The Lagrangian - From fields to events

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Dark Matter, Phase transitions and Gravitational Waves

The Lagrangian, the Action and Natural units

The classical Lagrangian

In the old days of Analytical Mechanics the Action (S) was given by

$$S = \int_{t_1}^{t_2} dt\, L(q_i(t), \dot{q}_i(t)) \qquad with \qquad L = \int d^3x \mathscr{L}(x)$$

where $\mathscr{L}$ is the Lagrangian density and $q_i(t), \dot{q}_i(t)$ are the generalised position and velocity, respectively. The minimisation of the action leads to Euler-Lagrange equations of motion. To go from just classical to a classical field theory we have to replace

$$q_i \to \phi(x^\mu) \equiv \phi(x); \qquad \dot{q}_i \to \partial_\mu \phi(x) \equiv \frac{\partial}{\partial x^\mu} \phi(x)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla} \right)$$

$$x_\mu \equiv \left(t, -\vec{x}\right) = \left(t, -x, -y, -z\right)$$

Our starting point is a Lagrangian density with an Action written as

$$S = \int d^4x\, \mathscr{L}(\phi(x), \partial_\mu \phi(x))$$

The Lagrangian density (only Lagrangian for us) will be our best friend during these lectures.

The fields

The fields are the basic entities in QFT. Particles are excitations of fields. Fields exist of course at the classical level. Once they are operators and quantised, and incorporating relativity in our description we will have a full QFT. Fields corresponding to already observed particles

ϕ Scalar or pseudoscalar - spin zero field (Higgs)

A_μ Vector or Axial vector - spin one field (gluons, weak gauge bosons, photon)

ψ Spinor - spin 1/2 field (Fermions - quarks and leptons)

Is dark matter one of these (kind of) particles? We don't know. Is one favoured over the others? Not that we know. Is any of them excluded by experiment. No.

Can we build models with any of them? Yes we can!

Building the Lagrangian - dimensions

$\hbar = c = 1$ - In Natural Units all quantities are measured in units of mass/energy to some power.

$$[p_\mu] = [\partial_\mu] = m \qquad [x_\mu] = m^{-1}$$

m stands for mass/energy

The action is now dimensionless (because Planck constant is 1)

$$S = \int \mathscr{L}\, d^4x \;\Rightarrow\; [\mathscr{L}] = m^4$$

The canonical dimension of the field is obtained from the free Lagrangian

$$\mathscr{L}^{KG}_{free} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \;\Rightarrow\; [\phi] = m$$

$$\mathscr{L}^{EM}_{free} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;\Rightarrow\; [F_{\mu\nu}] = m^2 \;\Rightarrow\; [A_\mu] = [F/\partial] = m$$

$$\mathscr{L}^{Dirac}_{free} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \;\Rightarrow\; [\psi] = m^{3/2}$$

$$\begin{aligned} 1\,\mathrm{GeV}^{-2} &= 0.389\,\mathrm{mb} \\ 1\,\mathrm{GeV}^{-1} &= 6.582 \cdot 10^{-25}\,\mathrm{s} \\ 1\,\mathrm{kg} &= 5.61 \cdot 10^{26}\,\mathrm{GeV} \\ 1\,\mathrm{m} &= 5.07 \cdot 10^{15}\,\mathrm{GeV}^{-1} \\ 1\,\mathrm{s} &= 1.52 \cdot 10^{24}\,\mathrm{GeV}^{-1} \end{aligned}$$

Building the Lagrangian - dimensions

Know your couplings; regarding interactions there are mainly three types (in what concerns dimensions)

- $\lambda_3 \phi^3 \;\Rightarrow\; [\lambda_3] = m$
- $\lambda_4 \phi^4 \;\Rightarrow\; [\lambda_4] = m^0 = 1$
- $\lambda_5 \phi^5 \;\Rightarrow\; [\lambda_5] = m^{-1}$

$$S = \int \mathscr{L}\, d^4x \;\Rightarrow\; [\mathscr{L}] = m^4$$

When is a coupling large? The theory is perturbative if:

If the coupling has mass dimensions, the mass has to be well below the energy scale probed.

If the coupling is dimensionless, the coupling has to be below 1 or below 4π or other possibilities (more later).

If the coupling has inverse mass dimensions, the mass has to be well above the energy scale probed (Fermi theory).

Building the Lagrangian - symmetries

Consider the function

$$f(x) = x^2 + x^3 + \sin x + \cos x$$

and now suppose you want this function to be invariant under $x \to -x$. The function becomes

$$f(x) = x^2 + \cos x$$

The world after Emmy Noether - invariance

Now you do this for a Lagrangian. The Lagrangian describes the world. The world has this symmetry.

Invariance under a given transformation leads to conserved quantities.

https://arxiv.org/abs/1902.01989

The world after Emmy Noether - conservation theorems

Invariant Variational Problems

I. If the integral $\mathcal{I}$ is invariant under a finite continuous group $\mathcal{G}_\rho$ with ρ parameters, then there are ρ linearly independent combinations among the Lagrangian expressions that become divergences—and conversely, that implies the invariance of $\mathcal{I}$ under a group $\mathcal{G}_\rho$.

I includes all the known theorems in mechanics, etc., concerning first integrals.

II. If the integral $\mathcal{I}$ is invariant under an infinite continuous group $\mathcal{G}^{\infty}{}_\rho$ depending on ρ arbitrary functions and their derivatives up to order σ, then there are ρ identities among the Lagrangian expressions and their derivatives up to order σ. Here as well the converse is valid.

II can be described as the maximum generalization in group theory of "general relativity."

Noether's Theorem (1915):

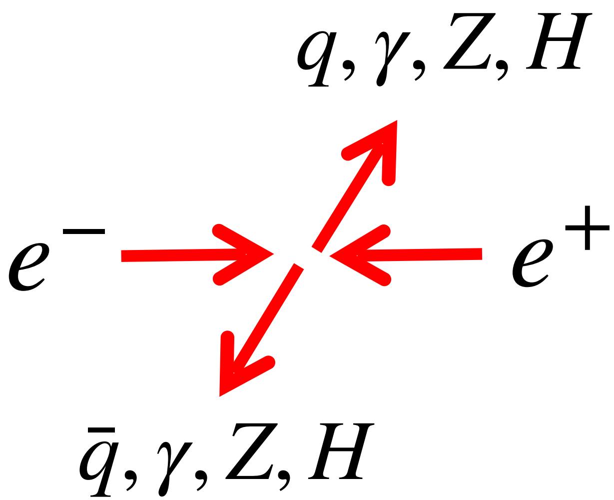
For every continuous symmetry in nature, there is a corresponding conservation law.

Every conservation law has a corresponding symmetry.

Elementary Consequences of Theorem I

Translation in space *No preferred location*	Momentum Conservation
Translation in time *No preferred time*	Energy Conservation
Rotational invariance *No preferred direction*	Angular Momentum Conservation

The world after Emmy Noether - internal symmetries



Why is charged conserved?

T1 - Global phase invariance implies a conserved quantity - we call it electric charge

T2 - Local phase (local gauge) invariance generate QED (more later).

Symmetries of the Lagrangian imply conservation laws.

$$\mathscr{L} = e\ \bar{\psi}_e \gamma_\mu \psi_e A^\mu$$

Conserved quantities ⟷ Electric charge

↕ ↕

Rules for interactions ⟷ Gauge symmetry

Examples of symmetries

Lagrangian built with a complex scalar field ϕ invariant under a U(1) transformation

$$\phi \to e^{i\theta}\phi$$

Terms in the Lagrangian that are invariant under this transformation

$$\phi^*\phi, \qquad (\phi^*\phi)^n, \ldots$$

Terms in the Lagrangian that break the symmetry

$\phi^2, \qquad \phi^3$ <u>Soft breaking terms</u>

ϕ^4 <u>Hard breaking term</u>

Examples of symmetries

Lagrangian built with two real scalar fields ϕ_1 and ϕ_2 invariant under a $\mathbb{Z}_2$ transformation

$$\phi_1 \to \phi_1, \qquad \phi_2 \to -\phi_2$$

Terms in the Lagrangian that are invariant under this transformation

$$\phi_1^2, \quad \phi_2^2, \quad \phi_1^3, \quad \phi_1^4, \quad \phi_1^2\phi_2^2, \ldots$$

Terms in the Lagrangian that would break the symmetry

$\phi_1^2\phi_2, \quad \phi_2^3$ — Soft breaking terms

$\phi_1^3\phi_2$ — Hard breaking term

P: Build the most general Lagrangian (up to dimension 4) with three fields ϕ_1, ϕ_2, ϕ_3 symmetries, the first two real and the third complex, with a symmetry $\phi_1 \to \phi_1, \quad \phi_2 \to -\phi_2, \quad \phi_3 \to e^{i\theta}\phi_3.$

The charge perspective

We can also look at the problem in terms of the charge of the fields. For the $\mathbb{Z}_2$ transformation we can think of one field with charge 1 (or even under the transformation) and the other with charge -1 (or odd under the transformation).

The electric charge is related to the U(1) invariance. An operator that commutes with the Hamiltonian can be defined in terms of creation and annihilation operators

$$Q = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_p} (a_p^\dagger a_p - b_p^\dagger b_p)$$

In this framework, charge (in a very general sense) is the rule to build the Lagrangian.

$$\mathscr{L} = a\, b\, c \quad \Longleftarrow \quad \begin{cases} Q(a) = -1 \\ Q(b) = 1 \\ Q(c) = 0 \end{cases}$$

Symmetries

In Quantum field theory symmetries can be:

Local - independent transformations at every point $\phi \to e^{i\theta(x)}\phi$

Global - transformation same at every point $\phi \to e^{i\theta}\phi$

Spacetime - symmetries acting on spacetime coordinates - Parity

Internal - intrinsic symmetries (independent of coordinates) - Charge Conjugation

Internal symmetries can be represented by a set (group) of unitary matrices denoted by U(n) or by a set (group) of special unitary matrices - with unit determinant denoted by SU(n). Here, n is the size/ dimension of the matrices.

The simplest internal symmetry group is U(1). Geometrically, it is the rotational symmetry of a circle. It is the one in QED.

SU(2) is the rotational symmetry of a sphere - a set of 2 x 2 matrices with unit determinant. They model the weak nuclear interactions - between pair of fermions (e.g. electron and neutrino) and a set of three bosons: Z and W (+,-)

The free scalar field

We will come back to this point later. For now you just have to know that a renormalisable theory (predictability) can only have terms in the Lagrangian up to dimension four.

For a field not to be static we need a kinetic term in the Lagrangian. It also has to be Lorentz invariant. So the kinetic term is

$$\mathscr{L} = \partial_\mu \phi \partial^\mu \phi$$

Mass dimension 4

If we have a plane wave this lead to a momentum squared term. Remember that in normal quantum mechanics one starts with

$$E = \frac{(\vec{p})^2}{2m} \qquad \vec{p} \to i\overrightarrow{\nabla}; \quad E \to i\frac{\partial}{\partial t}$$

to get Schrödinger's equation. Klein-Gordon equation is obtained with the same replacement but now on the relativistic version

$$E^2 = \vec{p}^2 + m^2 \quad \to \quad (\Box + m^2)\phi = 0 \qquad \Box \equiv \partial_\mu \partial^\mu$$

The free scalar field with mass

What is the relation between the Lagrangian and the Klein-Gordon equation? Starting with a Lagrangian

$$\mathscr{L} = \frac{1}{2}[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2]$$

Mass dimension 4

and using the equations of motion

$$\frac{\partial \mathscr{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) \quad \Rightarrow \quad -m^2 \phi = \partial_\mu \partial^\mu \phi$$

So we can write the Lagrangian that fits a given equation of motion. The solution of the Klein-Gordon equation, a free scalar field propagating in space is a plane wave

$$(\Box + m^2)\phi = 0 \quad \Rightarrow \quad \phi = N e^{-ip.x} \qquad E^2 = \vec{p}^2 + m^2$$

The scalar field with interactions

What if we include interactions? If there is a potential the Lagrangian is L=T-V. So we have

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

Mass dimension 4

and if we expand the potential in powers of ϕ

$$V(\phi) = a\phi + b\phi^2 + c\phi^3 + d\phi^4 + e\phi^5 + \dots$$

It the potential has a minimum at ϕ_0

$$\frac{\partial V}{\partial \phi}|_{\phi=\phi_0} = 0 \qquad and \qquad V(\phi_0) = V_0$$

And so

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2 + M\phi^3 + \lambda\phi^4 + e\phi^5 + \dots$$

(Auto) - Interaction terms

Mass term

Vacuum energy

P: What is the field is complex? Remember the Lagrangian is a real number.

The free vector field

To have a QFT for the photon, we start by writing everything in covariant form (in terms of the potential V and the potential vector $\overrightarrow{A}$. As a 4-vector

$$A_\mu \equiv \left(A_0, -\overrightarrow{A}\right) = \left(V, -\overrightarrow{A}\right)$$

And define the electromagnetic tensor as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & E_x & E_y & E_z . \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \qquad F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu}, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

And two more relations

$$F_{\mu\nu} F^{\mu\nu} \equiv F^2 = -2(\overrightarrow{E}^2 - \overrightarrow{B}^2)$$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} \equiv F^2 = -4\overrightarrow{E} \cdot \overrightarrow{B}$$

The free vector field

The equation of motion are Maxwell equations or the Lagrangian

$$\mathscr{L}_{Free} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^2 \qquad \frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\beta\left(\frac{\partial \mathscr{L}}{\partial(\partial_\beta A_\alpha)}\right) = 0$$

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

Maxwell equations

If we take $\alpha = 0$ we get

$$\partial_\mu F^{\mu 0} = \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \vec{\nabla}\cdot\vec{E} = 0$$

which is Poisson equation with no sources. With sources we would have $\vec{\nabla}\cdot\vec{E} = \rho$. How are the sources introduced in the Lagrangian?

$$\mathscr{L} = \mathscr{L}_{Free} + \mathscr{L}_{Int} = -\frac{1}{4}F^2 - A_\alpha J^\alpha \qquad \text{with} \qquad J^\alpha = (\rho, \vec{j})$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

Maxwell equations with sources

P: Do $\partial_\mu F^{\mu 1}$. Show equations with sources.

The massive vector field

The vector field with mass is described by the Proca lagrangian

$$\mathscr{L} = -\frac{1}{4}F^2 + \frac{1}{2}m^2A^2 \qquad \frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\beta\left(\frac{\partial \mathscr{L}}{\partial(\partial_\beta A_\alpha)}\right) = 0$$

The equations of motion for the Proca Lagrangian are

$$\boxed{\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0}$$

We can show that in this case, the Lorenz gauge always holds which is a consequence of the lost of gauge invariance

$$\partial_\mu A^\mu = 0$$

Remember that we want to describe a particle with two degrees of freedom using a vector with four components. We will come back to this point shortly.

The free spinor field - Dirac equation

The spinor field has four components. The Lagrangian that leads to Dirac equation

$$\mathscr{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi$$

Mass dimension 4

$$\frac{\partial \mathscr{L}}{\partial \bar{\psi}} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \bar{\psi})} \right) \quad \Rightarrow \quad (i\gamma_\mu \partial^\mu - m)\psi = 0$$

Dirac equation

Important definitions (in the Dirac basis)

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}; \; \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \qquad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \; \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \; \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$\bar{\psi} = \psi^\dagger \gamma^0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Know your spaces

Remember a field can live in multiple spaces.

$$\Psi_{i,j,k} \quad \Longleftarrow \quad \begin{cases} i \Rightarrow isospin \Rightarrow (2\times 2) \\ j \Rightarrow flavour \Rightarrow (3\times 3) \\ \qquad k\ldots \end{cases}$$

Take care of one space at a time. Suppose you have a field that lives both in the isospin two-dimensional space and in the flavour three-dimensional space. Suppose that you expand the SU(2) fields and they mix. You take care of the SU(2) mixing first, and leave the SU(3) vector for later

$$\begin{bmatrix} a_G & b_G \end{bmatrix} \begin{bmatrix} m_{11}^2 & m_{12}^2 \\ m_{21}^2 & m_{22}^2 \end{bmatrix} \begin{bmatrix} a_G \\ b_G \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_M & b_M \end{bmatrix} \begin{bmatrix} m_a^2 & 0 \\ 0 & m_b^2 \end{bmatrix} \begin{bmatrix} a_M \\ b_M \end{bmatrix}$$

And now expand the field in its SU(3) components

$$a_G = [a_1 \quad a_2 \quad a_3]$$

Gauge invariance

Is this Lagrangian invariant under some transformation?

$$\mathscr{L} = -\frac{1}{4}F^2 + \frac{1}{2}m^2A^2$$

If we try

$$A^\mu \to A^\mu + \partial^\mu \phi$$

It is easy to show that

$$F^{\mu\nu} \to F^{\mu\nu} \qquad\qquad A^2 \nrightarrow A^2$$

And therefore the spin 1 Lagrangian is invariant under this transformation but a mass term is not. What about the Maxwell Lagrangian with sources?

$$\mathscr{L} = \mathscr{L}_{Free} + \mathscr{L}_{Int} = -\frac{1}{4}F^2 - A_\alpha J^\alpha$$

P: Find this out. Write $A_\alpha J^\alpha \to (A_\alpha + \partial_\alpha \phi)$ and use $\partial_\alpha (J^\alpha \phi)$.

Gauge invariance

What is meant by gauge transformation?

The term *gauge* refers to any specific mathematical formalism to regulate redundant degrees of freedom in the Lagrangian of a physical system. The transformations between possible gauges, called *gauge transformations*, form a Lie group—referred to as the *symmetry group* or the *gauge group* of the theory.

What is the gauge principle?

A gauge principle specifies a procedure for obtaining an interaction term from a free Lagrangian which is symmetric with respect to a continuous symmetry—the results of localising (or gauging) the global symmetry group must be accompanied by the inclusion of additional fields (such as the electromagnetic field), with appropriate kinetic and interaction terms in the action, in such a way that the extended Lagrangian is covariant with respect to a new extended group of local transformations.

What is the U 1 symmetry?

The group U(1) corresponds to the circle group, consisting of all complex numbers with absolute value 1, under multiplication.

Local Gauge invariance - minimal coupling

Let us consider the free Klein-Gordon Lagrangian, but for a complex scalar field

$$\mathscr{L}_0 = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$ Invariant under $$\phi \rightarrow \phi e^{-i\epsilon}$$

Let us also suppose that the symmetry is local

$$\partial_\mu [\phi e^{-i\epsilon}] = (\partial^\mu \phi) e^{-i\epsilon} - i\phi((\partial^\mu \epsilon) e^{-i\epsilon}$$ not covariant

The fields and the derivatives transform as (infinitesimally)

$$\begin{cases} \delta\phi = -i\epsilon\phi \\ \delta(\partial_\mu \phi) = -i\epsilon(\partial_\mu \phi) - i(\partial_\mu \epsilon)\phi \end{cases}$$

P: Show these relations

The variation of the Lagrangian is

$$\delta \mathscr{L}_0 = \frac{\partial \mathscr{L}_0}{\partial \phi} \delta\phi + \frac{\partial \mathscr{L}_0}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) + (\phi \leftrightarrow \phi^*)$$

Local Gauge invariance - minimal coupling

Leading to

Noether current related to electric charge

$$\delta\mathscr{L}_0 = (\partial_\mu\epsilon)\boxed{i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)} = (\partial_\mu\epsilon)J^\mu$$

Let us add the term

$$\mathscr{L}_1 = -eJ^\mu A_\mu \qquad \text{with} \qquad A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\epsilon$$

The transformation for $\mathscr{L}_1$ reads

$$\delta\mathscr{L}_1 = -e(\delta J^\mu)A_\mu - eJ^\mu\delta A_\mu = -2e\phi^*\phi\partial^\mu\epsilon A_\mu - J_\mu\partial^\mu\epsilon$$

And one more term with the corresponding variation

$$\boxed{\mathscr{L}_2 = e^2A^2\phi^*\phi \quad \Rightarrow \quad \delta\mathscr{L}_2 = 2eA_\mu(\partial^\mu\epsilon)\phi^*\phi}$$

P: Show

And finally we add

$$\boxed{\mathscr{L}_3 = -\frac{1}{4}F^2 \quad \Rightarrow \quad \delta\mathscr{L}_3 = 0}$$

P: Show

Local Gauge invariance - minimal coupling

We can add all pieces to get

$$\mathscr{L} = (\partial_\mu \phi + ieA_\mu \phi)(\partial^\mu \phi^* - ieA^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F^2$$

and defining

$$D_\mu \phi \equiv (\partial_\mu + ieA_\mu)\phi \qquad \text{with} \qquad \delta(D_\mu \phi) = -i\epsilon(D_\mu \phi) \qquad \text{covariant derivative}$$

The Lagrangian is written as

$$\mathscr{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi^* \phi - \frac{1}{4} F^2$$

Lesson: <u>if you know the symmetry, you write the corresponding covariant derivative and you get the interaction term</u>. A term $m^2 A^2$ would not be invariant. <u>Enter the Higgs mechanism</u>.
For the SM the covariant derivative is

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2} Y B_\mu - i\frac{g_2}{2} \sigma_j W_\mu^j - i\frac{g_3}{2} \lambda_j G_\mu^j \longrightarrow \text{SU(3) Colour}$$

U(1) hypercharge

SU(2) Isospin

Covariant derivative and interactions

Let us suppose that the SM has a real scalar field and a photon. The free piece of the Lagrangian is

$$\mathscr{L}_{free} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) - \frac{1}{4} F^2$$

The covariant derivative is defined (if we consider the symmetries of the SM)

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2} Y B_\mu - i\frac{g_2}{2}\sigma_j W_\mu^j - i\frac{g_3}{2}\lambda_j G_\mu^j$$

The field ϕ has hypercharge 0, isospin 0 and colour 0. So

$$D_\mu = \partial_\mu$$

Lesson: <u>if such a field would exist in the SM it would have no interactions with the photon.</u>

P: What kind of interaction could this field have?

Covariant derivative and interactions

Let us now suppose that the SM has a complex scalar field and a photon. The free piece of the Lagrangian is

$$\mathscr{L}_{free} = (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) - \frac{1}{4} F^2$$

Again the covariant derivate is

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2} Y B_\mu - i\frac{g_2}{2} \sigma_j W_\mu^j - i\frac{g_3}{2} \lambda_j G_\mu^j$$

The field ϕ has now hypercharge 1, isospin 0 and colour 0. So

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2} B_\mu$$

And we recover the Lagrangian

$$\mathscr{L} = (\partial_\mu \phi - i\frac{g}{2} B_\mu \phi)(\partial^\mu \phi^* + i\frac{g}{2} B^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F^2$$

P: Write the most general Lagrangian up to dimension 4 with one real field ϕ_1 and one complex field ϕ_2.

Spontaneous symmetry breaking - The Higgs mechanism

Suppose we now start with the same Lagrangian

$$\mathscr{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi^* \phi - \frac{1}{4} F^2$$

but redefine the scalar field such that (this is SSB)

$$\phi \to \phi + v$$ where v is a real constant

The two last terms are not relevant, but the kinetic term has now the extra terms

$$e^2[B^2 \phi^* \phi + v^2 B^2 + v B^2 \phi + \ldots]$$

P: Show this is true.

A mass term for the vector was born

$$m_B^2 = e^2 v^2$$ this is SSB - giving a non-zero VEV to the field generates mass

The U(1) symmetry, present in the original Lagrangian was broken by the vacuum expectation value of the scalar field. We lost the U(1) symmetry and so the photon gains a mass. And electric charge is no longer conserved.

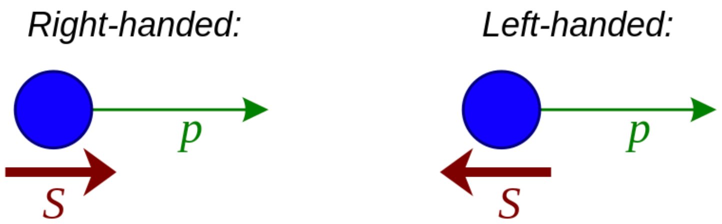
Gauge invariance

What is chirality?

A chiral phenomenon is one that is not identical to its mirror image. The spin of a particle may be used to define a handedness, or helicity, for that particle, which, in the case of a massless particle, is the same as chirality. A symmetry transformation between the two is called parity transformation. Invariance under parity transformation by a Dirac fermion is called chiral symmetry.

What is helicity?

The helicity of a particle is positive ("right-handed") if the direction of its spin is the same as the direction of its motion. It is negative ("left-handed") if the directions of spin and motion are opposite. Mathematically, helicity is the sign of the projection of the spin vector onto the momentum vector: "left" is negative, "right" is positive.



For massless particles – photons, gluons – chirality is the same as helicity.

SSB - Minimal requirements for electroweak symmetry breaking

Gauge structure

- $SU(2)_L \times U(1)_Y$ describes the electroweak interactions.

- $SU(2)_L$ factor is a left-handed isospin – an isospin charge carried by left-chirality fermions (eigenstate of left projector $\gamma_L = (1 - \gamma_5)/2$) .

- $U(1)_Y$ factor is related to the weak hypercharge.

- For each fermion type, left-handed and right-handed states are independent. Mass terms appear due to bilinear interactions which link different fields – they break electroweak symmetry.

- Electroweak symmetry also requires the gauge bosons to be massless but they do have a mass.

Interlude - Mass and gauge eigenstates

The simplest Lagrangian with two mixed fields we can write is

$$\mathscr{L} = a(\phi_1^g)^2 + 2b\phi_1^g\phi_2^g + c(\phi_2^g)^2 = \begin{bmatrix}\phi_1^g & \phi_2^g\end{bmatrix}\begin{bmatrix}a & b \\ b & c\end{bmatrix}\begin{bmatrix}\phi_1^g \\ \phi_2^g\end{bmatrix}$$

Rotate from the gauge eigenstates to the mass eigenstates

$$\Phi^m = R_\alpha \Phi^g \qquad \begin{bmatrix}\phi_1^m \\ \phi_2^m\end{bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\phi_1^g \\ \phi_2^g\end{bmatrix} \qquad \tan 2\alpha = \frac{2b}{a-c}$$

$$\mathscr{L} = (\Phi^g)^T M_g \Phi^g = (\Phi^m)^T R_\alpha M_g R_\alpha^T \Phi^m = (\Phi^m)^T M \Phi^m$$

Show that

$$\begin{cases} m_1^2 = \dfrac{a+c+\sqrt{(a-c)^2-4b^2}}{2} \\ m_2^2 = \dfrac{a+c-\sqrt{(a-c)^2-4b^2}}{2} \end{cases} \quad \Rightarrow \quad b=0 \quad \begin{cases} m_1^2 = a \\ m_2^2 = c \end{cases}$$

Interlude - Mass and gauge eigenstates

Consider the Lagrangian

$$\mathscr{L} = \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m_1^2 \phi_1^2 - m_2^2 \phi_2^2 - m_{12}^2 \phi_1 \phi_2 + \lambda \phi_1^2 \phi_2^2$$

which is not complete.

P: Find the mass eigenstates. Write the Lagrangian in terms of the mass eigenstates.

SSB in the SM

The difference to the SM is that the Higgs field in an SU(2) doublet. The electroweak covariant derivative is a 2 by 2 matrix and therefore we need a doublet field

$$D_\mu \equiv \partial_\mu - i\frac{g_1}{2}YB_\mu - i\frac{g_2}{2}\sigma_j W_\mu^j \qquad \Phi_{SM} = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \qquad <\Phi_{SM}> = \begin{pmatrix} v_1 + iv_2 \\ v_3 + iv_4 \end{pmatrix}$$

Vacuum expectation values of the Higgs doublet

If there is SSB, the gauge boson mass terms always come from the covariant derivative term.

The Higgs doublet has four components but the fact that the field is invariant under SU(2) allows to rotate fields away (it is like choosing a reference frame). The Higgs field has isospin 1/2 and hypercharge 1 (we choose). So we choose

$$<\Phi_{SM}> = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \qquad Q_{SM} <\Phi_{SM}> = \left(I_3 + \frac{Y}{2}\right) <\Phi_{SM}> = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = 0$$

P: Expand $(D_\mu \Phi)^\dagger (D^\mu \Phi)$

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Mass eigenstates - gauge bosons

- Gauge bosons

$$D_\mu \Phi = \partial_\mu \Phi - \frac{i}{2}\begin{bmatrix} gW_\mu^3 + g'B_\mu & \sqrt{2}gW_\mu^+ \\ \sqrt{2}gW_\mu^- & -gW_\mu^3 + g'B_\mu \end{bmatrix}\Phi$$

$$< \Phi_{SM} > = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$$

$$D_\mu \equiv \partial_\mu - i\frac{g'}{2}YB_\mu - i\frac{g}{2}\sigma_j W_\mu^j - i\frac{g_3}{2}\lambda_j G_\mu^j$$

$$\left\langle \left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right) \right\rangle_{\text{mass}} = \frac{g^2 v^2}{2} W_\mu^+ W^{\mu -} + \frac{v^2}{8}\begin{bmatrix} W_\mu^3 & B_\mu \end{bmatrix}\begin{bmatrix} g & -g'g \\ -g'g & g'^2 \end{bmatrix}\begin{bmatrix} W^{3\mu} \\ B^\mu \end{bmatrix}$$

$$= \frac{g^2 v^2}{2} W_\mu^+ W^{\mu -} + \frac{(g^2 + g'^2)v^2}{4} Z_\mu Z^\mu$$

$$\begin{cases} Z_\mu = c_W W_\mu^3 - s_W B_\mu \\ A_\mu = s_W W_\mu^3 + c_W B_\mu \end{cases} \qquad c_W = \frac{g}{\sqrt{g^2 + g'^2}} = \frac{M_W}{M_Z}$$

Charge Breaking (CB) in the SM

What would happen if we started with

$$< \Phi_{SM} > = \begin{pmatrix} v_1 + i v_2 \\ v_3 + i v_4 \end{pmatrix}$$

We would find the same mass spectrum (for the gauge bosons)

$$m_1^2 = m_2^2 = \frac{g^2 v^2}{4}$$

$$v^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2$$

$$m_3^2 = \; = \frac{v^2}{4}(g^2 + g'^2 Y^2)$$

$$m_4^2 = 0$$

It's the photon!

So U(1) survives and charge is always conserved independently of the pattern of symmetry breaking - a consequence of gauge invariance.

Parity (P), Charge conjugation (C), and CP

Parity is an operation that works like a reflection relative to an axis. If $\vec{r}$ is the particle's position $P(\vec{r}\,) = -\,\vec{r}$ and $P[F(\vec{r}\,)] = \pm\, F(-\vec{r}\,)$.

Observables are even (if invariant) and odd (if not invariant) under P. A space derivative is odd under P. Angular momentum is even.

https://radiolab.org/podcast/122382-desperately-seeking-symmetry

Parity is violated by the weak interactions

Chien-Shiung Wu proved in 1956 that Parity was not conserved in weak interactions

In QED and QCD parity is conserved.

^{60}Co atoms aligned by a uniform magnetic field and cooled to near T=0 K so that thermal motions did not ruin the alignment. Cobalt-60 is unstable and decays beta to ^{60}Ni. During this decay, one of the neutrons in the cobalt-60 nucleus decays to a proton emitting an e^- and an ν_e (weak). The resulting nickel nucleus is an excited state and promptly decays to its ground state by emitting two gammas (electromagnetic).

QED conserves parity - the distribution of the emitted electrons could be compared to the distribution of the emitted photons in order to compare whether they too were being emitted isotropically. Distribution of gammas acted as a control for the distribution of the electrons.

Results: photons anisotropy was approximately 0.6. Wu observed that electrons were emitted in a direction preferentially opposite to that of photons with an asymmetry significantly greater than the photons. That is, most of the electrons favoured a very specific direction of decay, specifically opposite to that of the nuclear spin.

C is also violated in the weak interactions. And CP is also violated in the weak interactions.

Parity (P), Charge conjugation (C), and CP

Charge conjugation (C) is an operation that transforms a particle in its anti-particle.

If the theory conserves C (the Lagrangian is invariant for a C transformation) the states are eigenstates of C (C is a good quantum number).

In a theory without fermions C is a good quantum number and the photon and Z boson have.

$$C[e^-] = e^+; \qquad C[h] = h; \qquad C[\gamma] = -\gamma$$

The symmetries can be applied sequentially and in that case is called CP. The SM does not conserve CP as well.

CB (and CP) in the SM

You can use the SU(2) freedom to perform the rotation

C - Charge Conjugation

P - Parity

CP - C+P

$$< \Phi_{SM} > = \begin{pmatrix} v_1 + i v_2 \\ v_3 + i v_4 \end{pmatrix} \quad \rightarrow \quad < \Phi_{SM} > = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

Using a more general vacuum would just mean to redefine the charge operator.

For the same reason, any phase in the vacuum can be rotated away. This means that no spontaneous CP can occur. And the potential is also explicitly CP conserving.

Explicit breaking - if the Lagrangian is not invariant under a given symmetry

Spontaneous breaking - if the Lagrangian is invariant under a given symmetry but the vacuum is not

The SM has no CB and no CP violation in the potential.

C and P number without fermions

Suppose we have some extension of the SM but with no fermions. Also let us assume for the moment that the theory conserves C and P separately. The C and P quantum numbers of the photon and Z boson are

$$C(A_\mu) = P(A_\mu) = -1 \qquad C(Z_\mu) = P(Z_\mu) = -1$$

Not being sloppy

$$CZ_\mu C^{-1} = -Z_\mu; \quad PZ_\mu P^{-1} = Z^\mu \qquad P\partial_\mu P^{-1} = \partial^\mu$$

If the theory has the vertices hhh and HHH,

$$P(h) = P(H) = 1;\ C(h) = C(H) = 1 \qquad P(A) = 1;\ C(A) = -1$$

What about the terms below? Are they C invariant, P invariant, CP-invariant?

$$Z_\mu Z^\mu h; \qquad \partial \cdot Zh^2; \qquad \partial \cdot Zh; \qquad \partial \cdot ZAh$$

Parity is violated by the weak interactions

The discovery set the stage for the development of the Standard Model, as the model relied on the idea of symmetry of particles and forces and how particles can sometimes break that symmetry.

We know how to build the fermionic currents. We can ask if they invariant under P and C transformations.

	C	P	CP
$\bar{\psi}_i \, \psi_j$	$\bar{\psi}_j \, \psi_i$	$\bar{\psi}_i \, \psi_j$	$\bar{\psi}_j \, \psi_i$
$\bar{\psi}_i \, \gamma_5 \, \psi_j$	$\bar{\psi}_j \, \gamma_5 \, \psi_i$	$-\bar{\psi}_i \, \gamma_5 \, \psi_j$	$-\bar{\psi}_j \, \gamma_5 \, \psi_i$
$\bar{\psi}_i \, \partial\!\!\!/ \, \psi_j$	$\bar{\psi}_j \, \partial\!\!\!/ \, \psi_i$	$\bar{\psi}_i \, \partial\!\!\!/ \, \psi_j$	$\bar{\psi}_j \, \partial\!\!\!/ \, \psi_i$
$\bar{\psi}_i \, \partial\!\!\!/ \, \gamma_5 \psi_j$	$-\bar{\psi}_j \, \partial\!\!\!/ \, \gamma_5 \, \psi_i$	$-\bar{\psi}_i \, \partial\!\!\!/ \, \gamma_5 \, \psi_j$	$\bar{\psi}_j \, \partial\!\!\!/ \, \gamma_5 \, \psi_i$

The way to deal with the weak interactions is to treat left and right components of the fermions as independent. What is left and right?

$$\psi_L = \gamma_L \psi = \frac{1-\gamma_5}{2}\psi; \qquad \psi_R = \gamma_R \psi = \frac{1+\gamma_5}{2}\psi;$$

P: Let us build a theory invariant under C with two scalars with $C(\phi_1)$=1 and $C(\phi_2)$=-1 and one fermion. Assuming that the scalar fields are even under P, build one P invariant and another one CP invariant. Add a photon and redo the exercise.

Particles and their quantum numbers

Fermions

$$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix};\ L_L^i = \begin{pmatrix} \nu_L^i \\ l_L^i \end{pmatrix};\ u_R^i;\ d_R^i;\ l_R^i;\ \ldots$$

Hypercharge is chosen to give the fermions the right electric charge.

Gauge bosons

$$(W_\mu^1,\ W_\mu^2,\ W_\mu^3);\ B_\mu$$

Hypercharge is chosen such that the charged particle is on the top.

Scalars

$$\Phi_i = \begin{pmatrix} a^- \\ b_i + i\, c_i \end{pmatrix} \qquad \langle \Phi_i \rangle = \begin{pmatrix} v_1^i + i v_2^i \\ v_3^i + i v_4^i \end{pmatrix} \qquad x_i = \Phi_i^\dagger \Phi_i$$

$$\omega_i = \alpha_i + i\beta_i \qquad \langle \omega_i \rangle = \sigma_1^i + i\sigma_2^i \qquad y_i = \omega_i^* \omega_i$$

If Y ≠ 0 the fields have electric charge and couple to the gauge bosons Z and photon.

The Yukawa Lagrangian - scalars quantum numbers

$$\Phi \;\Rightarrow\; Y = 1 \qquad I_3 \to \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \qquad Q \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Phi^* \;\Rightarrow\; Y = -1 \qquad I_3 \to \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \qquad Q \to \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a^* \\ b^* \end{bmatrix}$$

$$i\sigma_2\Phi \;\Rightarrow\; Y = 1 \qquad I_3 \to \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} \qquad Q \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} b \\ -a \end{bmatrix}$$

$$\tilde{\Phi} = i\sigma_2\Phi^* \;\Rightarrow\; Y = -1 \qquad I_3 \to \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \qquad Q \to \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} b^* \\ -a^* \end{bmatrix}$$

$$Q = I_3 + \frac{Y}{2} \;\Leftarrow\; \text{Remember}$$

Note: note the difference in isospin in the up and down components is what gives a unit difference of charge. Hipercharge then fixes the actual value of the charge.

The Yukawa Lagrangian - Yukawa invariants

	T	T_3	$Y/2$	Q
ν_{eL}	1/2	1/2	-1/2	0
e_L	1/2	-1/2	-1/2	-1
u_L	1/2	1/2	1/6	2/3
d_L	1/2	-1/2	1/6	-1/3
e_R	0	0	-1	-1
u_R	0	0	2/3	2/3
d_R	0	0	-1/3	-1/3

$$\mathscr{L}_{lep} = -Y_e L_L \phi e_R = -Y_e (\bar{\nu} \quad \bar{e}_L) \begin{pmatrix} a \\ b \end{pmatrix} e_R$$

$$Y = 1 + 1 - 2$$

$$(T_3)_{up} = -1/2 + 1/2 + 0$$

$$\mathscr{L}_{qd} = -Y_d Q_L \phi d_R = -Y_d (\bar{u} \quad \bar{d}) \begin{pmatrix} a \\ b \end{pmatrix} d_R \qquad Y = -1/3 + 2 - 2/3$$

$$\mathscr{L}_{qu} = -Y_u Q_L \tilde{\phi} u_R = -Y_u (\bar{u} \quad \bar{d}) \begin{pmatrix} b^* \\ -a^* \end{pmatrix} u_R \qquad Y = -1/3 - 1 + 4/3 \qquad \tilde{\Phi} = i\sigma_2 \Phi^*$$

Add the hermitian conjugate to all terms. This is a version of the Lagrangian where the neutrinos are massless. As we now know this is not true. More later.

The Yukawa Lagrangian - mass eigenstates

$$-L_Y = \begin{bmatrix} \bar{U} & \bar{D} \end{bmatrix}_L \Phi Y_d D_R + \begin{bmatrix} \bar{U} & \bar{D} \end{bmatrix}_L \tilde{\Phi} Y_u U_R + \begin{bmatrix} \bar{N} & \bar{E} \end{bmatrix}_L \Phi Y_e E_R + h.c$$

where the gauge eigenstates are

$$U = \begin{bmatrix} u_g & c_g & t_g \end{bmatrix};\ D = \begin{bmatrix} d_g & s_g & b_g \end{bmatrix};\ N = \begin{bmatrix} \nu_e & \nu_\mu & \nu_\tau \end{bmatrix};\ E = \begin{bmatrix} e & \mu & \tau \end{bmatrix}$$

and Y are matrices in flavour space. To get the mass terms we just need the vacuum expectation values of the scalar fields

$$\langle \Phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}; \quad \langle \tilde{\Phi} \rangle = i\tau_2 \Phi^* = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}$$

The mass terms are

$$-L_Y^{mass} = \frac{v}{\sqrt{2}} \bar{U}_L Y_u U_R + \frac{v}{\sqrt{2}} \bar{D}_L Y_d D_R + \frac{v}{\sqrt{2}} \bar{E}_L Y_e E_R + h.c.$$

which have to be diagonalised. We are still dealing with the group eigenstates.

The Yukawa Lagrangian - mass eigenstates

So we define

$$D_R \to N_R^{-1} D_R ; D_L \to N_L^{-1} D_L ; U_R \to K_R^{-1} U_R ; U_L \to K_L^{-1} U_L$$

and the mass matrices are

$$-\frac{v}{\sqrt{2}} N_L^\dagger Y_d N_R = M_d; \qquad -\frac{v}{\sqrt{2}} K_L^\dagger Y_u K_R = M_u$$

and finally the CKM matrix revealed

$$\frac{v}{\sqrt{2}} \bar{U}_L K_L^\dagger Y_d N_R D_R \ldots + \frac{v}{\sqrt{2}} \bar{D}_L N_L^\dagger Y_u K_R U_R \ldots =$$

$$\frac{v}{\sqrt{2}} \bar{U}_L K_L^\dagger N_L M_D D_R \ldots + \frac{v}{\sqrt{2}} \bar{D}_L N_L^\dagger K_L M_U U_R \ldots =$$

$$\frac{v}{\sqrt{2}} \bar{U}_L V_{CKM} M_D D_R \ldots + \frac{v}{\sqrt{2}} \bar{D}_L V_{CKM}^\dagger M_U U_R \ldots$$

and flavour changing neutral currents (FCNC)?

$$\Phi = \begin{pmatrix} 0 \\ (h+v)/\sqrt{2} \end{pmatrix}$$

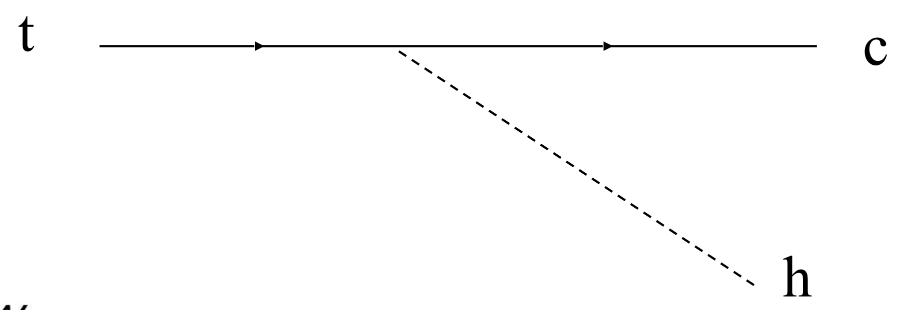
$$-L_Y^{\text{interactions}} = \frac{h}{\sqrt{2}} \bar{D}_L Y_d D_R \propto \frac{v}{\sqrt{2}} \bar{D}_L Y_d D_R$$

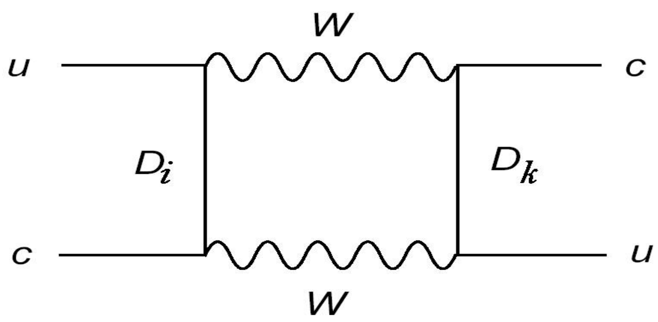
The Yukawa Lagrangian - FCNC

$$-\mathcal{L}_Y^{interactions} = \frac{h}{\sqrt{2}}\,\bar{D}_L Y_d D_R \propto \frac{v}{\sqrt{2}}\,\bar{D}_L Y_d D_R \qquad\qquad -\frac{v}{\sqrt{2}}\,N_L^{\dagger} Y_d N_R = M_d$$

Higgs couplings to the fermions are proportional to the fermions mass! No tree-level FCNC.

$\bar{t}ch$

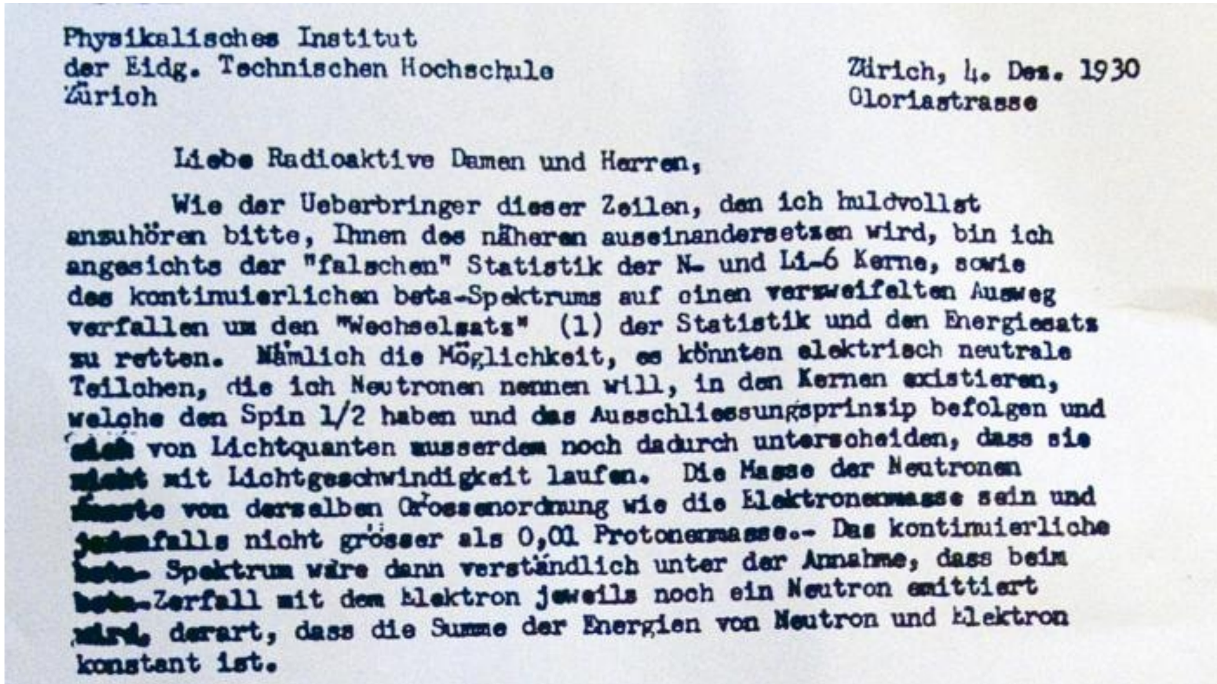




D-Mixing in the Standard Model is very small

Tree level couplings fine tuned to be unusually small

Dear radioactive Ladies and Gentlemen,



Physikalisches Institut
der Eidg. Technischen Hochschule
Zürich

Zürich, 4. Dez. 1930
Gloriastrasse

Liebe Radioaktive Damen und Herren,

Wie der Ueberbringer dieser Zeilen, den ich huldvollst anzuhören bitte, Ihnen des näheren auseinandersetzen wird, bin ich angesichts der "falschen" Statistik der N- und Li-6 Kerne, sowie des kontinuierlichen beta-Spektrums auf einen verzweifelten Ausweg verfallen um den "Wechselsatz" (1) der Statistik und den Energiesatz zu retten. Nämlich die Möglichkeit, es könnten elektrisch neutrale Teilchen, die ich Neutronen nennen will, in den Kernen existieren, welche den Spin 1/2 haben und das Ausschliessungsprinzip befolgen und sich von Lichtquanten ausserdem noch dadurch unterscheiden, dass sie nicht mit Lichtgeschwindigkeit laufen. Die Masse der Neutronen müsste von derselben Grossenordnung wie die Elektronenmasse sein und jedenfalls nicht grösser als 0,01 Protonenmasse.- Das kontinuierliche beta- Spektrum wäre dann verständlich unter der Annahme, dass beim beta-Zerfall mit dem Elektron jeweils noch ein Neutron emittiert wird, derart, dass die Summe der Energien von Neutron und Elektron konstant ist.

1930
Pauli's neutrino hypothesis

December 4th, 1930
Letter to his colleagues in Tübingen

Neutrino masses

Zürich, Dec. 4, 1930

Physics Institute of the ETH

Gloriastrasse

Zürich

Dear Radioactive Ladies and Gentlemen,

As the bearer of these lines, to whom I graciously ask you to listen, will explain to you in more detail, because of the "wrong" statistics of the N- and Li-6 nuclei and the continuous beta spectrum, I have hit upon a desperate remedy to save the "exchange theorem" of statistics and the law of conservation of energy. Namely, the possibility that in the nuclei there could exist electrically neutral particles, which I will call neutrons, that have spin 1/2 and obey the exclusion principle and that further differ from light quanta in that they do not travel with the velocity of light.

(.../...)

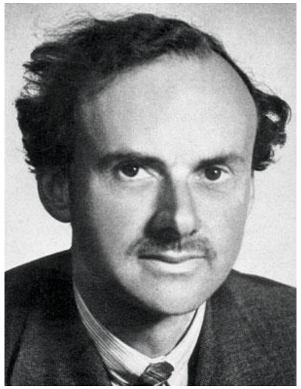
But so far I do not dare to publish anything about this idea, and trustfully turn first to you, dear radioactive people, with the question of how likely it is to find experimental evidence for such a neutron if it would have the same or perhaps a 10 times larger ability to get through [material] than a gamma-ray.

I admit that my remedy may seem almost improbable because one probably would have seen those neutrons, if they exist, for a long time. (.../...) Thus, dear radioactive people, scrutinize and judge. - Unfortunately, I cannot personally appear in Tübingen since I amindispensable here in Zürich because of a ball on the night from December 6 to 7. With my best regards to you, and also to Mr. Back, your humble servant

signed W. Pauli

Neutrino masses

Paul Dirac



Ettore Majorana

$$m_D \overline{f}_L f_R + \text{h.c.} = m_D \overline{f} f$$

$$\boxed{f \neq f^c}$$

$$\frac{1}{2} m_M \overline{f^c_X} f_X + \text{h.c.}$$

$$\boxed{f = f^c}$$

Only neutral fermions can be Majorana

Note:
$f^c \equiv C\overline{f}^T$

Pontecorvo, the "father" of neutrino oscillations, recalls the origin of Majorana neutrinos in the following way: Dirac discovers his famous equation describing the evolution of the electron; Majorana goes to Fermi to point out a fundamental detail: " I have found a representation where all Dirac γ matrices are real. In this representation it is possible to have a real spinor that describes a particle identical to its antiparticle."

It would be interesting though to know if neutrino are Dirac or Majorana particles. However, there is no problem with the neutrino masses. Just extend the SM with right-handed neutrinos.

The Gauge groups

Here the important conventions are for the field strengths and the covariant derivatives. We have

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu, \quad a = 1, \ldots, 8 \tag{D.1}$$

where f^{abc} are the group structure constants, satisfying

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{D.2}$$

and T^a are the generators of the group. The covariant derivative of a (quark) field q in some representation T^a of the gauge group is given by

$$D_\mu q = \left(\partial_\mu - i\, g\, G^a_\mu T^a\right) q \tag{D.3}$$

In QCD the quarks are in the fundamental representation and $T^a = \lambda^a/2$ where λ^a are the Gell-Mann matrices. A gauge transformation is given by a matrix

$$U = e^{-iT^a \alpha^a} \tag{D.4}$$

and the fields transform as

$$\begin{aligned} & q \to e^{-iT^a\alpha^a} q && \delta q = -iT^a \alpha^a q \\ & G^a_\mu T^a \to U G^a_\mu T^a U^{-1} - \frac{i}{g} \partial_\mu U U^{-1} && \delta G^a_\mu = -\frac{1}{g} \partial_\mu \alpha^a + f^{abc} \alpha^b G^c_\mu \end{aligned} \tag{D.5}$$

where the second column is for infinitesimal transformations. With these definitions one can verify that the covariant derivative transforms like the field itself,

$$\delta(D_\mu q) = -i\, T^a \alpha^a (D_\mu q) \tag{D.6}$$

ensuring the gauge invariance of the Lagrangian.

Feynman Rules - Prof. Jorge Romão.

Feynman Rules and signs - Romão e Silva https://arxiv.org/pdf/1209.6213.pdf.

D.2.2 Gauge Group $SU(2)_L$

This is similar to the previous case. We have

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu, \quad a = 1, \ldots, 3 \tag{D.7}$$

where, for the fundamental representation of $SU(2)_L$ we have $T^a = \sigma^a/2$ and ϵ^{abc} is the completely anti-symmetric tensor in 3 dimensions. The covariant derivative for any field ψ_L transforming non-trivially under this group is,

$$D_\mu \psi_L = \left(\partial_\mu - i\, g\, W^a_\mu T^a\right) \psi_L \tag{D.8}$$

D.2.3 Gauge Group $U(1)_Y$

In this case the group is abelian and we have

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{D.9}$$

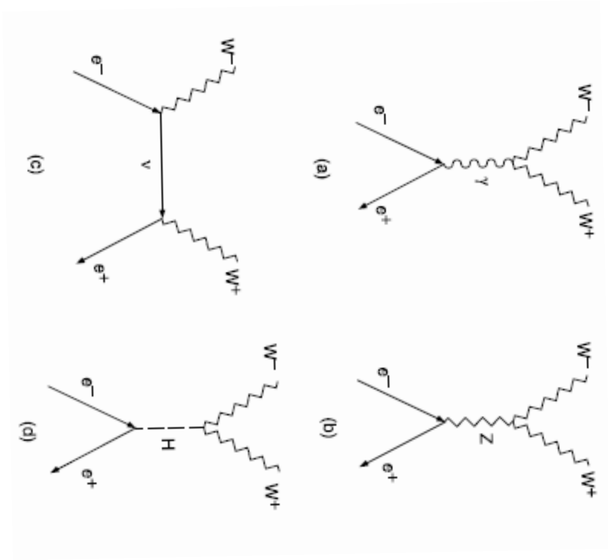
with the covariant derivative given by

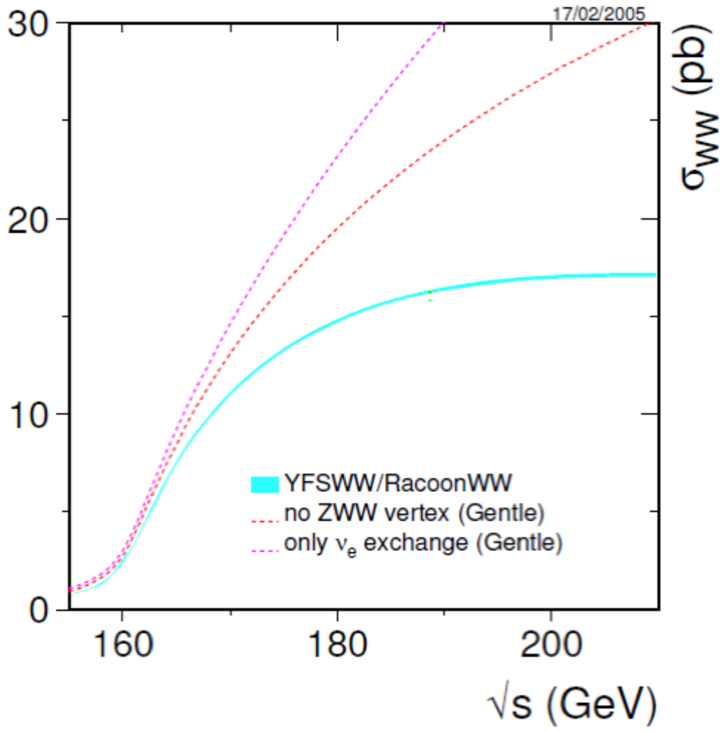
$$D_\mu \psi_R = \left(\partial_\mu + i\, g'\, Y\, B_\mu\right) \psi_R \tag{D.10}$$

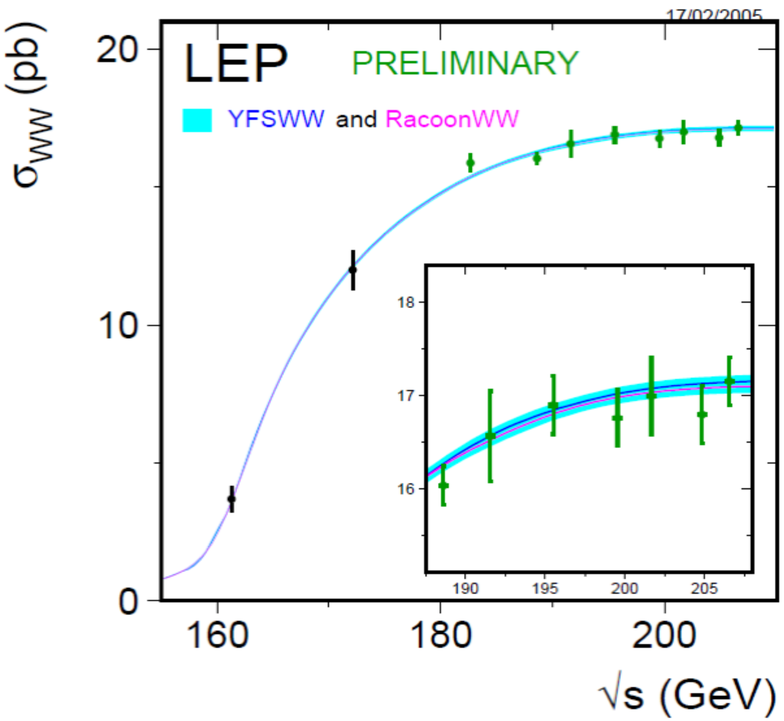
where Y is the hypercharge of the field. Notice the different sign convention between Eq. (D.8) and Eq. (D.9). This is to have the usual definition[1]

$$Q = T_3 + Y\ . \tag{D.12}$$

Where is the gauge structure?







The gauge fixing Lagrangian in QED

In order to quantise the theory we still need to fix the gauge and therefore to introduce the gauge fixing Lagrangian

$$\mathscr{L} = \mathscr{L}_{free} + \mathscr{L}_{GF} = -\frac{1}{4}F^2 - \frac{1}{2\xi}(\partial \cdot A)^2$$

and the equations of motion are now

$$\Box A_\mu + (\frac{1}{\xi} - 1)\partial_\mu(\partial \cdot A) = 0$$

Choosing $\xi = 1$ the components of the photon field obey a Klein-Gordon equation and can be treated as a scalar field. Using the parameter a Lagrange multiplier we get

$$\partial_\mu A^\mu = 0 \qquad\qquad k_\mu \epsilon^\mu = 0$$

Note: note the gauge fixing Lagrangian is not unique and (underline twice now) the physical observables cannot depend on the gauge fixing parameter.

The gauge fixing Lagrangian in the SM

In the case of the SM the gauge fixing Lagrangian has the form

$$\mathscr{L}_{GF} = -\frac{1}{2\beta_A}(\partial \cdot A - \gamma_A M_Z G_0) - \frac{1}{2\beta_Z}(\partial \cdot Z - \gamma_Z M_Z G_0) - \frac{1}{\beta_W}\,|\,\partial \cdot W^+ - \gamma_W M_W G^+\,|$$

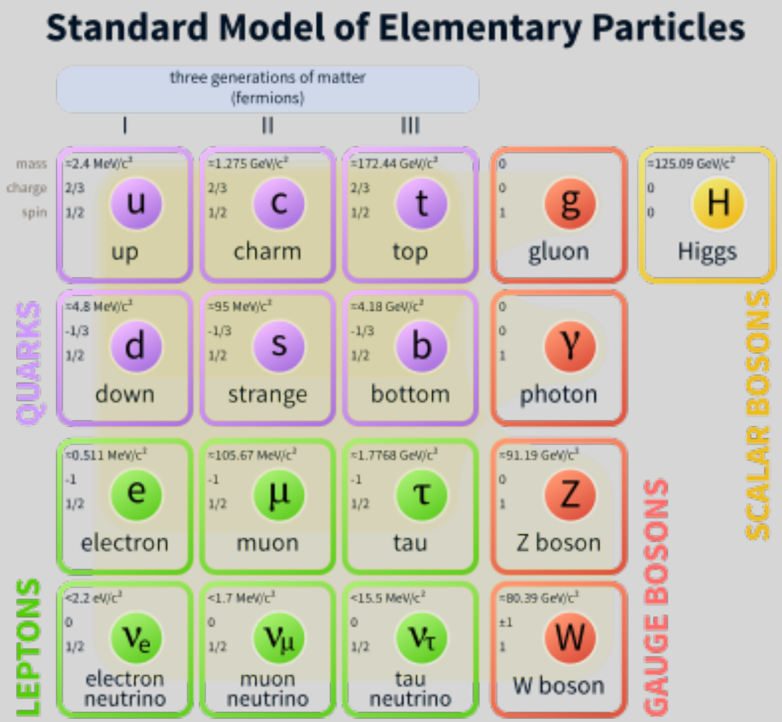
one fixes the gauge with the added bonus of cancelling the bilinear term below allowing us to define the propagator.

$$\mathscr{L}_S = (D_\mu \phi)^\dagger (D^\mu \phi) = -\tfrac{1}{2} h \partial^2 h - \tfrac{1}{2} G_0 \partial^2 G_0 - G^+ h \partial^2 G^-$$
$$+ M_W^2 W^+ W^- + M_W (i W_m u^- \partial^\mu G^+ + h\,.\,c.) + \tfrac{1}{2} M_Z^2 Z^2 - M_Z (Z_m u \partial^\mu G_0)$$

The Fadeev-Popov Lagrangian

Go home and study the FP Lagrangian.

The SM (and beyond) for people in a hurry



$$G = SU(3)_c \times SU(2)_L \times U(1)_Y$$

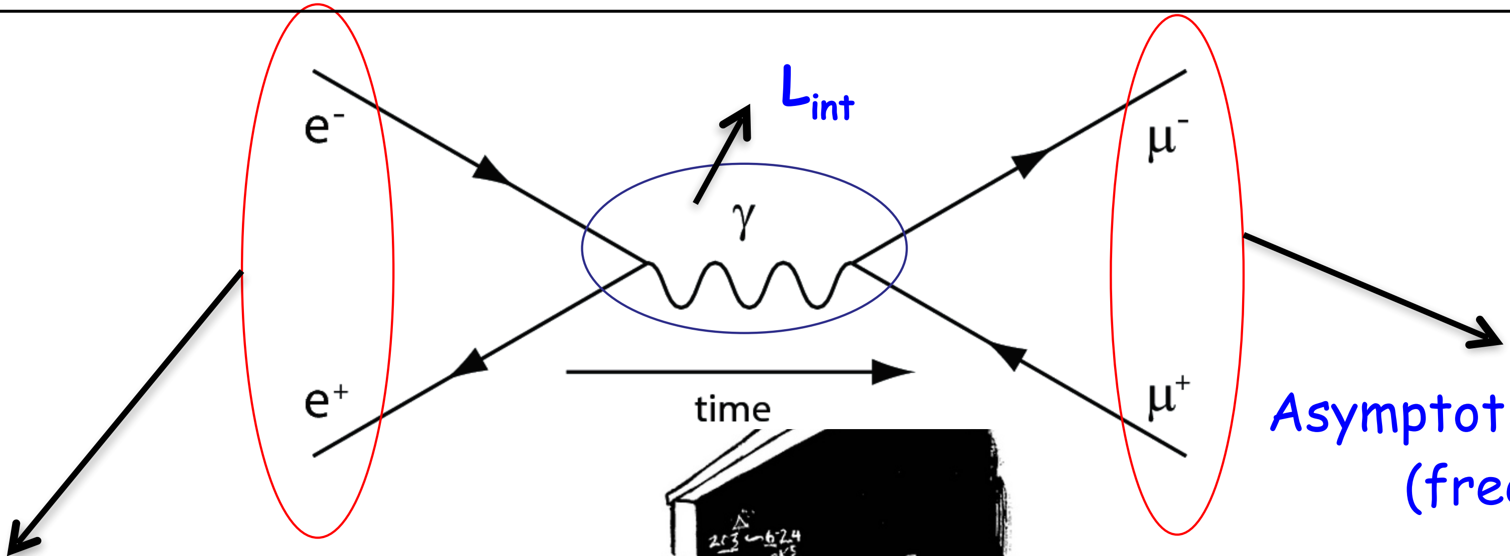
$$\Downarrow$$

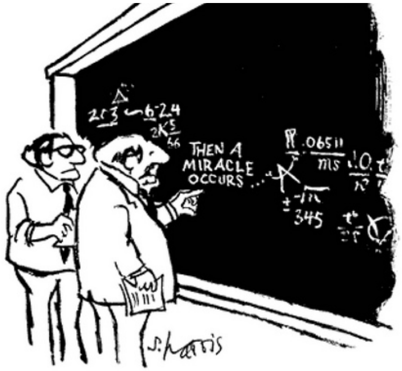
$$G' = SU(3)_c \times U(1)_Q$$

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}}$$

Cross Sections and Branching ratios - circumventing QFT

Feynman diagrams





"I think you should be more explicit here in step two."

An electron and a positron collide, exchange a virtual photon, and create a pair of muon and anti-muon (there are more diagrams).

The time arrow tells us that the diagram has to be read from left to right. This is the most common definition now.

The initial and final states are free states obeying the Dirac equation.

Asymptotic states

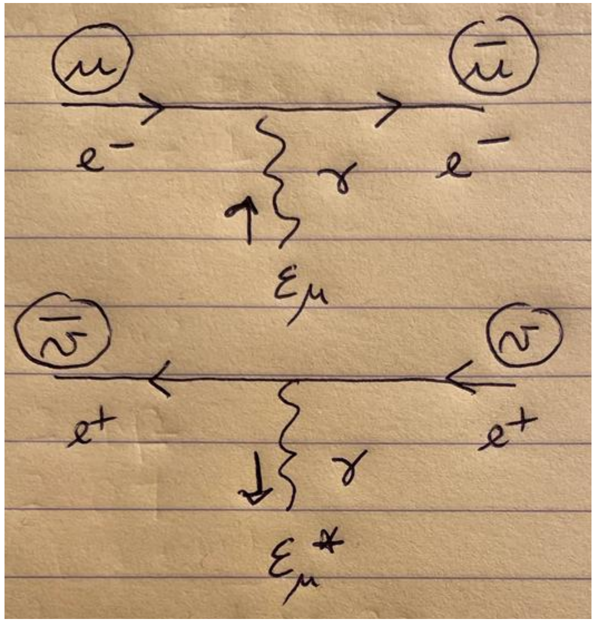
The way calculations in QFT work are as follows. Particles colide in free states and move away from the interaction in free states as well (asymptotic). For the type of particles

1 Scalar

ϵ_μ Vector -polarisation vector

$u, v, \bar{u}, \bar{v}$ Spinor

All solutions are multiplied by the corresponding solution of the equation of motion - a plane wave solution.

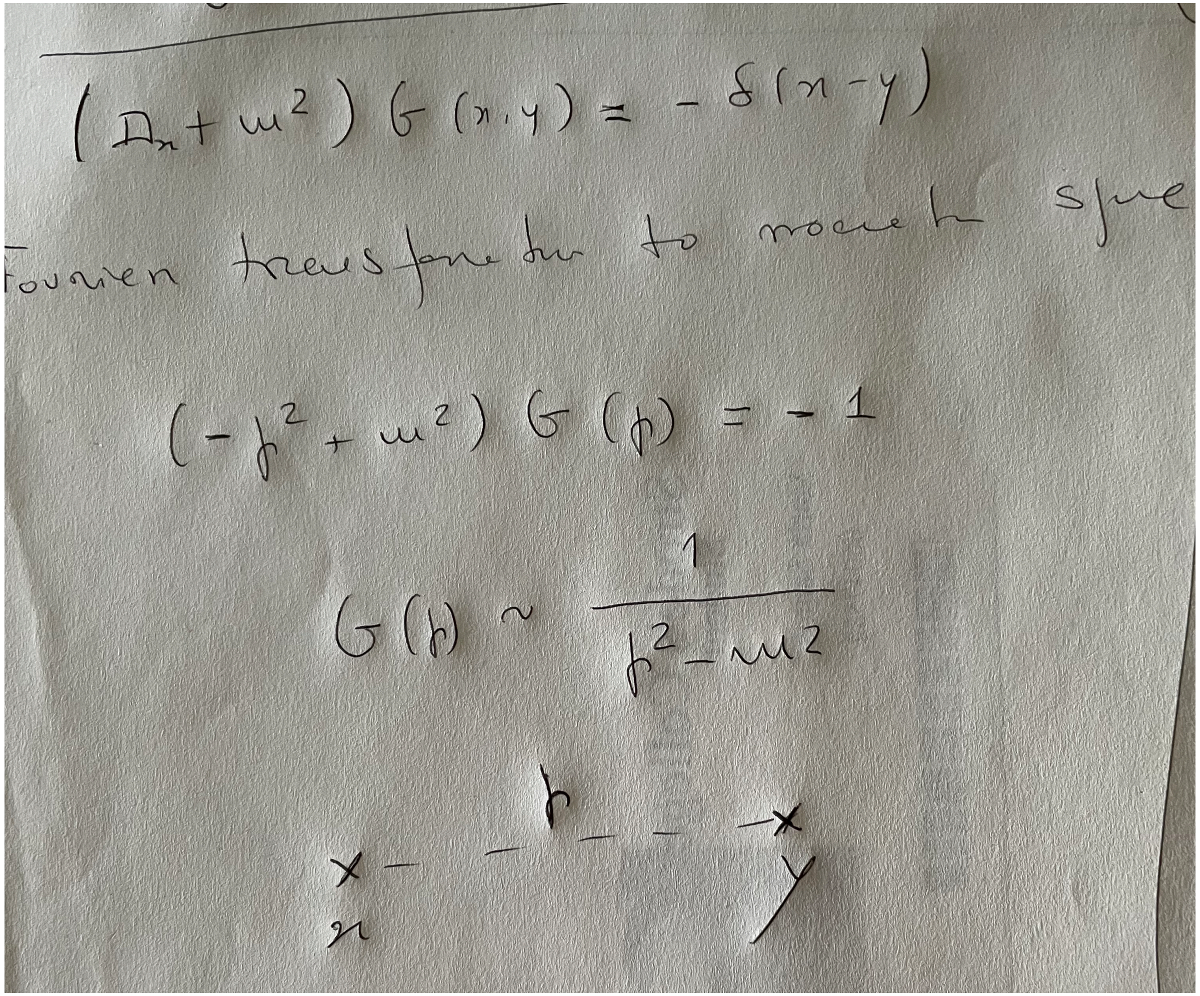


Processes in QED with electrons, positrons and photons.

Propagators

$$\mathscr{L}_{free}^{KG} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \Rightarrow \frac{i}{p^2 - m^2}$$

Scalar field



Propagators

$$\mathscr{L}^{KG}_{free} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) \Rightarrow \frac{i}{p^2 - m^2}$$ Scalar field

The photon is described by a 4-component vector. Yet it has only two degrees of freedom. In classical electromagnetism we choose a gauge. The Lorenz gauge is

$$\partial_\mu A^\mu = 0 \rightarrow p_\mu \epsilon^\mu = 0$$

What is done in QFT is to add a gauge fixing Lagrangian

$$\mathscr{L}_{GF} = -\frac{1}{4}F^2 + \frac{1}{2}\lambda(\partial \cdot A)^2 \quad \Rightarrow \quad \Box A^\mu + (\lambda - 1)\partial^\mu(\partial_\nu A^\nu) = 0$$

$$-\frac{ig_{\mu\nu}}{p^2 - m^2}$$ Vector field in the 't Hooft-Feynman Gauge

$$i\frac{p_\mu \gamma^\mu + m}{p^2 - m^2}$$ Spinor 4 by 4 matrix

Interactions

$$\mathscr{L}_{int} = \frac{\lambda}{4!}\phi^4 \;\Rightarrow\; i\lambda$$ (Real) Scalar theory with self-interactions

$$\mathscr{L}_{int} = \frac{\lambda}{4}(\phi^\dagger\phi)^2 \;\Rightarrow\; i\lambda$$ (Complex) Scalar theory with self-interactions

$$\mathscr{L}_{int} = e\overline{\psi}\gamma_\mu A^\mu \psi \;\Rightarrow\; ie\gamma_\mu$$ QED

$$\mathscr{L}_{int} = -e^2 g_{\mu\nu} A^\mu A^\nu \phi^\dagger \phi \;\Rightarrow\; -2ie^2 g_{\mu\nu}$$ Scalar QED

Cross Sections

The S-matrix (scattering matrix) is the unitary operator S that determines the evolution of the initial state |i> at t=-∞ to state |f> at t =+∞.

The probability amplitude for a transition between initial state |i> and state |f> is

$$S_{fi} = \langle f | S | i \rangle$$

and S is the scattering matrix. S is expanded at each order and it depends on the interaction Lagrangian

$$S \Leftarrow \mathscr{L}_{int}$$

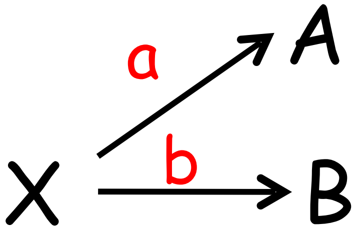
The cross section is proportional to the transition amplitude

$$\sigma \propto |\langle f | S | i \rangle|^2$$

and is the probability of a given process to occur.

Decay width and branching ratios

Here again the interaction Lagrangian appears in the calculation of the decay width just like with a cross section. When a particle decays it may decay to different sets of particles. The decay width is like the decay constant in Nuclear Physics



Nucleus X decays to A with decay constant λ_a and to B with decay constant λ_b.

$$\begin{cases} \left(\frac{dN}{dt}\right)_a = -\lambda_a N & \text{to A} \\ \left(\frac{dN}{dt}\right)_b = -\lambda_b N & \text{to B} \end{cases} \Rightarrow \frac{dN}{dt} = \left(\frac{dN}{dt}\right)_a + \left(\frac{dN}{dt}\right)_b = -(\lambda_a + \lambda_b) N$$

$$N(t) = N_0 \; e^{-\lambda_T t} \quad \text{com} \quad \lambda_T = \lambda_a + \lambda_b$$

In particle physics the notation is

$$\lambda \rightarrow \Gamma$$

Particle lifetime is the time taken for the sample to reduce to 1/e of original sample.

Decay width and branching ratios

In natural units (more later) the decay width is the decay constant which is the inverse of the lifetime of the particle. For instance for the Z boson the total width is

$$\Gamma_Z = \Gamma_{ee} + \Gamma_{\mu\mu} + \Gamma_{\tau\tau} + \Gamma_{\nu_i\nu_i} + \Gamma_{q_i q_i}$$

with the measured value of

$$\Gamma_Z = 2.4952 \pm 0.0023 \text{ GeV}$$

and a lifetime of about 2.7 × 10^{-25} s. Each of the terms are called partial width for the corresponding channel.

The branching ratio for a specific channel

$$\text{BR}_X = \frac{\Gamma_X}{\Gamma_Z}$$

Fraction of electrons coming from the Z decay

$$\text{BR}_{ee} = \frac{\Gamma_{ee}}{\Gamma_Z}$$

Z DECAY MODES

	Mode		Fraction (Γ_i/Γ)	Scale factor/ Confidence level
Γ_1	e^+e^-	[a]	(3.3632±0.0042) %	
Γ_2	$\mu^+\mu^-$	[a]	(3.3662±0.0066) %	
Γ_3	$\tau^+\tau^-$	[a]	(3.3696±0.0083) %	
Γ_4	$\ell^+\ell^-$	[a,b]	(3.3658±0.0023) %	
Γ_5	$\ell^+\ell^-\ell^+\ell^-$	[c]	(3.5 ±0.4) × 10^{-6}	S=1.7
Γ_6	invisible	[a]	(20.000 ±0.055) %	
Γ_7	hadrons	[a]	(69.911 ±0.056) %	
Γ_8	$(u\overline{u}+c\overline{c})/2$		(11.6 ±0.6) %	
Γ_9	$(d\overline{d}+s\overline{s}+b\overline{b})/3$		(15.6 ±0.4) %	
Γ_{10}	$c\overline{c}$		(12.03 ±0.21) %	
Γ_{11}	$b\overline{b}$		(15.12 ±0.05) %	
Γ_{12}	$b\overline{b}b\overline{b}$		(3.6 ±1.3) × 10^{-4}	
Γ_{13}	ggg		< 1.1 %	CL=95%
Γ_{14}	$\pi^0\gamma$		< 2.01 × 10^{-5}	CL=95%
Γ_{15}	$\eta\gamma$		< 5.1 × 10^{-5}	CL=95%
Γ_{16}	$\omega\gamma$		< 6.5 × 10^{-4}	CL=95%
Γ_{17}	$\eta'(958)\gamma$		< 4.2 × 10^{-5}	CL=95%
Γ_{18}	$\phi\gamma$		< 8.3 × 10^{-6}	CL=95%
Γ_{19}	$\gamma\gamma$		< 1.46 × 10^{-5}	CL=95%

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A generalised concept of luminosity

When particles colide at a lepton collider you have to count the number of electrons and positrons that collide. Suppose we are calculated the process

$$e^+e^- \to \gamma\gamma$$

The number of $\gamma\gamma$ events produced at the collider is

$$N_\gamma = 2\, N_{e^+}\, \sigma_{e^+e^- \to \gamma\gamma}$$

and N_{e^+} is the luminosity. Same principles apply to any lepton collider. Two notes: a) the number of electrons that actually count to the collision is not the number of electrons produced; b) the number of photons actually detected depend on the detectors and on the specific experimental analysis.

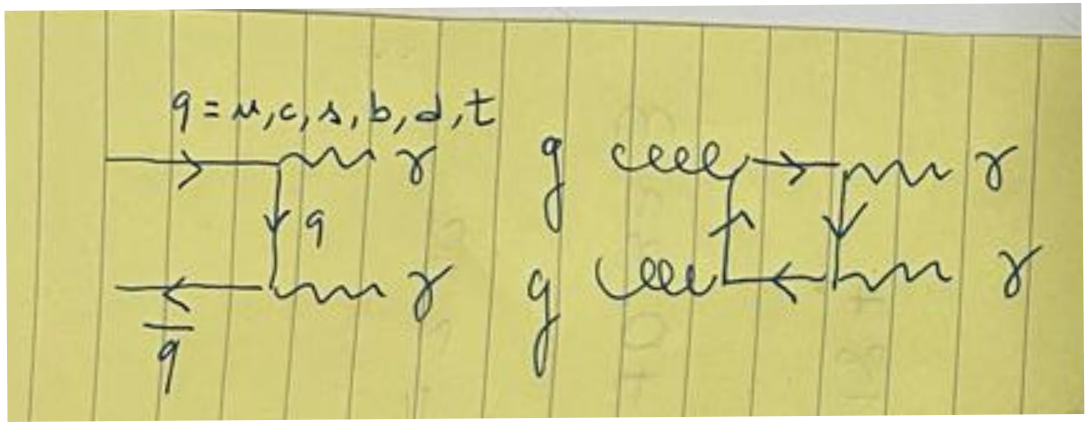
Let us now move to a photon collider. There are various methods of e → γ conversion but the best one is the Compton scattering of laser light on high-energy electrons. In this method a laser photon is scattered backward taking from the high-energy electron a large fraction of its energy. The scattered photon travels along the direction of the initial electron. Here, the CM energy is calculated to be a fraction of the initial leptons.

A generalised concept of luminosity

And finally <u>hadron colliders</u>. Hadron colliders are pp (LHC - Large Hadron Collider) or $p\bar{p}$ (Tevatron) colliders. The protons are made of quarks and gluons. Further, when they collide there is a lot of energy and many elementary particles can be present. Considering again the production of two photons

$$pp \to \gamma\gamma$$

From the point of view of the particles (partons) that constitute the protons we have the following set (just one of each type) of Feynman diagrams



Depending on the energy scale of the problem we may consider only the lightest quarks and the gluon, or the heavier ones. At the LHC there are two schemes: the 4F scheme where the b-quark is not considered and the 5F scheme where it is taken into account. The top quark is too heavy to play a role at the LHC. The u and d are called valence quarks.

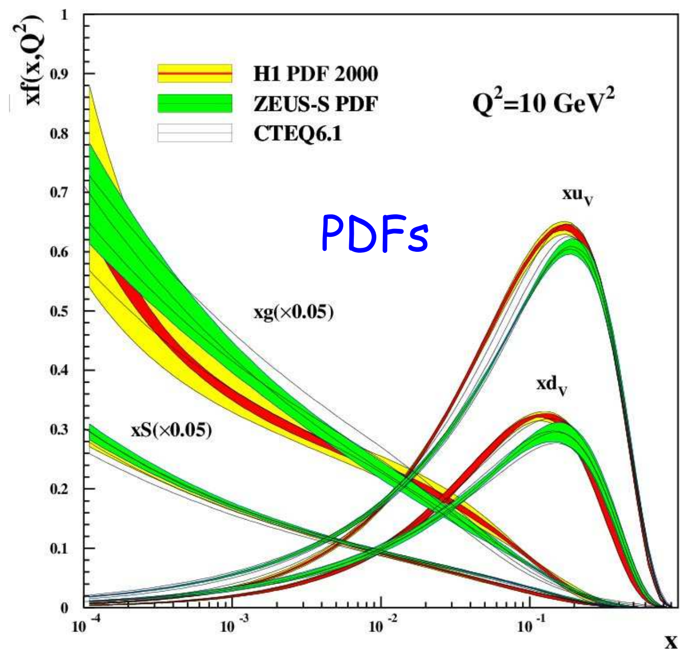
A generalised concept of luminosity

To calculate the cross section $\sigma(pp \to \gamma\gamma)$ we first need to calculate the corresponding Parton level cross section $\sigma(q\bar{q} \to \gamma\gamma)$ for all quarks but the top and $\sigma(gg \to \gamma\gamma)$ for the gluon. The total cross section is calculated using the Parton Distribution Functions (PDF).

The PDF is defined as the probability density for finding a particle with a certain longitudinal momentum fraction x at some energy scale. Because of the non-perturbative nature of partons, which cannot be observed as free particles, parton densities cannot be calculated using perturbative QCD. They are therefore calculated using fits from all collider data available so far. The total cross section for $pp \to \gamma\gamma$ is given by

$$\sigma_{pp\to\gamma\gamma} = \int dx_1\, f_{q/p}(x_1,\mu^2) \int dx_2\, f_{q/p}(x_2,\mu^2)\, \hat{\sigma}(x_1 p_1, x_2 p_2, \mu^2)$$

PDFs combine theoretical knowledge from QCD with experiment. Complicated stuff! Still experiment shows that the proton is indeed a (uud), that the sea quarks play a role and that the gluons carry about 50% of the momentum.



Number of events

To find the number of events in a given process you need: (a) the cross section; (b) the branching ratio; (c) the luminosity; (d) the efficiency. Suppose we are looking for a Higgs in the process

$$pp \to h \to \gamma\gamma$$

The total number of events for a luminosity of 25 fb^{-1} and a center-of-mass energy of 8 TeV is

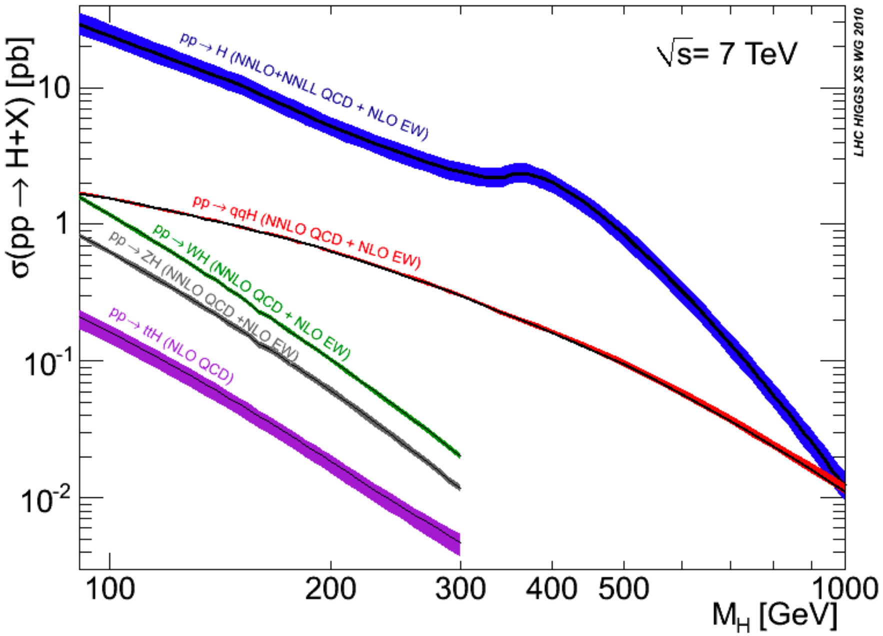
$$N_{Higgs} = \sigma(pp(gg) \to h)\, BR(h \to \gamma\gamma)\, L\, \epsilon$$

$$N_{Higgs} = (21.4 \times 10^{3}) \times (2.28 \times 10^{-3}) \times (25) \times (1) \approx 1220$$

with the efficiency set to 1.

This is the maximum number of events. We then need to take into account the background and the fact that all apparatus and analysis have a specific efficiency.

Profiling the Higgs



Cross section for Higgs production at the LHC
For a center of mass energy of 7 TeV.

Total width of the Higgs as a function of the mass.

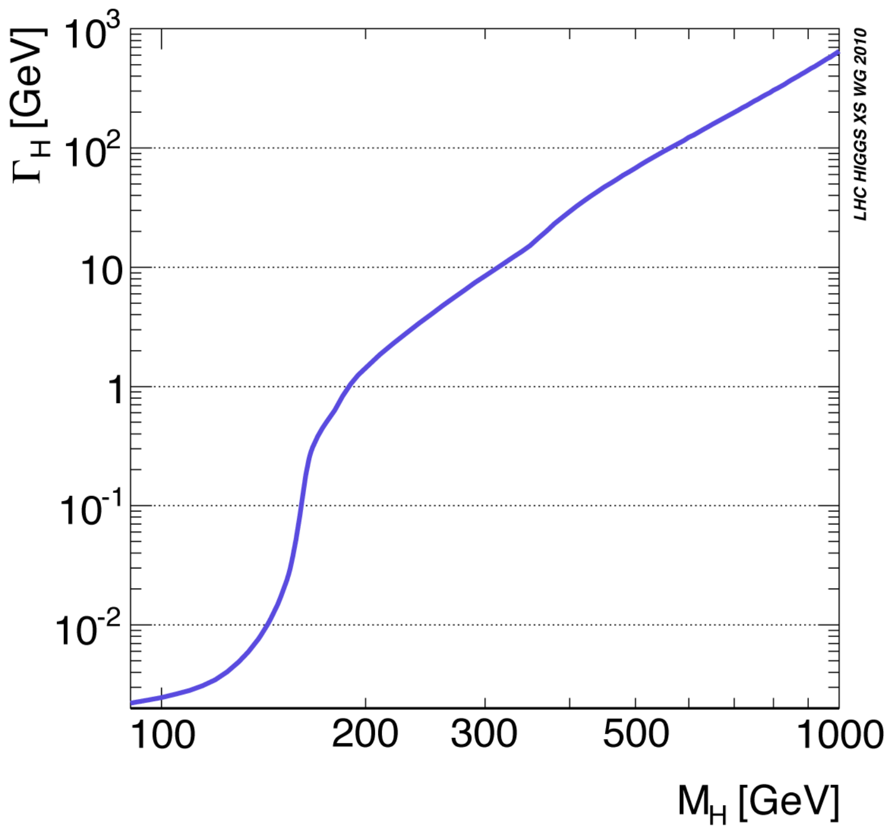
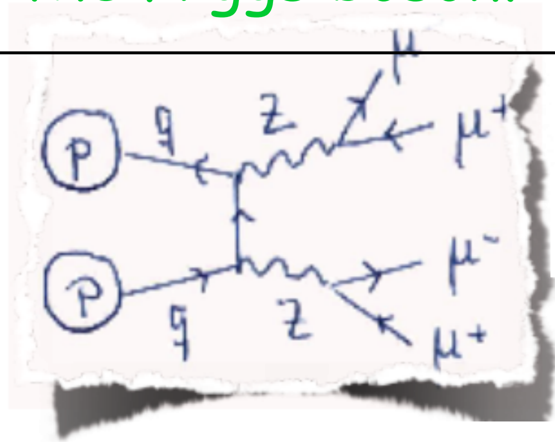


Table 11.3: The branching ratios and the relative uncertainty [43,44] for a SM Higgs boson with $m_H = 125\,\text{GeV}$.

Decay channel	Branching ratio	Rel. uncertainty
$H \to \gamma\gamma$	2.27×10^{-3}	2.1%
$H \to ZZ$	2.62×10^{-2}	±1.5%
$H \to W^+W^-$	2.14×10^{-1}	±1.5%
$H \to \tau^+\tau^-$	6.27×10^{-2}	±1.6%
$H \to b\bar{b}$	5.82×10^{-1}	$^{+1.2\%}_{-1.3\%}$
$H \to c\bar{c}$	2.89×10^{-2}	$^{+5.5\%}_{-2.0\%}$
$H \to Z\gamma$	1.53×10^{-3}	±5.8%
$H \to \mu^+\mu^-$	2.18×10^{-4}	±1.7%

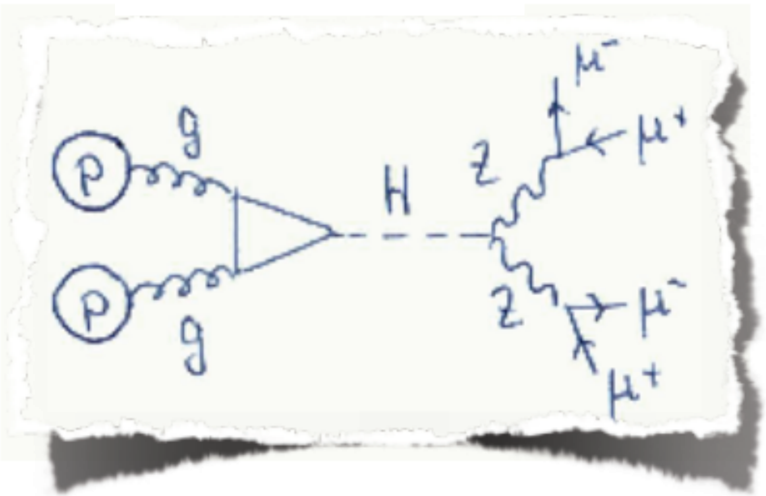
How do we search for the Higgs boson?

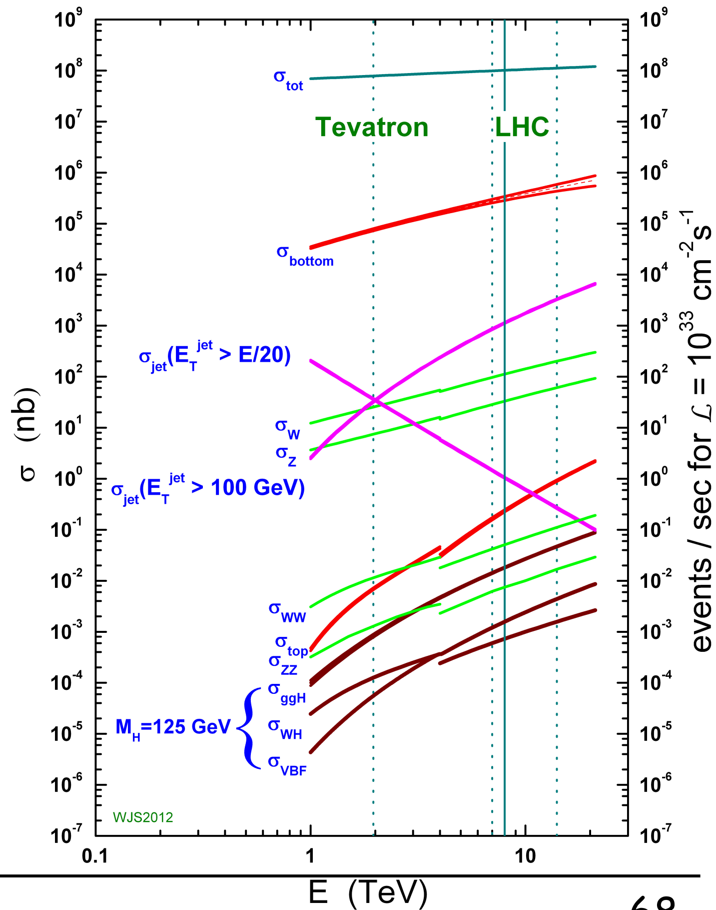
Signal and background



a lot

a few

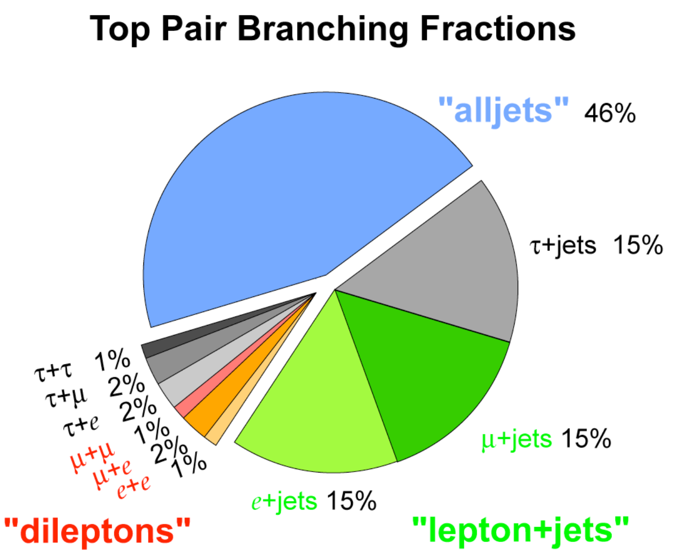


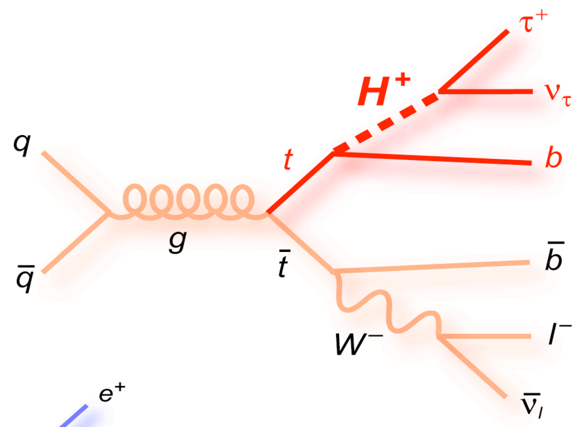


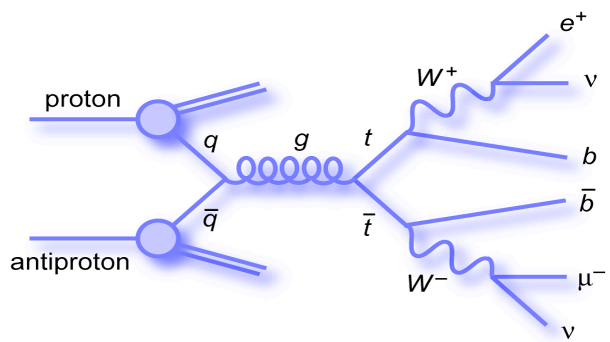
Example of an LHC search

- Suppose we are searching for a charged Higgs at the LHC
- The charged Higgs comes from a top-quark (t -> H^+ b)
- There are two top-quarks in the event
- The charged Higgs will decay to a tau and a neutrino (H^+ -> τ^+ υ)
- The tau will decay to an electron or a muon and neutrinos (τ^+-> e^+ υ υ)
- Experimentalists will look for a an electron and missing energy (from the neutrinos).

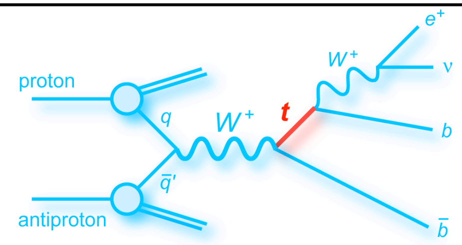
- Do they come from a charged Higgs?







Levels of the search - folklore in traditional approach



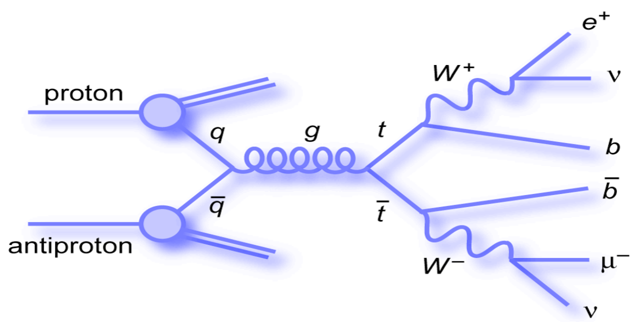
Parton Level pp -> e^+ υ υ υ b + X

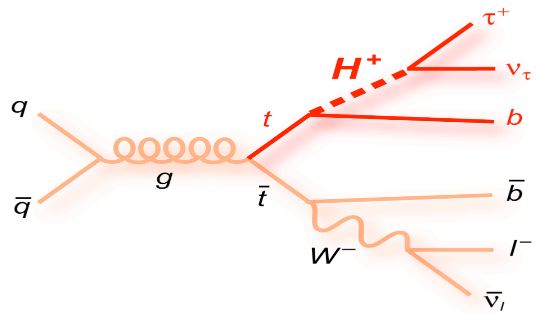
a) Introduce background (again there are several levels)

Irreducible - pp -> t + X -> W^+ b + X-> e^+ υ υ b + X

Reducible - pp -> t t -> W^+ b W^- b

- Number of reducible backgrounds is virtually infinite (a jet has some probability of being misidentified as an electron).





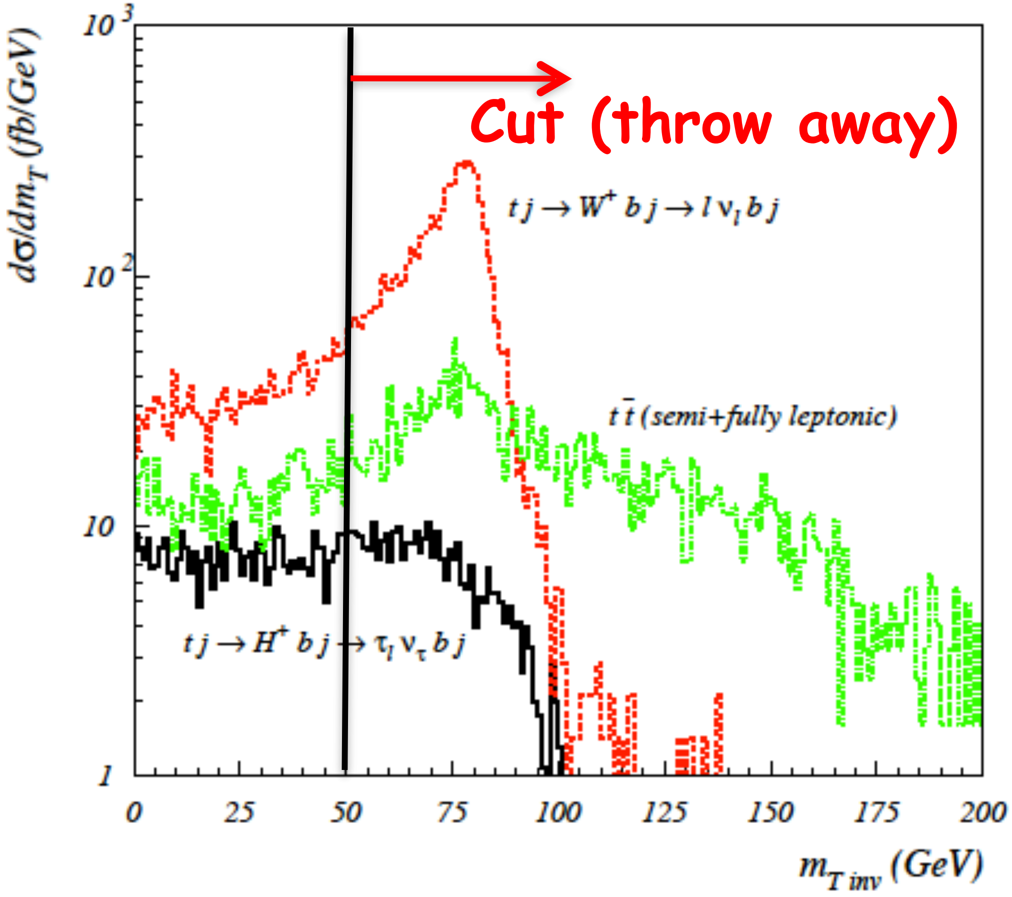
b) Pretend you understand what is happening (by mimicking the experimental analysis)

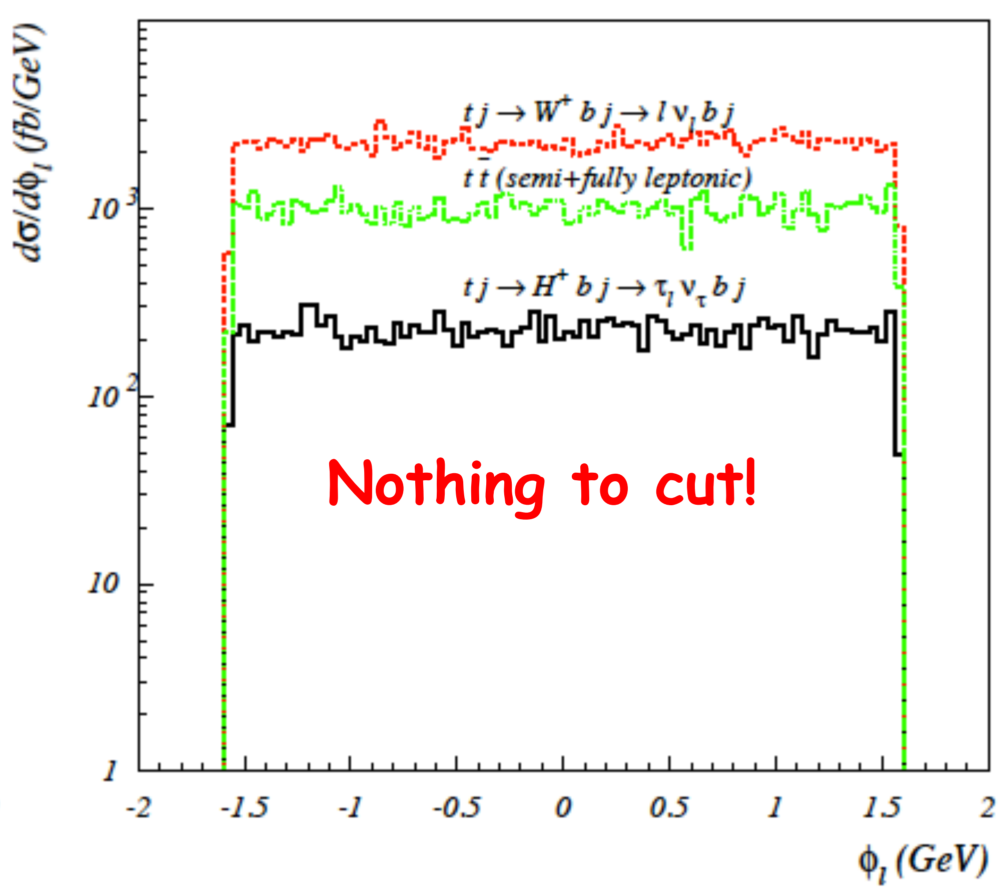
Trigger – how efficiently are "our" events recorded? One lepton!

Electron – how efficient is electron recognition?

Levels of the search - folklore in traditional approach

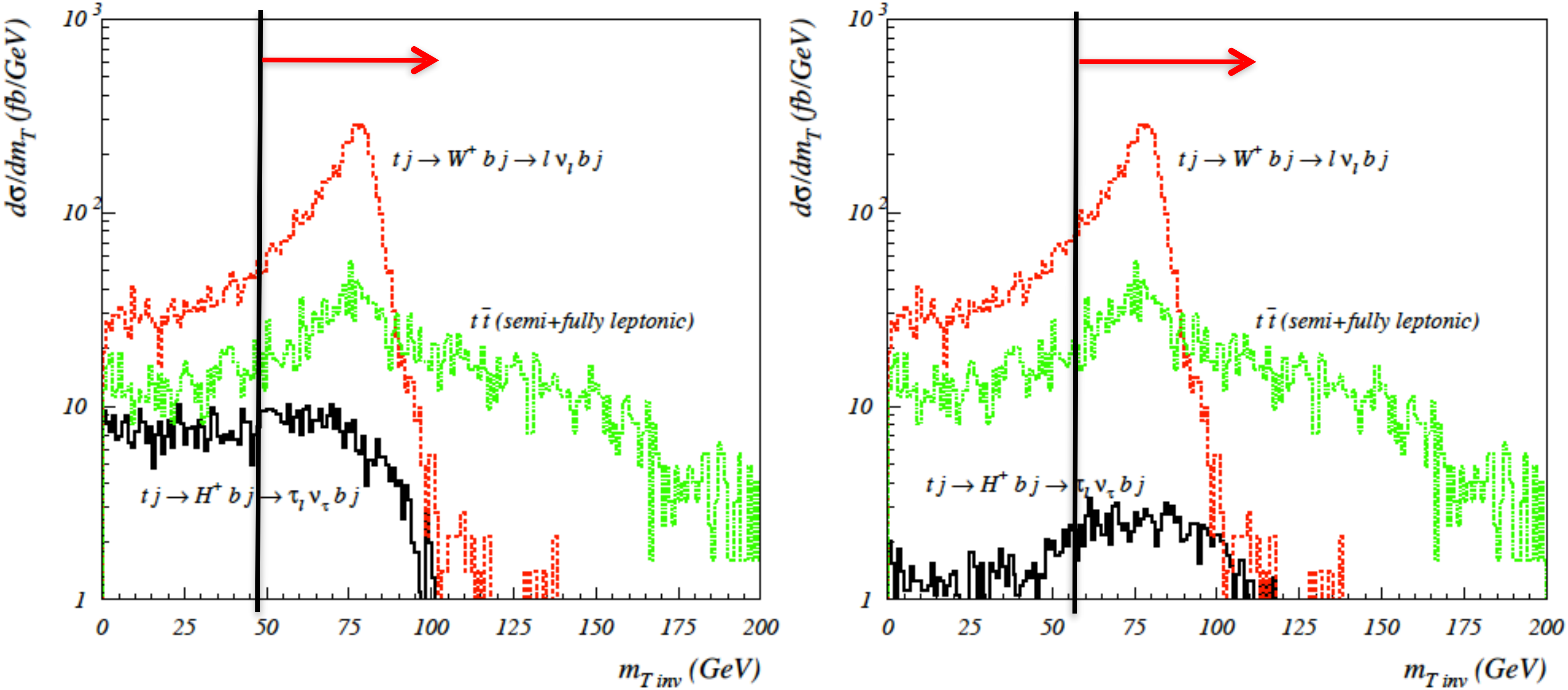
c) Plot distributions and spot the differences (loose signal but loose even more background)





Two types of variables: the transverse mass and the lepton azimuthal angle (cut-based).

Levels of the search - folklore in traditional approach



Distributions depend on the model parameters. The higher the charged Higgs mass the lower the cross section. There is also a "peak" shift.

d) e) f) Radiation, Detector and finally Data.

Levels of the search - folklore in traditional approach

	$\sigma_{m_{H^\pm}=100\,\mathbf{GeV}}$	$\sigma_{m_{H^\pm}=110\,\mathbf{GeV}}$	$\sigma_{m_{H^\pm}=120\,\mathbf{GeV}}$	$\sigma_{m_{H^\pm}=130\,\mathbf{GeV}}$	$\sigma_{m_{H^\pm}=140\,\mathbf{GeV}}$
Process					
Signal	379.4 fb	274.4 fb	202.7 fb	118.9 fb	65.5 fb
Bg (single-top)	1705.4 fb				
Bg ($t\bar{t}$ semi-leptonic)	683.1 fb				
Bg ($t\bar{t}$ leptonic)	393.6 fb				
σ_S/σ_B	0.14	0.098	0.073	0.042	0.023
$\sigma_S/\sqrt{\sigma_B}$ (fb$^{1/2}$)	7.19	5.20	3.84	2.25	1.24

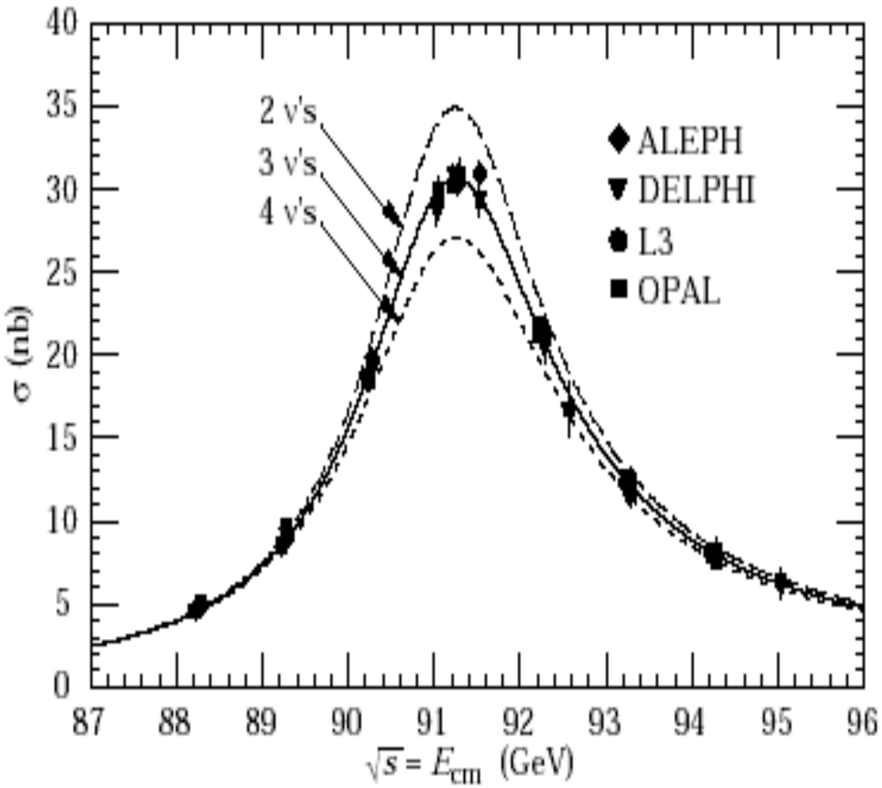
The analysis is done. We now have, for a given luminosity, S signal events and B background events.

Discovery - $S/B^{1/2}$ > 5

An exclusion (absence of signal) is usually shown for 95 % C.L.

We see mass!

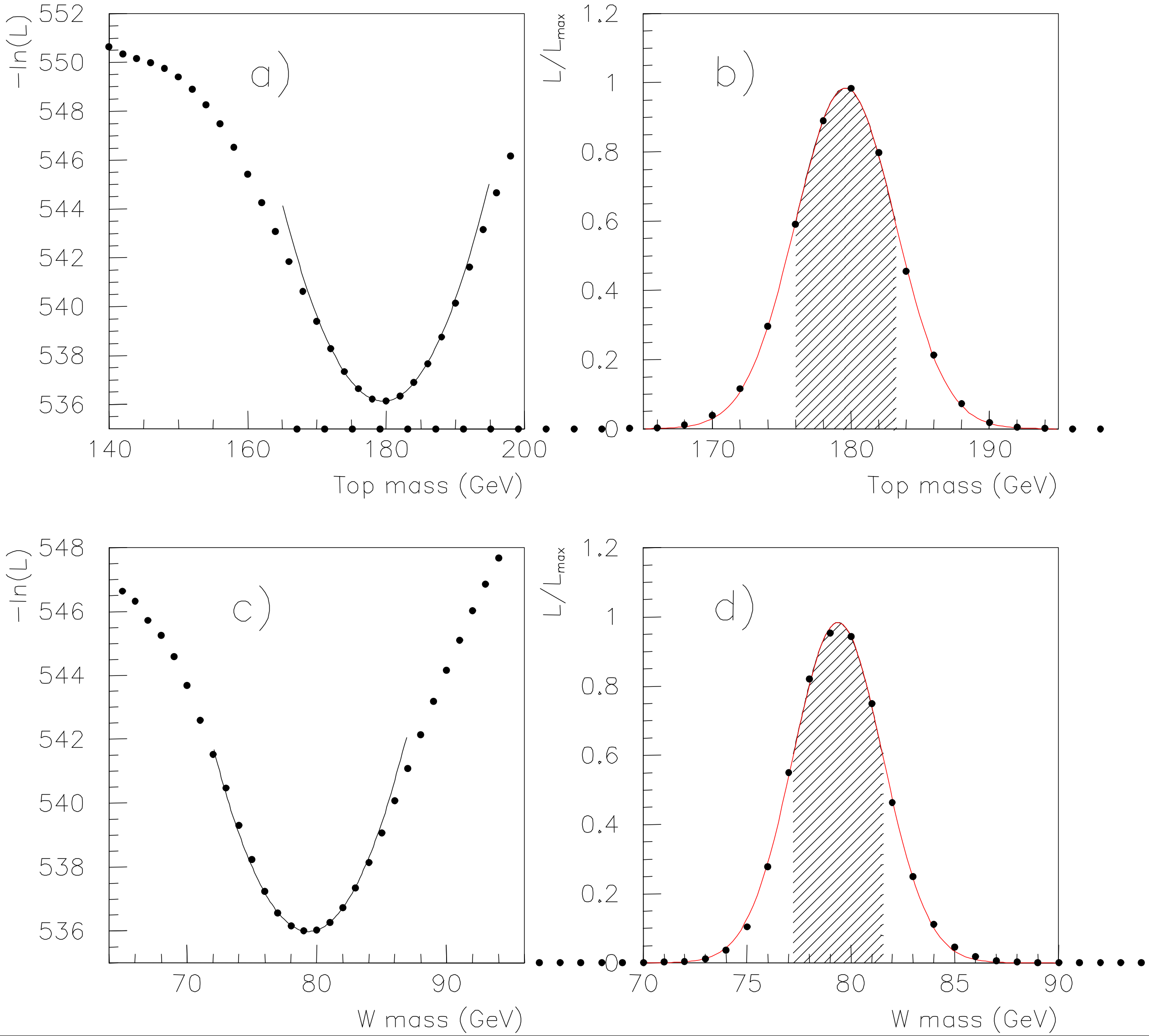
Weak gauge bosons <u>have</u> mass (LEP). The number of neutrinos can be counted in a given model



$$e^+e^- \rightarrow Z^0$$

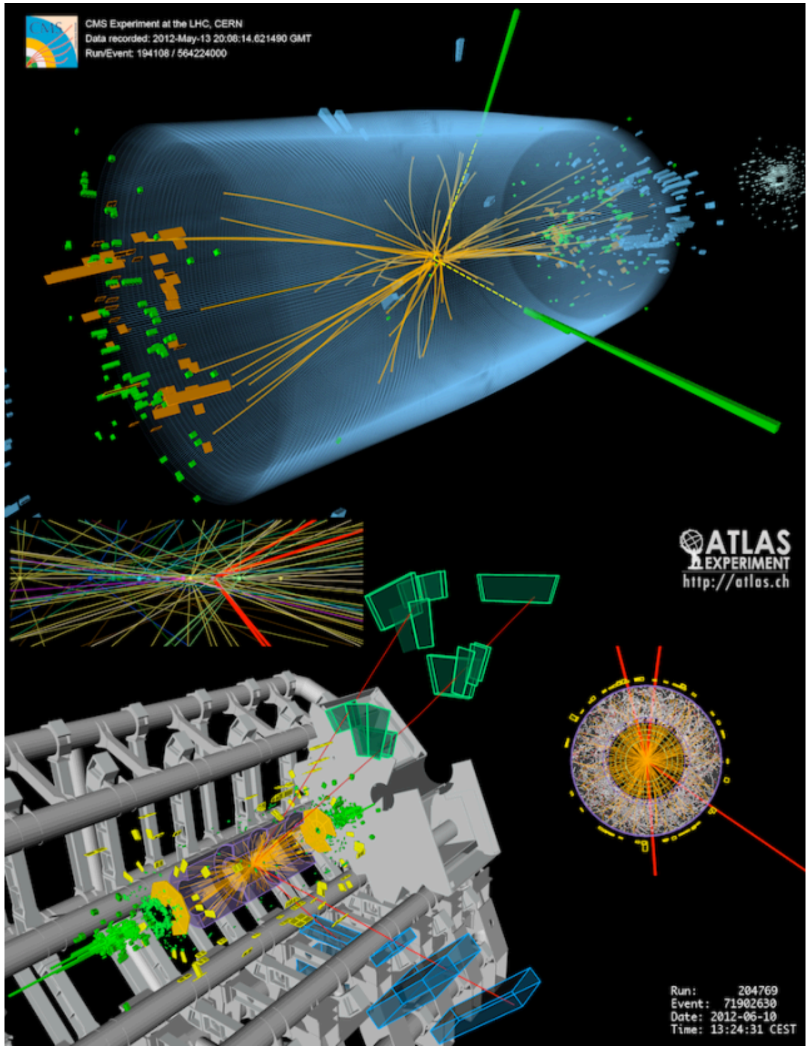
We see mass!

- Fermions <u>have</u> mass



top and W mass measurements by D0 (Tevatron Run I)

Some history - The Higgs discovery



There is a model, the Standard Model, that is based on symmetries.

With the symmetries, all particles emerge without a mass. But most particles have mass. Brout, Englert and Higgs proposed a mechanism that gives mass to the particles via the interaction with a field we now call the "Higgs" field.

Just after the Big Bang the Higgs field was zero but as the temperature fell below a critical value, it spontaneously grew and particles interacting with it got a mass. The larger the interaction the heavier the particle. No coupling to the photon.

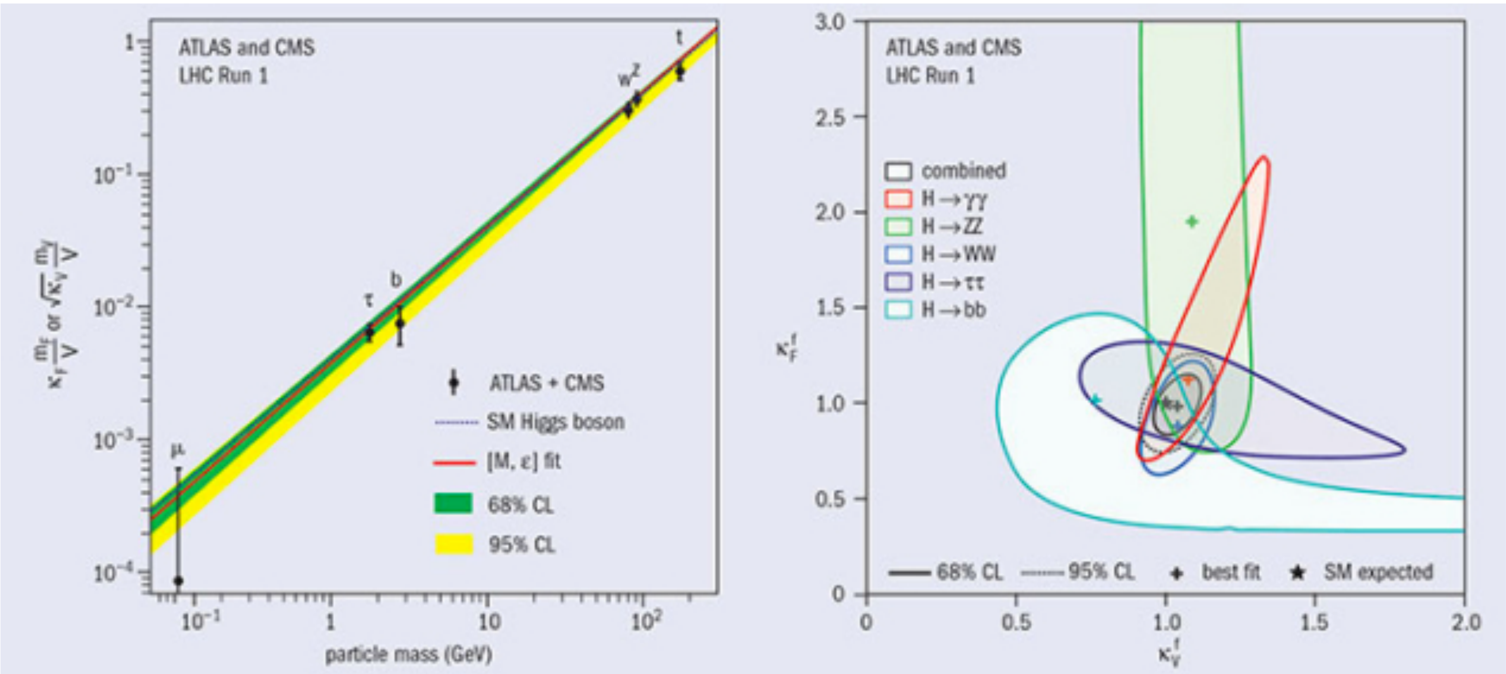
On July 4 2012, the ATLAS and CMS experiments at CERN's Large Hadron Collider observed a new particle in the mass region around 125 GeV, consistent with the Standard Model Higgs boson. Is it the Higgs boson predicted by the Standard Model?

Some history - The Higgs discovery

There is a potential - the Higgs potential

$$V_{SM} = m_{11}^2(\Phi_1^\dagger\Phi_1) + \frac{\lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2$$

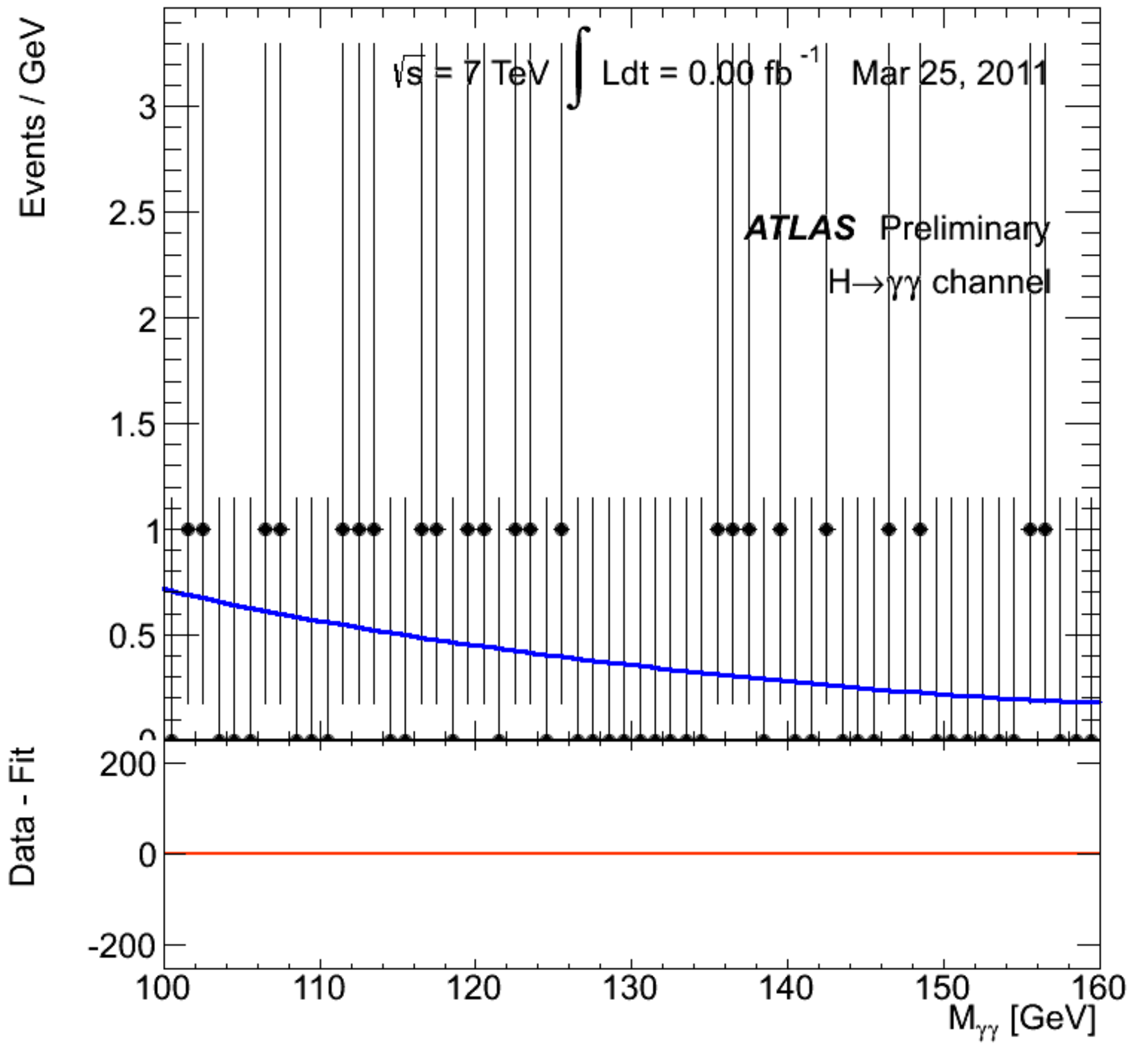
- The Higgs field
- Couples to boson fields - mass of the bosons
- Couples to fermion fields - mass of the fermions



$$g_{NP}^{hVV} = \kappa_V g_{SM}^{hVV}$$

So, 8 years after the discovery, the 125 GeV scalar looks very much like the SM Higgs

Some history - The Higgs discovery

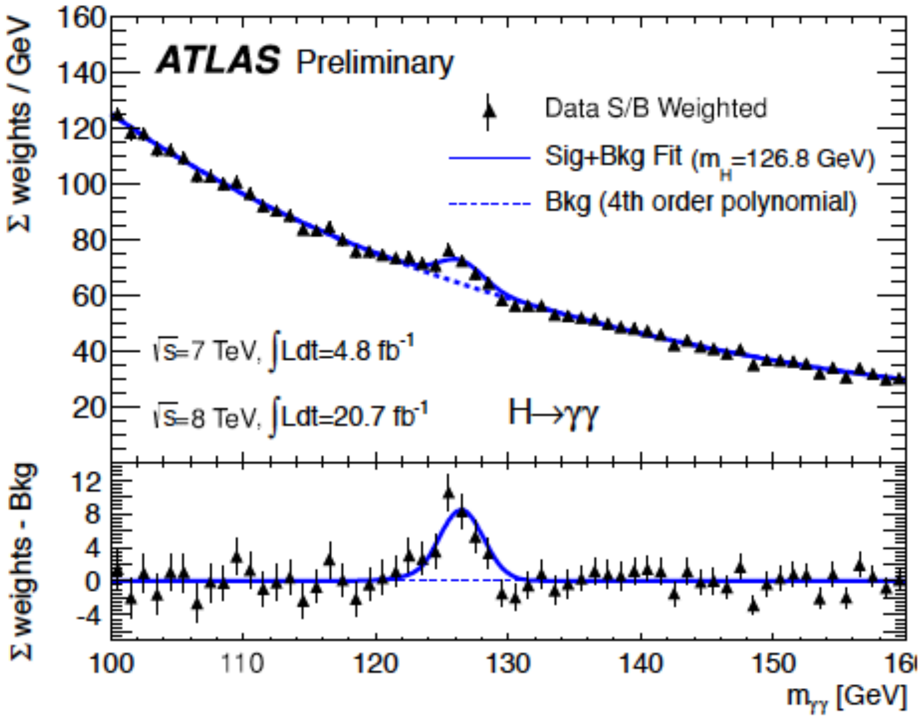


5 sigma and the Higgs discovery!

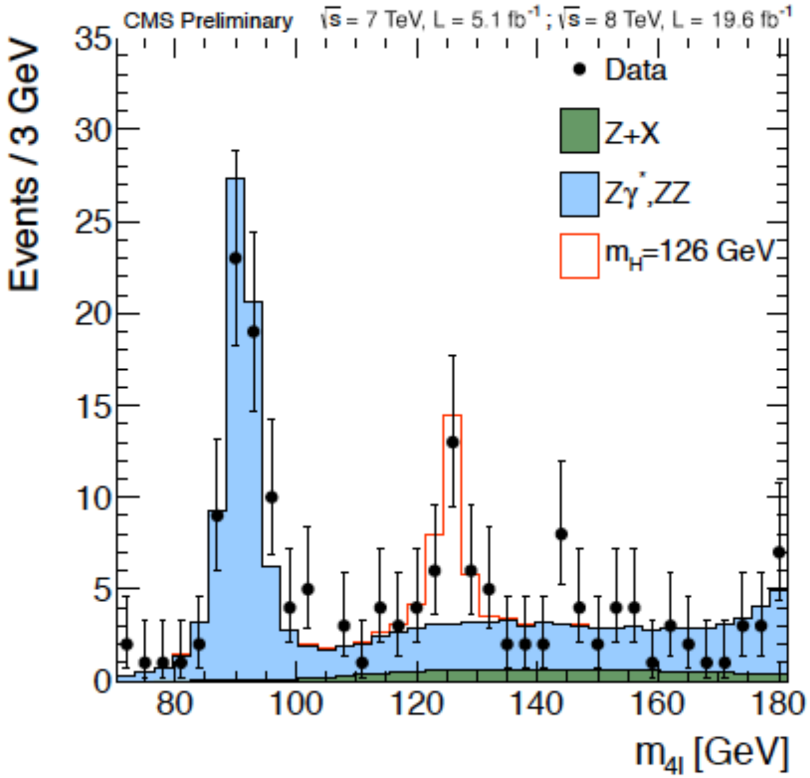
LHC collects a huge number of collisions data and counts how many times two given photons have the Higgs mass. When 5 sigma was reached the Higgs was considered to have been discovered.

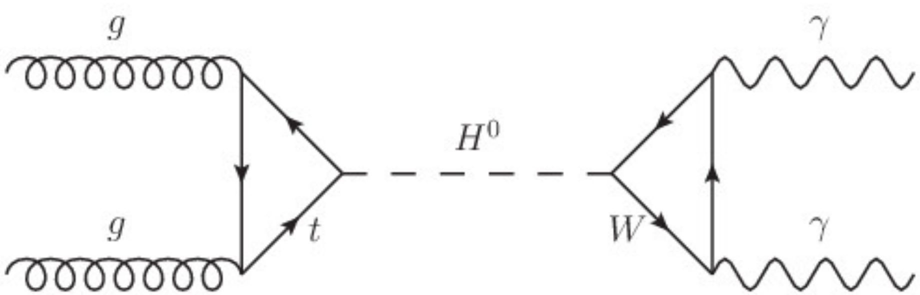
Some history - The Higgs discovery

ATLAS-CONF-2013-12



CMS-PAS-HIG-13-002





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New results indicate that particle discovered at CERN is a Higgs boson

14 Mar 2013

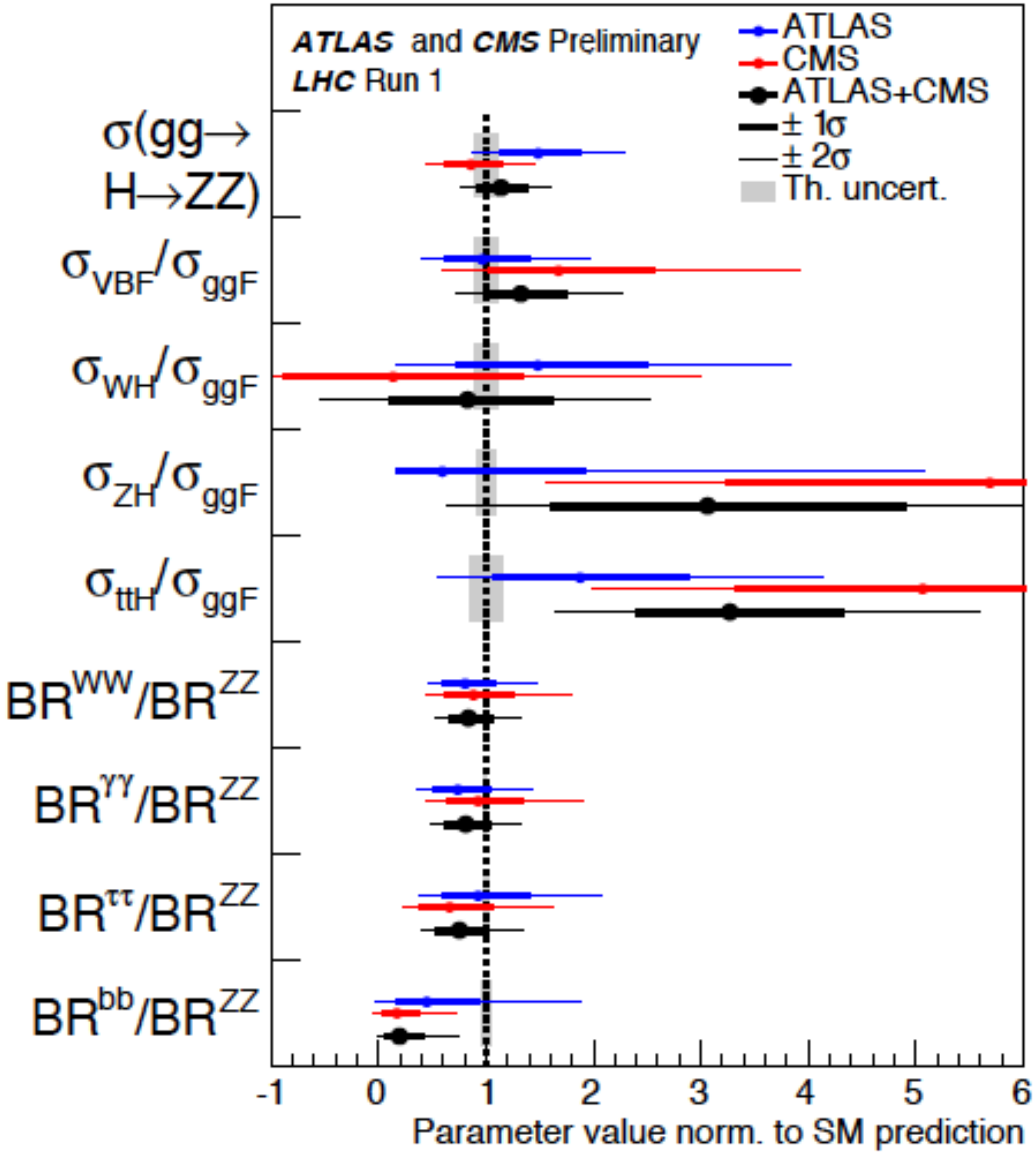
Geneva, 14 March 2013. At the Moriond Conference today, the ATLAS and CMS collaborations at CERN[1]'s Large Hadron Collider (LHC) presented preliminary new results that further elucidate the particle discovered last year. Having analysed two and a half times more data than was available for the discovery announcement in July, they find that the new particle is looking more and more like a Higgs boson, the particle linked to the mechanism that gives mass to elementary particles. It remains an open question, however, whether this is the Higgs boson of the Standard Model of particle physics, or possibly the lightest of several bosons predicted in some theories that go beyond the Standard Model. Finding the answer to this question will take time.

Whether or not it is a Higgs boson is demonstrated by how it interacts with other particles, and its quantum properties. For example, a Higgs boson is postulated to have spin 0, and in the Standard Model its parity – a measure of how its mirror image behaves – should be positive. CMS and ATLAS have compared a number of options for the spin-parity of this particle, and these all prefer no spin and positive parity. This,

1964
Brout-Englert-Higgs-
Mechanism

2013
Nobel Prize for Physics

The Standard Model
is complete.
Now what?



ATLAS/CMS combination with all run1 data.

ATLAS-CONF-2015-044
CMS-PAS-HIG-15-002

15th September 2015

The Higgs looks very SM-like because all couplings are well within the SM predictions. And there is no hint of new physics so far.

There are essentially two ways of showing we need new physics:

a) deviations from the SM predictions

b) finding something new (like with dark matter if it is indeed a particle)

Feynman Rules, Cross Sections and Branching ratios - calculations

See extra set of slides

Infinities in QFT - renormalisation

Can an infinite quantity be made finite and measured?

"Excuse me, is this the Society for Asking Stupid Questions?"

Infinities occur when integrals coming from the computation of cross sections and decay widths, using Feynman diagrams, give rise to terms that lead to infinities in the high energy (short distance) or low energy (long distance) limits.

Infinities from low energy physics are called infrared divergences and occur when there are massless particles.

Infinities from high energy physics are called ultraviolet divergences and arise in the limit of high energy.

Can an infinite quantity be made finite and measured?

We start with the infinite integral

$$f(x) = \int_1^{+\infty} \frac{1}{x+y} dy = [\ln(x+y)]_1^{+\infty} = \infty$$

and now we regularise it in an obviously non-unique way

$$\bar{f}(x) = f(x) - f(0) = \int_1^{+\infty} \frac{-x}{y(x+y)} dy = -\ln x$$

and we get the finite quantity

$$f(x) = \bar{f}(x) + f(0)$$

$\bar{f}(x)$ → Renormalised (measured) quantity.

$f(x)$ → High order contribution.

$f(0)$ → Tree-level value.

The infinity will be hidden in the experimental measured quantity, like mass.

Regularisation - IR regulator and cut-off

We now start with the infinite integral

$$f(0) = \int_0^{+\infty} \frac{1}{y^2} dy = \left[-\frac{1}{y} \right]_0^{+\infty} = \infty$$

When an infrared regulator x is introduced, the integral is now finite for a non-zero regulator

$$f(x) = \int_0^{+\infty} \frac{1}{y^2 + x^2} dy$$

When a cut-off is introduced the integral will depend on the cut-off

$$f(\Lambda) = \int_\Lambda^{+\infty} \frac{1}{y^2} dy$$

And the cut-off can also be introduced to handle ultraviolet divergences

Dimensional Regularisation

The simplest divergences for large momentum behave as $|p|^{-2}$ or as $|p|^{-4}$. The momentum integrals are finite only for dimensions D < 2 and D < 4 respectively.

The idea is to calculate a Feynman integral for a continuous-valued number of dimensions D for which convergence is assured. The simplest integral is

$$I(D) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - m^2} = \frac{S_D}{(2\pi)^D} \int_0^\infty dp\, p^{D-1} \frac{1}{p^2 - m^2}$$

where

$$S_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}; \qquad \Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$$

is the surface of a unit sphere in D dimensions.

Dimensional Regularisation

By setting D=4-ε, the integral can be written as

$$I(D)=\int\frac{d^D p}{(2\pi)^D}\frac{1}{p^2-m^2}=\frac{m^2}{(4\pi)^2}\left(\frac{4\pi}{m^2}\right)^{\varepsilon/2}\Gamma(\varepsilon/2-1)$$

and expanding in ε,

$$\left(\frac{4\pi}{m^2}\right)^{\varepsilon/2}=1+\frac{\varepsilon}{2}\ln\left(\frac{4\pi}{m^2}\right)+O(\varepsilon^2) \qquad \Gamma(\varepsilon/2-1)=\frac{2}{\varepsilon}+\psi(2)+O(\varepsilon)$$

we obtain the result,

$$I(D)=\frac{m^2}{(4\pi)^2}\left\{\frac{2}{\varepsilon}+\psi(2)+\ln\left(\frac{4\pi}{m^2}\right)\right\}$$

Note the wrong dimensions in the argument of the log. The introduction of a mass scale related to coupling constant will solve this problem.

Infrared divergences

Let us consider the integral that represents the electron self-energy (I will come back to this later). It is clear that when $k^2 \to 0$ the integral diverges

$$\int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{q} - \not{k} + m)}{(q-k)^2 - m^2 + i\varepsilon} \gamma_\mu \frac{1}{k^2}$$

Electron propagator: $\frac{i(\not{q} - \not{k} + m)}{(q-k)^2 - m^2 + i\varepsilon}$

Photon propagator: $\frac{1}{k^2}$

To know where the infinities are we add a mass for the photon and when the calculation is properly done, term proportional to this mass will cancel

$$\int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{q} - \not{k} + m)}{(q-k)^2 - m^2 + i\varepsilon} \gamma_\mu \frac{1}{k^2 - m_\gamma^2}$$

IR regulator: m_γ^2

Infrared divergences

IR divergences arise from not considering all the factors in a cross-section. Although both $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to \mu^+\mu^-(\gamma)$ are IR divergent, their sum does not depend on the photon mass.

When we consider the virtual corrections

$$\sigma_V = \frac{e^2}{8\pi^2}\sigma_0\left\{-4\ln^2\frac{m_\gamma^2}{Q^2}+\frac{\pi^2}{3}-\frac{7}{2}\right\}$$

together with the real emission contribution

$$\sigma_R = \frac{e^2}{8\pi^2}\sigma_0\left\{4\ln^2\frac{m_\gamma^2}{Q^2}-\frac{\pi^2}{3}+5\right\}$$

The measured cross section does not depend on the regulator.

Renormalisable theories

If a theory is renormalisable all the infinities that arise in the calculation of physical observables can be absorbed in the parameters of the theory.

All terms compatible with the symmetry of the Lagrangian must be included.

Terms with mass dimension above 4 should be discarded. Let us consider the Lagrangian for a scalar field

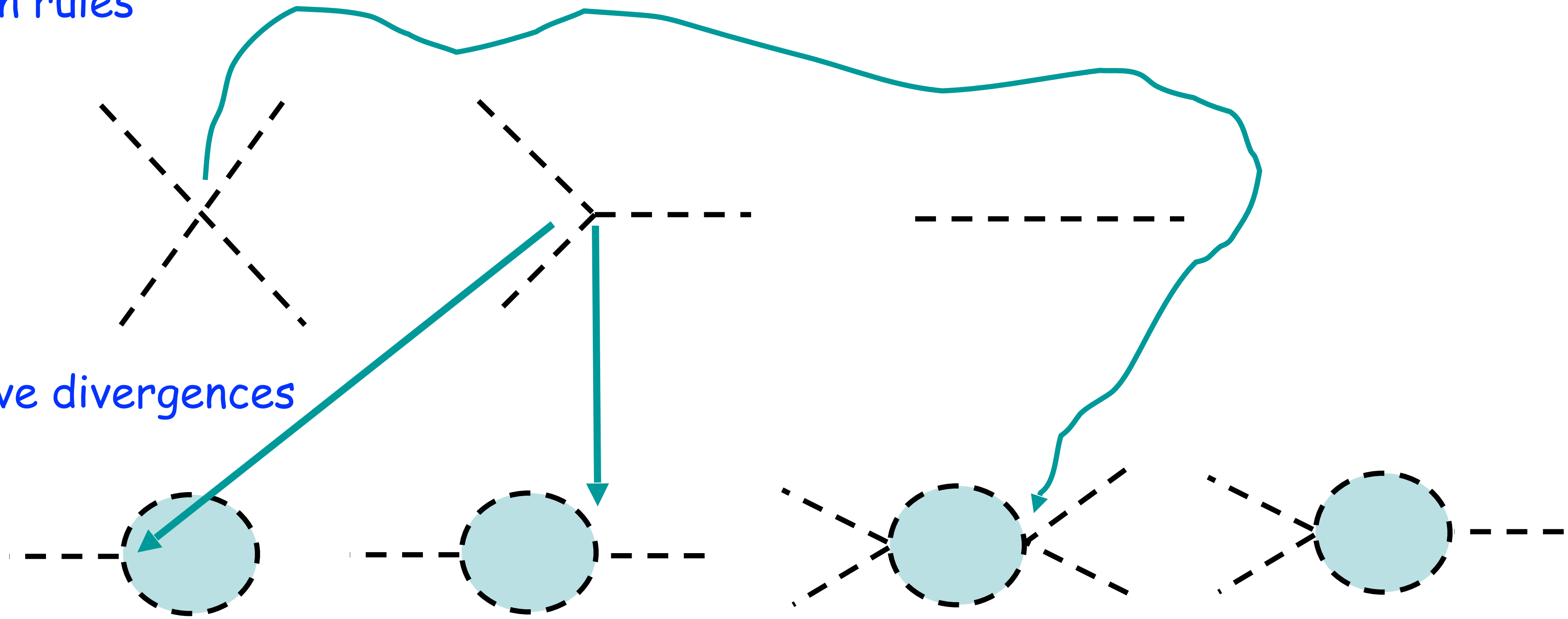
$$L = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \sum_{n \geq 3} \frac{C_n}{n!}\Phi^n \quad \Rightarrow \quad [C_n] = M^{4-n}$$

In particular the cubic coupling C_3 has dimension M, the C_4 term is dimensionless while for n above 4, C_n has a negative mass dimension.

Renormalisable theories - Mass dimension in pictures

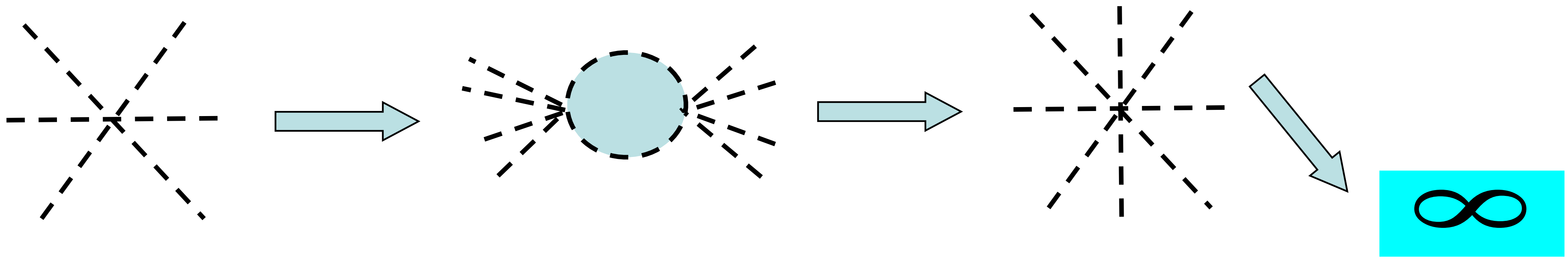
Feynman rules

Primitive divergences



Stop at ϕ^4

Higher order terms



Still…

… theories with n > 4 may be used as approximate effective theories (without the divergent loop graphs) for low-energy processes. The classic example is the Fermi theory of weak interactions, with Lagrangian

$$\mathscr{L}_{int} = \frac{G_F}{\sqrt{2}} \sum_{i,j} \bar{\psi}_i \gamma_\mu (1 - \gamma_5) \psi_i \;\; \bar{\psi}_i \gamma_\mu (1 - \gamma_5) \psi_i$$

where $[G_F] = M^{-2}$ ($G_F = 1.17 \times 10^{-5}$ GeV^{-2}). This is a good effective theory for low-energy interactions, but it cannot be used for energies above $1/(G_F)^{1/2} \approx 300$ GeV. In fact it only works for and energy well below the W boson mass. At higher energies one should use the proper electroweak theory. In QFTs which are valid for all energies, all coupling must have zero or positive energy dimensions.

Renormalisation and our favourite finite pieces

Again: renormalization is a procedure by which we make infinities of loops to disappear by absorbing them into the parameters of the Lagrangian. The subtraction scheme is a way to choose what part of the finite contribution we want to keep. Most typical are:

- Minimal Subtraction (MS): Only the divergence is removed.

- Modified MS: The divergence along with the some constant factors are removed. Of course, this is only used with dimensional regularisation.

- On-shell subtraction: Finites parts are removed such that the renormalized parameters are observables themselves.

Renormalisation of the ϕ^4 theory

We will now consider the simplest theory with all the necessary ingredients for renormalisation

$$\mathscr{L} = \frac{1}{2}[\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2] - \frac{\lambda_0}{4!}\phi_0^4$$

The procedure is then the same for any theory: we redefine the parameters in such a way that the delta quantity will absorb the infinity from the loop

$$\rho_{i,0} = \rho_i + \delta\rho_i$$ **for the parameters.**

$$\phi_{j,0} = \sqrt{Z_{\phi_j}}\,\phi_j \approx \left(1 + \frac{\delta Z_{\phi_j}}{2}\right)\phi_j$$ **for the fields.**

Note the there is no need to renormalise the wave function. You just need to do it if you want finite Green functions.

Renormalisation of the ϕ^4 theory

Starting with the bare Lagrangian

$$\mathscr{L} = \frac{1}{2}[\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2] - \frac{\lambda_0}{4!}\phi_0^4$$

and redefine the parameter as

$$\phi_0 = \sqrt{Z}\phi; \qquad m_0^2 = m^2 + \delta m^2; \qquad \lambda_0 = \lambda + \delta\lambda$$

which after expanding Z leads to the renormalised Lagrangian at 1-loop

$$\mathscr{L} = \frac{1}{2}[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2] - \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}\partial_\mu \phi \partial^\mu \phi - \frac{\delta m^2}{2}\phi^2 - \frac{\delta\lambda}{4!}\phi^4$$

We now need to calculate the counterterms. We have only two independent infinities – primitive divergences.

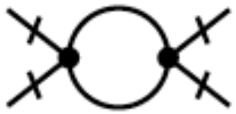
Renormalisation of the ϕ^4 theory

Power counting: we have discussed the renormalisability of the theory in terms of the dimension of the coupling. The dimension of the coupling determines the dimension of the integrand. Therefore,

$$
\begin{aligned}
[\lambda] > 0 &\quad \text{super-renormalizable,} \\
[\lambda] = 0 &\quad \text{renormalizable,} \\
[\lambda] < 0 &\quad \text{non-renormalizable.}
\end{aligned}
$$

Primitive divergences in Φ^4 theory

$$\sim \int^{\Lambda} \frac{d^4k}{k^2} \sim \Lambda^2,$$



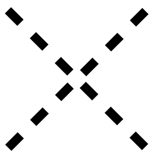
$$\sim \int^{\Lambda} \frac{d^4k}{k^4} \sim \ln \Lambda$$

$$\sim \int^{\Lambda} \frac{d^4k}{k^6} = \text{finite}$$

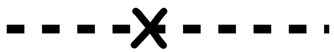
In the case of a super-renormalisable theory there is some order above which all diagrams are finite.

The Feynman rules for the renormalised Lagrangian are

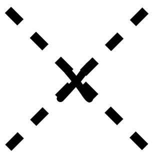
$$\frac{i}{p^2 - m^2}$$



$$-i\lambda$$



$$i(p^2 \delta Z - \delta m^2)$$

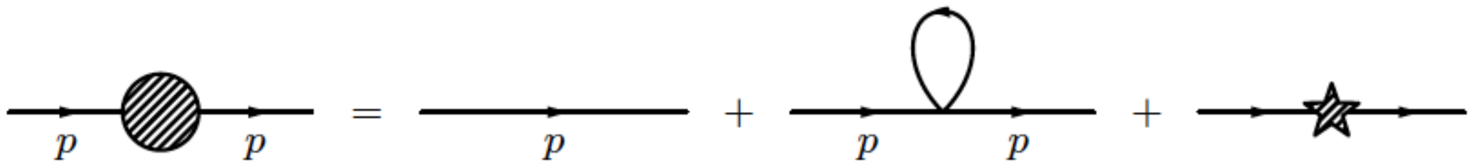


$$-i\delta\lambda$$

The last two are the counterterm diagrams. Let us see now how to build the primitive divergences.

Renormalisation of the two point function

The diagrams contributing to the two point function are



and the amplitude

$$i\mathcal{M} = \frac{i}{p^2 - m_R^2 + i\epsilon} + \frac{i}{p^2 - m_R^2 + i\epsilon}\left[\frac{1}{2}(-i\lambda_R)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m_R^2 + i\epsilon}\right]\frac{i}{p^2 - m_R^2 + i\epsilon}$$

$$+ \frac{i}{p^2 - m_R^2 + i\epsilon}\left[i\left(p^2\delta_Z - \delta_m\right)\right]\frac{i}{p^2 - m_R^2 + i\epsilon}$$

$$= \frac{i}{p^2 - m_R^2 + i\epsilon} + \left(\frac{i m_R^2 \lambda_R}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \log\left(\frac{4\pi e^{-\gamma_E}\mu^2}{m_R^2}\right)\right] + i\left(p^2\delta_Z - \delta_m\right)\right)\left(\frac{i}{p^2 - m_R^2 + i\epsilon}\right)^2$$

- MS: Only cancel out the finite part. Thus

$$\delta_Z = 0, \quad \delta_m = \frac{m_R^2 \lambda_R}{16\pi^2 \epsilon}$$

In this scheme,

$$i\mathcal{M} = \frac{i}{p^2 - m_R^2 + i\epsilon} + \frac{i m_R^2 \lambda_R}{32\pi^2}\left[1 + \log\left(\frac{4\pi e^{-\gamma_E}\mu^2}{m_R^2}\right)\right]\left(\frac{i}{p^2 - m_R^2 + i\epsilon}\right)^2$$

- $\overline{\mathrm{MS}}$: Cancel out the inifinite part and the 4π's and γ_E. Thus

$$\delta_Z = 0, \quad \delta_m = \frac{m_R^2 \lambda_R}{32\pi^2}\left[\frac{2}{\epsilon} + \log\left(4\pi e^{-\gamma_E}\right)\right]$$

In this scheme

$$i\mathcal{M} = \frac{i}{p^2 - m_R^2 + i\epsilon} + \frac{i m_R^2 \lambda_R}{32\pi^2}\left[1 + \log\left(\frac{\mu^2}{m_R^2}\right)\right]\left(\frac{i}{p^2 - m_R^2 + i\epsilon}\right)^2$$

Renormalisation of the two point function

- **On-shell subtraction:** In this scheme, we define δ_m by giving a physical interpretation to the parameter m_R. The usual thing to do would be to say that the **physical mass** of the particle is given by m_R. This is the condition that

$$= \frac{i}{p^2 - m_R^2 + i\epsilon} + \text{terms that are regular at } p^2 = m_R^2$$

with residue i. This implies

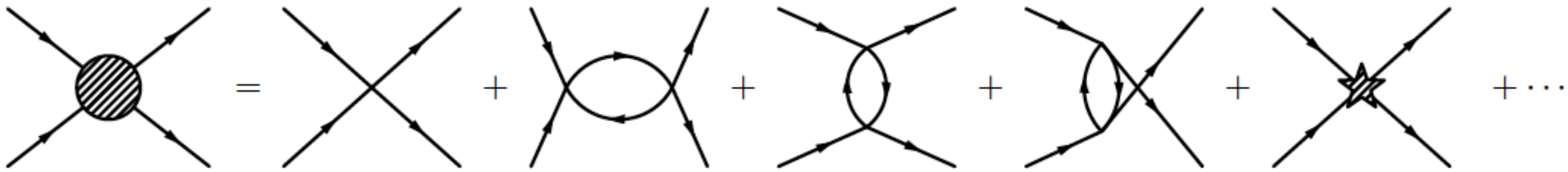
$$\delta_Z = 0, \quad \delta_m = \frac{m_R^2 \lambda_R}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \log\left(\frac{4\pi e^{-\gamma_E}\mu^2}{m_R^2}\right)\right] \implies i\mathcal{M} = \frac{i}{p^2 - m_R^2 + i\epsilon}$$

Notice that now that we have a physical interpretation of the parameter m_R, all dependence on the unphysical parameter has dropped out.

We have seen that in all subtraction schemes $\delta_Z = 0$ at this order. This is not true when higher order corrections are imposed.

Renormalisation of the four point function

The diagrams contributing to the four point function are



and the amplitude is

$$
\begin{aligned}
i\mathcal{M} = -i\lambda_R &+ \frac{1}{2}(-i\lambda_R)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_R^2 + i\epsilon} \frac{i}{(p_1+p_2+k)^2 - m_R^2 + i\epsilon} \\
&+ \frac{1}{2}(-i\lambda_R)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_R^2 + i\epsilon} \frac{i}{(p_1-p_3+k)^2 - m_R^2 + i\epsilon} \\
&+ \frac{1}{2}(-i\lambda_R)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_R^2 + i\epsilon} \frac{i}{(p_1-p_4+k)^2 - m_R^2 + i\epsilon} - i\delta_\lambda \\
= -i\lambda_R &+ (-i\lambda_R)^2 \left[iV(s) + iV(t) + iV(u) \right] - i\delta_\lambda
\end{aligned}
$$

$$
i\mathcal{M} = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2} \int_0^1 dx \left[\frac{6}{\epsilon} + \log\left(\frac{\left(4\pi e^{-\gamma_E}\mu^2\right)^3}{[m_R^2 - x(1-x)s][m_R^2 - x(1-x)t][m_R^2 - x(1-x)u]} \right) \right] - i\delta_\lambda
$$

Renormalisation of the four point function

- **MS**: Only cancel out the finite part. Thus

$$\delta_\lambda = \frac{\lambda_R^2}{32\pi^2}\int_0^1 dx \frac{6}{\epsilon} = \frac{3\lambda_R^2}{16\pi^2\epsilon}$$

In this scheme

$$i\mathcal{M} = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\int_0^1 dx \log\left(\frac{\left(4\pi e^{-\gamma_E}\mu^2\right)^3}{[m_R^2 - x(1-x)s][m_R^2 - x(1-x)t][m_R^2 - x(1-x)u]}\right)$$

- $\overline{\textbf{MS}}$: Cancel out the inifinite part and the 4π's and γ_E. Thus

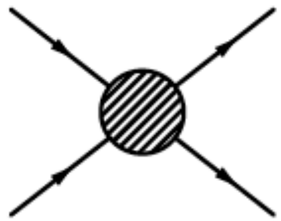
$$\delta_\lambda = \frac{\lambda_R^2}{32\pi^2}\int_0^1 dx \left[\frac{6}{\epsilon} + 3\log\left(4\pi e^{-\gamma_E}\right)\right] = \frac{\lambda_R^2}{32\pi^2}\left[\frac{6}{\epsilon} + 3\log\left(4\pi e^{-\gamma_E}\right)\right]$$

In this scheme

$$i\mathcal{M} = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\int_0^1 dx \log\left(\frac{\left(\mu^2\right)^3}{[m_R^2 - x(1-x)s][m_R^2 - x(1-x)t][m_R^2 - x(1-x)u]}\right)$$

Renormalisation of the four point function

- **On-shell subtraction**: In this scheme, we choose δ_λ such that the renormalized parameters are themselves the observable quantities. To define the observable quantity, we must define a **Renormalization Condition**. The usual thing to do is to perform an experiment at some scale $s = s_0,\ t = t_0,\ u = u_0$, and define the measure amplitude at that scale as the value of the coupling.



$$\xrightarrow{s=4m_P^2,\ t=u=0} -i\lambda_R$$

We then get the equation

$$-i\lambda_R = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\int_0^1 dx \left[\frac{6}{\epsilon} + \log\left(\frac{\left(4\pi e^{-\gamma_E}\mu^2\right)^3}{m_R^4[m_R^2 - 4x(1-x)m_P^2]}\right)\right] - i\delta_\lambda$$

$$\implies \delta_\lambda = \frac{\lambda_R^2}{32\pi^2}\int_0^1 dx \left[\frac{6}{\epsilon} + \log\left(\frac{\left(4\pi e^{-\gamma_E}\mu^2\right)^3}{m_R^4[m_R^2 - 4x(1-x)m_P^2]}\right)\right]$$

In this scheme,

$$i\mathcal{M} = -i\lambda_R + \frac{i\lambda_R^2}{32\pi^2}\int_0^1 dx \log\left(\frac{m_R^4[m_R^2 - 4x(1-x)m_P^2]}{[m_R^2 - x(1-x)s][m_R^2 - x(1-x)t][m_R^2 - x(1-x)u]}\right)$$