of a certain scale gets large, **$\delta \sim 1$**, that scale becomes **non-linear**.

More precisely, it is considered that **the transition happens when the growing dimensionless matter power spectrum reaches Δ^2 (k) = 1**

That scale decouples from the expansion (i.e. stops following the expansion) and starts to decrease in proper size. The density constrast starts increasing very fast, resulting in a gravitationally bound structure → there is a collapse and structure is formed (in reality linear evolution does not really produces structure, just slowly growing values of the density contrast).

Linearized perturbed Einstein equations are no longer valid to compute the non-linear evolution.

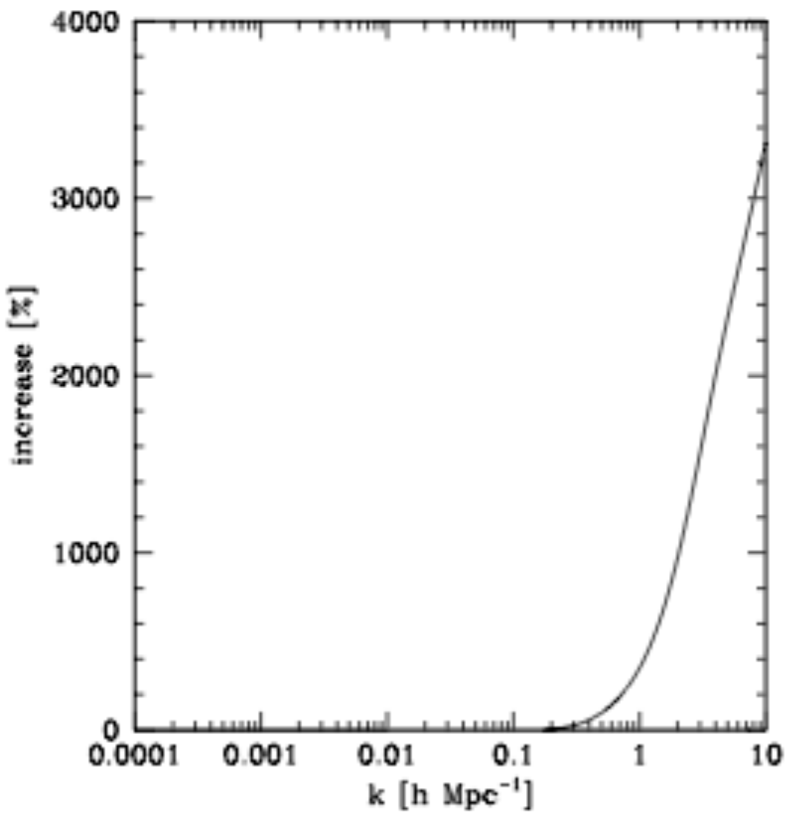
Different approaches are needed → higher-order perturbation theory (renormalization methods, 'Feynman diagrams'), the Halo model approximation, N-body simulations.

In ΛCDM cosmology with standard values of the cosmological parameters (concordance model), the non-linear scales are the smallest ones:

$k > 0.2$ h/Mpc

The amplitude of the non-linear matter power spectrum is much larger than the amplitude that would be computed for those same scales assuming linear evolution.

This figure shows the percentage increase of power (P_{NL}/P_L) as function of scale:



In addition, when the evolution is non-linear, the statistical distribution of the density contrast (Gaussian) is not preserved and non-Gaussianities arise.

Hierarchical structure formation

As we will see, the result of the power spectrum evolution in the standard ΛCDM model is a **hierarchical structure formation**, i.e., scales move from the clustering phase to the collapsing phase in different times and sequentially → smaller scales collapse first than larger scales:

Very small scales (stars, star clusters, satellite galaxies) are not cosmological scales. They are below the free-streaming limit and do not form from the structure formation process of cosmological evolution → they are sub-clumps of galactic-size dark matter halos.

Small scales (galactic scale) are the first to reach non-linearity and to decouple from the expansion. They are the first to form structures (the galaxies). On these scales $\Delta^2 > 1$ today.

Intermediate scales (clusters scale)
are the scales that are collapsing today at z=0 (meaning that only today did Δ^2 at this scale reach the value ~1.

In the standard model, the theoretical calculation finds that this scale is

$k = 0.78$ h/Mpc → R ~ 8 Mpc/h.

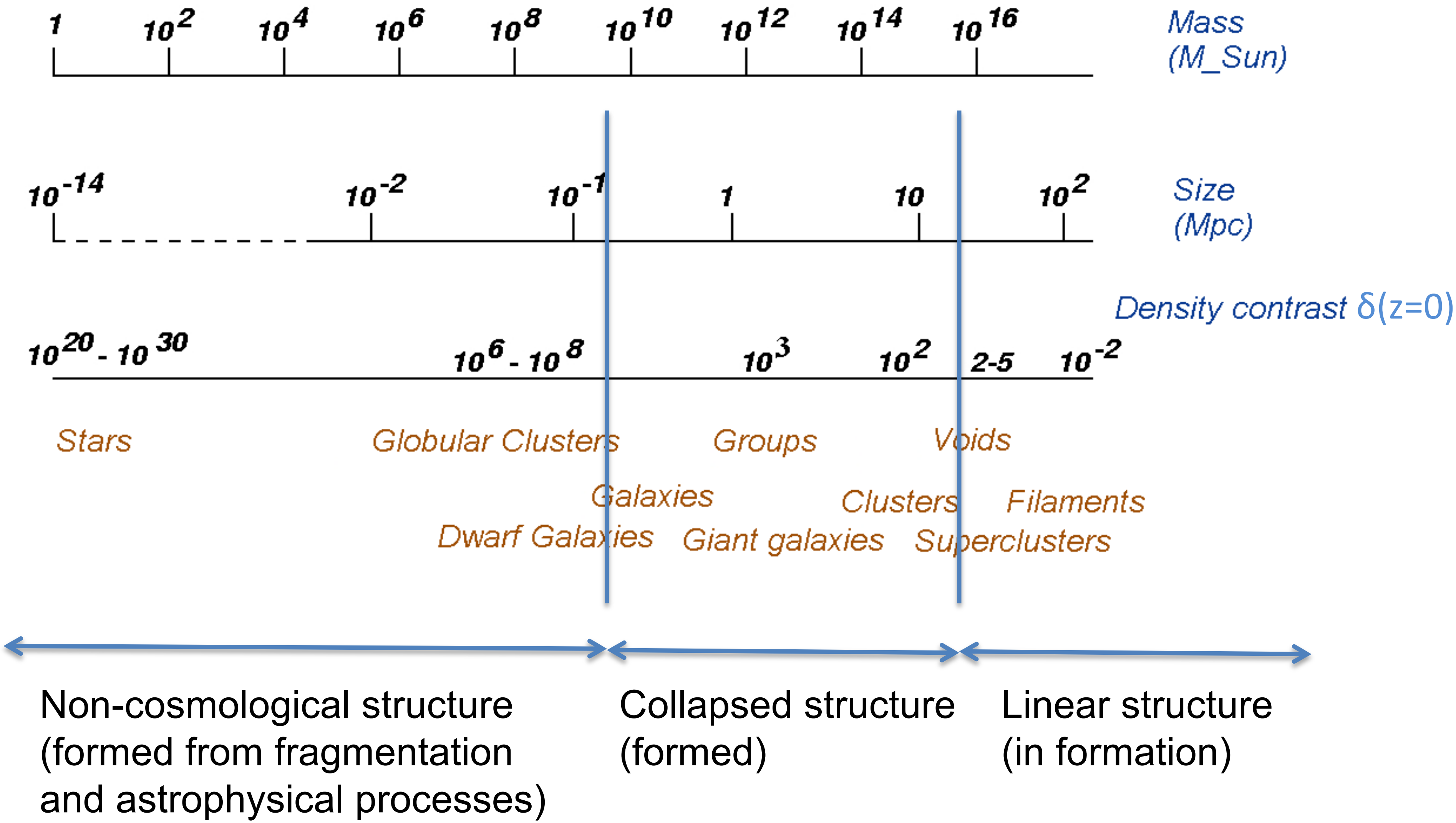
Consider a homogeneous sphere of this size with density twice the mean density of the universe (i.e. $\delta = 1$). Inserting the standard values for the critical density and matter density, the mass of this region is

$$M = 4/3\pi\ 8^3\ \rho_c\ \Omega_m\ (1+\delta) \sim 10^{15}\ M\_Sun / h$$

which is a typical mass of a cluster, thus confirming that galaxy clusters are the largest collapsed structures.

Large scales (R > 8 Mpc/h)
for example filaments, are still linear today ($\Delta^2 < 1$) and did not form bound collapsed structures yet → not dense enough for star formation → they are non-luminous.
But they continue to evolve and will form large bound structures in the future.

Very large scales (R > 500 Mpc)
δ is so small that perturbations are still negligible today.
If $\Delta^2 \to 0$ on large scales, the Universe has a homogeneity scale.



1
10^2
10^4
10^6
10^8
10^10
10^12
10^14
10^16
Mass (M_Sun)
10^-14
10^-2
10^-1
1
10
10^2
Size (Mpc)
Density contrast δ(z=0)
10^20 - 10^30
10^6 - 10^8
10^3
10^2
2-5
10^-2
Stars
Globular Clusters
Groups
Voids
Galaxies
Clusters
Filaments
Dwarf Galaxies
Giant galaxies
Superclusters
Non-cosmological structure (formed from fragmentation and astrophysical processes)
Collapsed structure (formed)
Linear structure (in formation)

The dark matter power spectrum today (z=0)

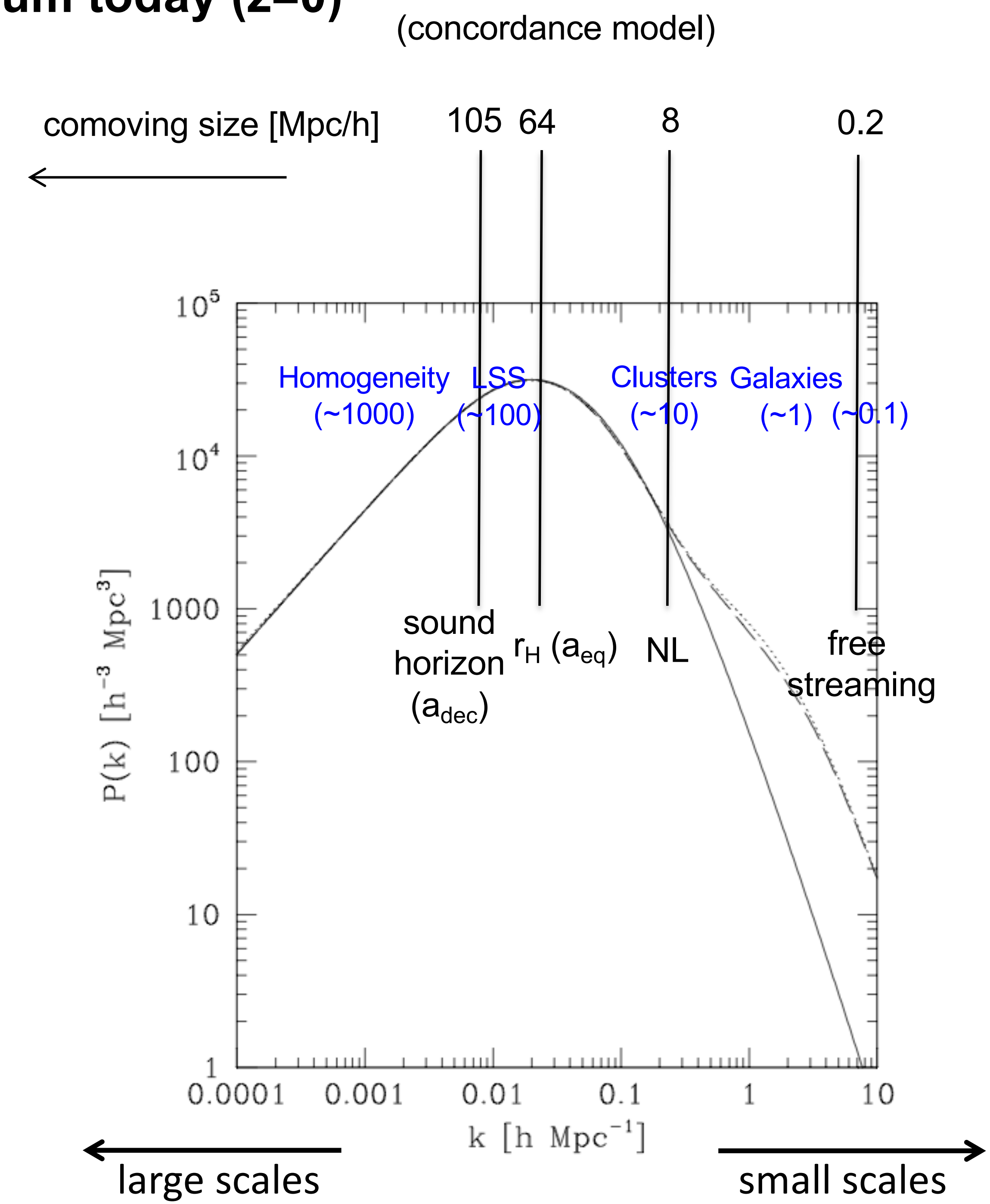
The structure formation process transforms the primordial matter power spectrum of the early Universe into the current matter power spectrum at z=0,

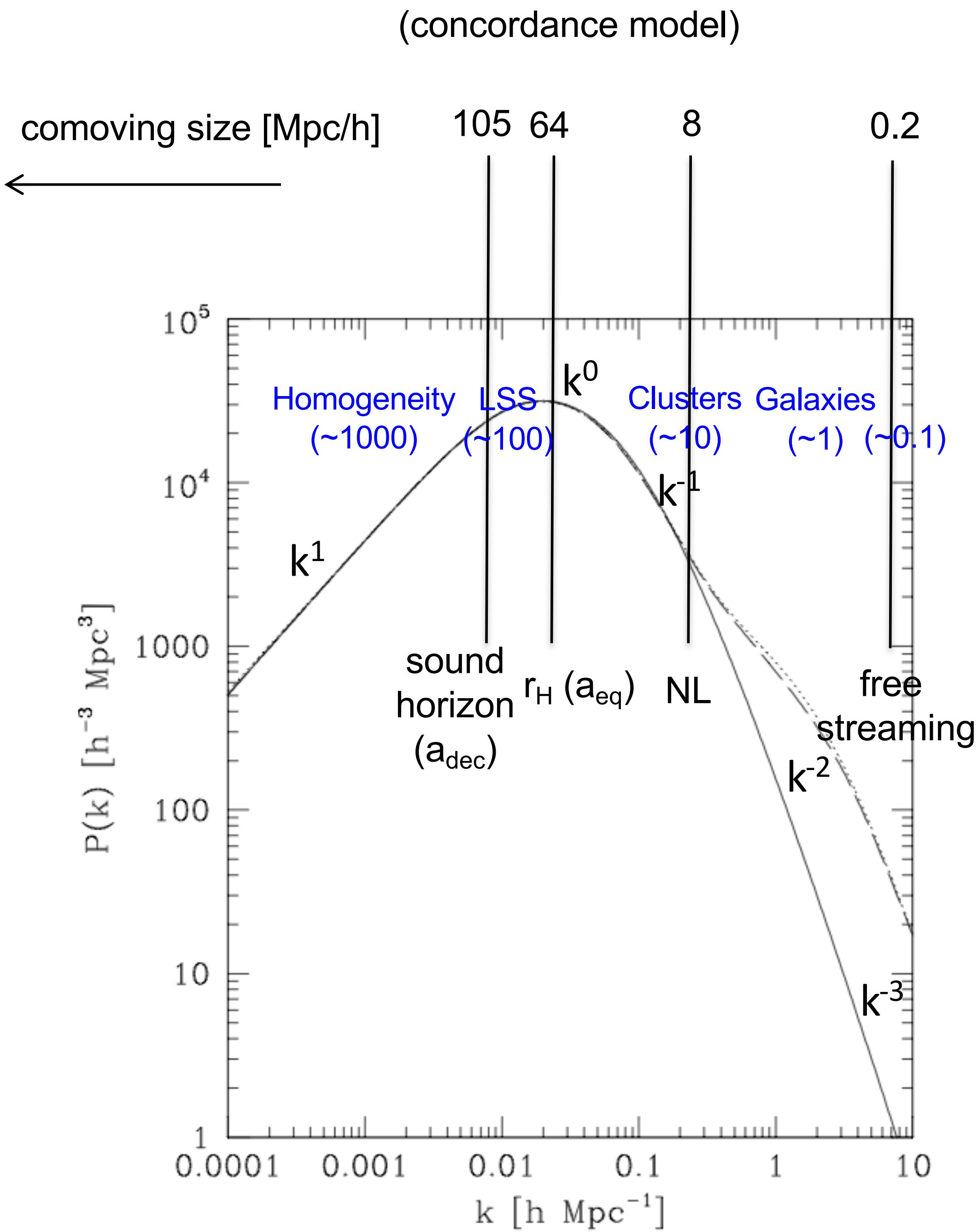
by making an overall increase of the amplitude (by a **growth factor** $D_+(a)$) and a change in shape (T(k)):

$$P_\delta(k,a) = A k^{n_s} \, T^2(k) \, D_+^2(a)$$

the change in shape is called the **transfer function**.

The figure shows the **theoretical dark matter power spectrum** at z=0, computed for the concordance model.





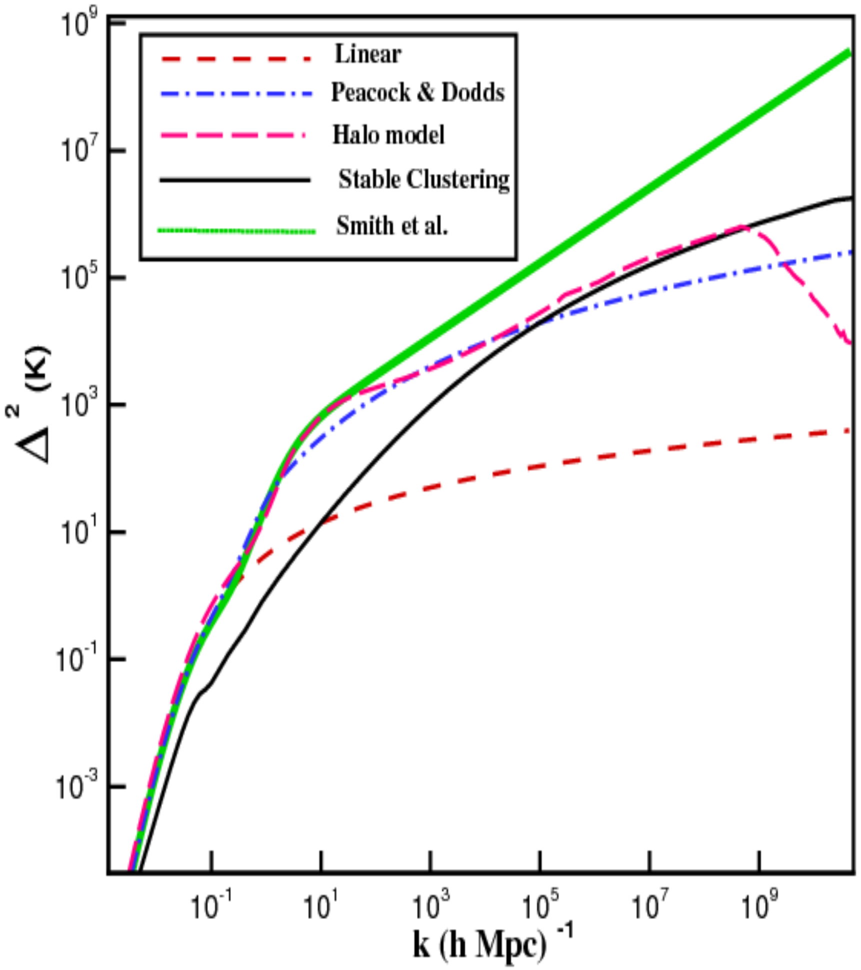
The shape changed from the original power-law k^1 to a sequence of power-laws with indexes 1,0,-1,-2,-3

Some important scales are shown:

- sound horizon at decoupling (z=1100)
- Hubble radius at radiation-matter equality (z ~3500)
- non-linear scale at z=0
- free-streaming scale at z=0

The dashed line shows the non-linear power spectrum. It deviates from the linear one for scales smaller than the NL scale.

The figure shows the **theoretical dimensionless dark matter power spectrum** at z=0, up to an extremely small scale.



Its shape is a a sequence of power-laws with indexes 4,3,2,1,0

The dimensionless power spectrum increases for small scales, while the power spectrum has a peak.

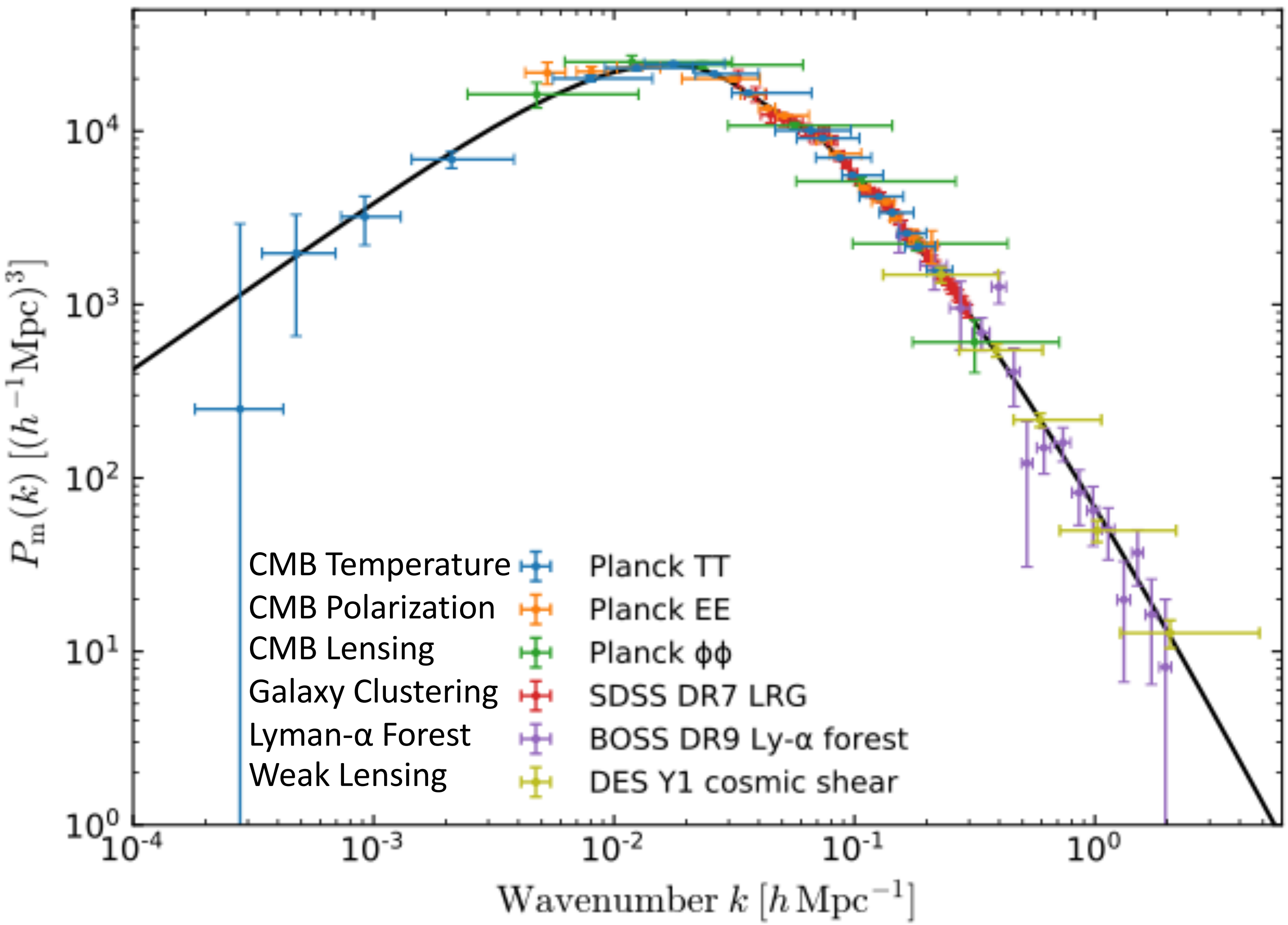
The amplitude of clustering is directly seen in the dimensionless power spectrum and not in the power spectrum. The reason is that the power spectrum has dimensions of volume (inverse of k-volume), so the amplitude it shows on small scales is greatly decreased by a factor of k^3.

The linear Δ^2 is the dashed red line. All the other lines show the non-linear Δ^2 computed in different ways. Notice that:

-The ratio between non-linear and linear is very large on these small scales.
- Different methods to compute the non-linear solution give quite different results.

The figure shows the **measured dark matter power spectrum** by various **cosmological probes of the inhomogeneous universe,** measured in 4 different **surveys**

Each survey may observe at a different redshift (but all results are transposed for z=0)



The various probes observe different scales:

- Probes of high redshift (CMB) measured in surveys of high resolution (Planck) may measure correlations on small angular scales, but even a small angular scale corresponds to a large size (given the high z) , and so to a large scale (small k) when transposed to z=0.
- Probes of lower redshift - DES measured correlations of WL on smaller scales than the GC measurements made by SDSS that covered a larger area of the sky.