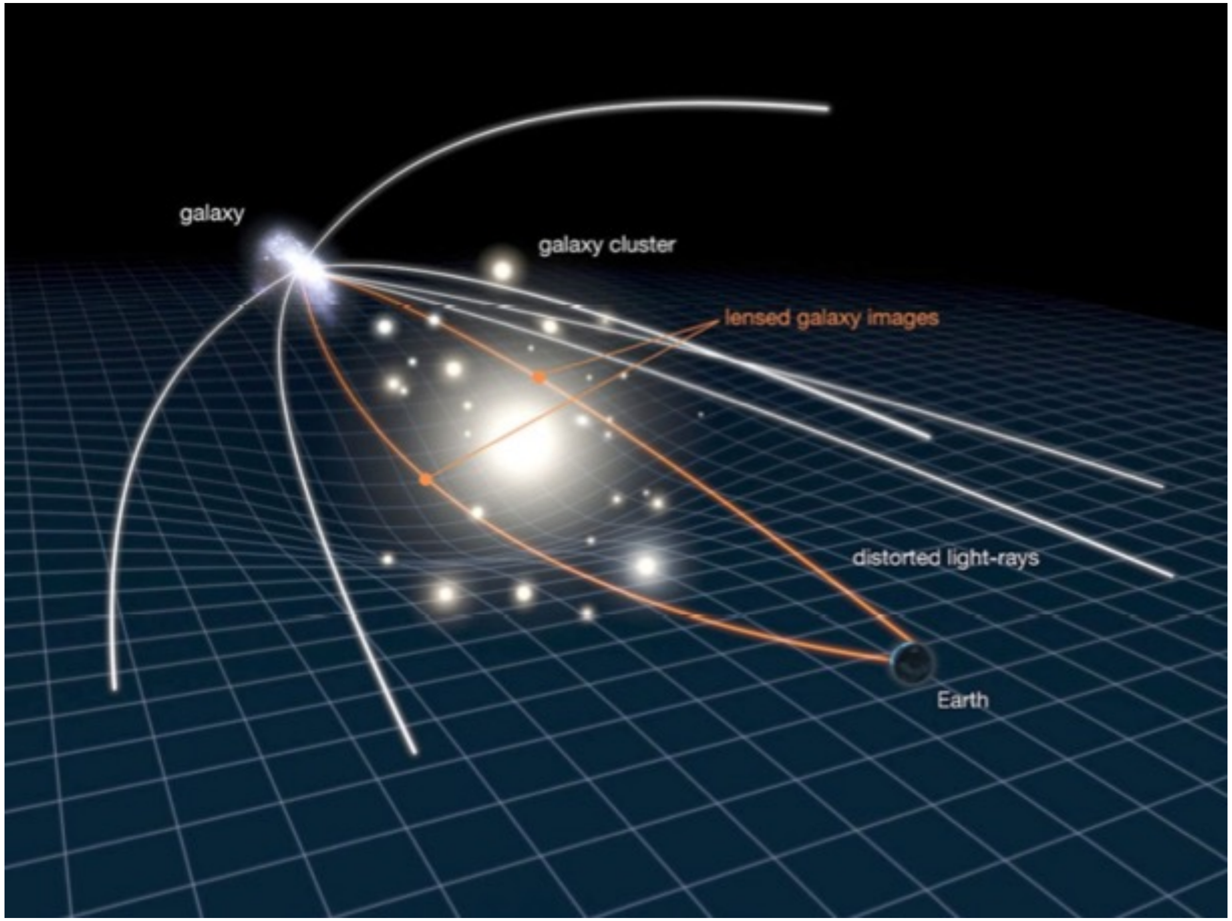
Cosmological Observations

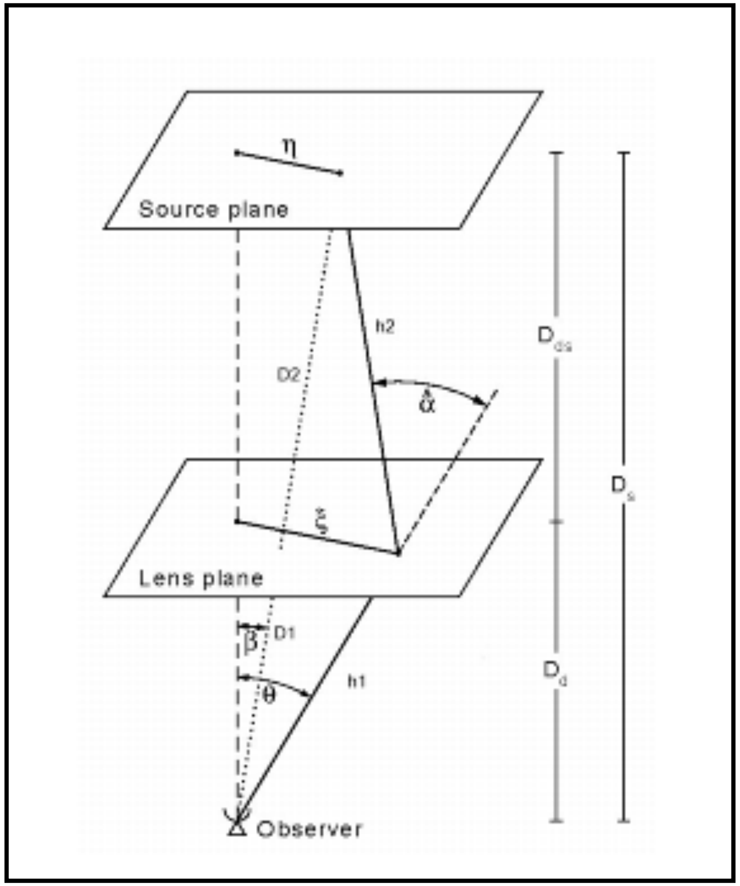
Gravitational lensing

Deflection of light

The basis of gravitational lensing is the effect of deflection of light caused by gravity.



In general, we define a **source - lens - observer system**



source position in the source plane

deflection angle

impact parameter in the lens plane

image position in the image plane

optical axis

Light from a point emitted at an angular position β is observed at a different angular position θ.

It is deflected by a deflection vector α induced by gravity.

The lens equation, relating source and lens planes can be found from the diagram above, by using simple trigonometry (vector addition on the source plane):

$$D_s\vec{\theta} = D_s\vec{\beta} + 2D_{ds}\frac{\hat{\vec{\alpha}}}{2}$$

α is determined by the properties of the lens : it contains the physical (gravitational field) information we want to find out.

Measuring the change between θ and β we can find α if we know the distances (there is a degeneracy with the distance).

How does the deflection angle relate to the lens gravitational potential?

Let us consider **light propagation from source to observer** in the Universe described by the Robertson-Walker metric (here written in conformal time) with a small inhomogeneity representing the lensing potential:

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \left[dx_1^2 + dx_2^2 + dx_3^2\right]$$

The deflection may be derived using the principle of Fermat**: light follows a path of extremal time.**

Light follows null geodesics, and setting $ds^2 = 0$ we can immediately write the speed of light when travelling in the gravitational field of the lens.

It is:

$$c' = \frac{|\mathrm{d}\vec{x}|}{\mathrm{d}t} = c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \approx c\left(1 + \frac{2\Phi}{c^2}\right)$$

We can think of the gravitational field as a “change of medium” since it effectively changes the speed of light propagation.

This medium is thus associated to an effective index of refraction, given by:

$$n = c/c' = \frac{1}{1 + \frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$

In terms of properties of light propagation, the perturbed metric is like a medium where the speed of light is v < c → it bends the light with respect to the homogeneous spacetime where v = c.

Now, let x(l) be a light path crossing the medium.

The light travel time is then proportional to:
(since the refraction index is basically dt/dx)

$$\int_A^B n[\vec{x}(l)]\mathrm{d}l$$

and we want to find the path of extremal (minimum) time, i.e.,

$$\delta \int_A^B n[\vec{x}(l)]\mathrm{d}l = 0$$

This is a standard **variational problem**, that as we know will lead to the Euler-Lagrange equations.

The extremal light path verifies:

$$\delta \int_A^B n(\vec{x})dx = 0 = \delta \int_{\lambda_A}^{\lambda_B} n(\vec{x}(\lambda))\frac{dx}{d\lambda}d\lambda = \delta \int_{\lambda_A}^{\lambda_B} n(\vec{x}(\lambda))|\dot{\vec{x}}|d\lambda = \delta \int_{\lambda_A}^{\lambda_B} L(x, \dot{x}; \lambda)d\lambda,$$

where λ is an arbitrary affine parameter, labeling the positions along the path,

and we found out that $n[\vec{x}(\lambda)]\left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda}\right|$ has the role of a Lagrangian.

Having found the Lagrangian we can now describe the light path using the Euler-Lagrange equations:

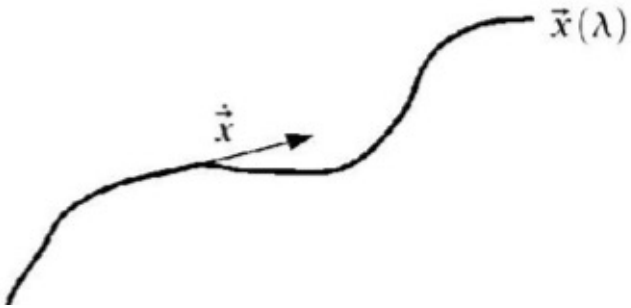
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{\vec{x}}} - \frac{\partial L}{\partial \vec{x}} = 0$$

From our Lagrangian, we compute:

$$\frac{\partial L}{\partial \dot{\vec{x}}} = n\frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$$

$$\frac{\partial L}{\partial \vec{x}} = |\dot{\vec{x}}|\frac{\partial n}{\partial \vec{x}} = (\vec{\nabla} n)|\dot{\vec{x}}|$$

This means that the Euler-Lagrange equation is an equation for the evolution of $\dot{\vec{x}}$, which is a vector tangent to the light path.



$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = \qquad \frac{d}{d\lambda}(n(\vec{x})\vec{u}_x) - \vec{\nabla} n|\dot{\vec{x}}| = n\dot{\vec{u}}_x + (\vec{\nabla} n.\vec{u}_x)\vec{u}_x - \vec{\nabla} n|\dot{\vec{x}}|$$

(u is the normalised vector tangent to the path) and so, the Euler-Lagrange equation is:

$$n\dot{\vec{u}}_x + (\vec{\nabla} n.\vec{u}_x)\vec{u}_x - \vec{\nabla} n. |\dot{\vec{x}}| = 0$$

$$\Leftrightarrow \quad \dot{\vec{u}}_x = \frac{1}{n(\vec{x})}\left(\vec{\nabla} n - (\vec{\nabla} n.\vec{u}_x)\vec{u}_x\right) |\dot{\vec{x}}|$$

this is the gradient of n perpendicular to the light path

$$\Leftrightarrow \quad \dot{\vec{u}}_x = \frac{1}{n(\vec{x})}\vec{\nabla}_\perp n(\vec{x}) = (1 + \frac{2\Phi}{c^2})(-\frac{2}{c^2}\vec{\nabla}_\perp \Phi) \approx -\frac{2}{c^2}\vec{\nabla}_\perp \Phi$$

and therefore, the gradient of the potential.

Now, the derivative of the tangent vector is by definition the deflection.

So **we found that the deflection is the gradient of the lens potential in the plane orthogonal to the tangent to the path (i.e. on the lens plane).**

Notice the minus sign, meaning the gradient of the potential points away from the lens centre and the deflection angle points toward the lens (light is pulled towards the lens).

The potential changes from point to point along the light path, so the **total deflection** is the integral over the "pull" of the gravitational potential perpendicular to the light path:

$$\vec{\alpha} = -\frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \dot{\vec{u}}_x \, d\lambda = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \, d\lambda,$$

Note that:

- The integral should be made over the actual light path

(a priori unknown before computing the deflection → so it is a recursive problem).

However, given the smallness of the potential $\Phi/c^2 \ll 1$, the deflection angle is usually small and in practice we can integrate over the unperturbed light path. (This is called the Born approximation, also used in scattering theory).

- Since the speed of light is effectively slowed down in the gravitational field, the travel time to cross a given length is larger than it would be in the absence of a lens. This is called the Shapiro delay.

$$\Delta t = \int_C \frac{n(\vec{x})}{c'(\vec{x})}\, dl - \int_C \frac{n(\vec{x})}{c}\, dl = \frac{1}{c}\int_C [n(\vec{x}) - 1]\; dl = -\frac{2}{c^3}\int_C \Phi\, dl$$

- The deflection angle can also be computed for **Newtonian gravity**

This can be done in two ways

- Classical approach - A particle emitted with velocity v=c at infinity, follows a hyperbolic trajectory (corpuscular theory of light) - result derived by von Soldner in 1801

- Special relatvity approach - gravitational field equivalent to acceleration of the reference system (equivalence principle)

Minkowski space-tme in an accelerated frame

$$ds^2 = -\left(1 + \frac{2\Psi}{c^2}\right) c^2 dt^2 + dx^2$$

Doing the same derivation with the refraction index, the result is different by a factor of 2:

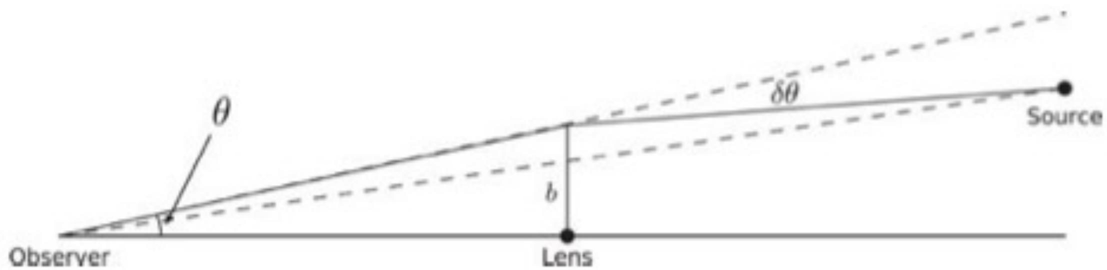
$$\vec{\alpha} = -\frac{1}{c^2} \int_{\lambda_A}^{\lambda_B} \dot{\vec{u}}_x \, d\lambda = \frac{1}{c^2} \int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \, d\lambda,$$

Having found the relation between deflection angle and gravitational potential, we can compute the deflection of the light emitted by a **point source** when passing by a lens.

Let us consider a point mass lens, with potential

$$\Phi = -\frac{GM}{r}$$

Light from the source travels along the z-axis towards the observer and crosses the lens plane (i.e., the plane x,y orthogonal to the z-axis), at a distance b from the point mass. b is called the impact parameter.



$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{b^2 + z^2},\ b = \sqrt{x^2 + y^2}$$

The potential on the lens plane is

$$\vec{\nabla}_\perp \phi = \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \end{pmatrix} = \frac{GM}{r^3} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\begin{pmatrix} x \\ y \end{pmatrix} = b \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$

and the resulting deflection vector is:

$$\begin{aligned} \hat{\vec{\alpha}}(b) &= \frac{2GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{(b^2+z^2)^{3/2}} \\ &= \frac{4GM}{c^2} \begin{pmatrix} x \\ y \end{pmatrix} \left[\frac{z}{b^2(b^2+z^2)^{1/2}} \right]_0^\infty = \frac{4GM}{c^2 b} \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \end{aligned}$$

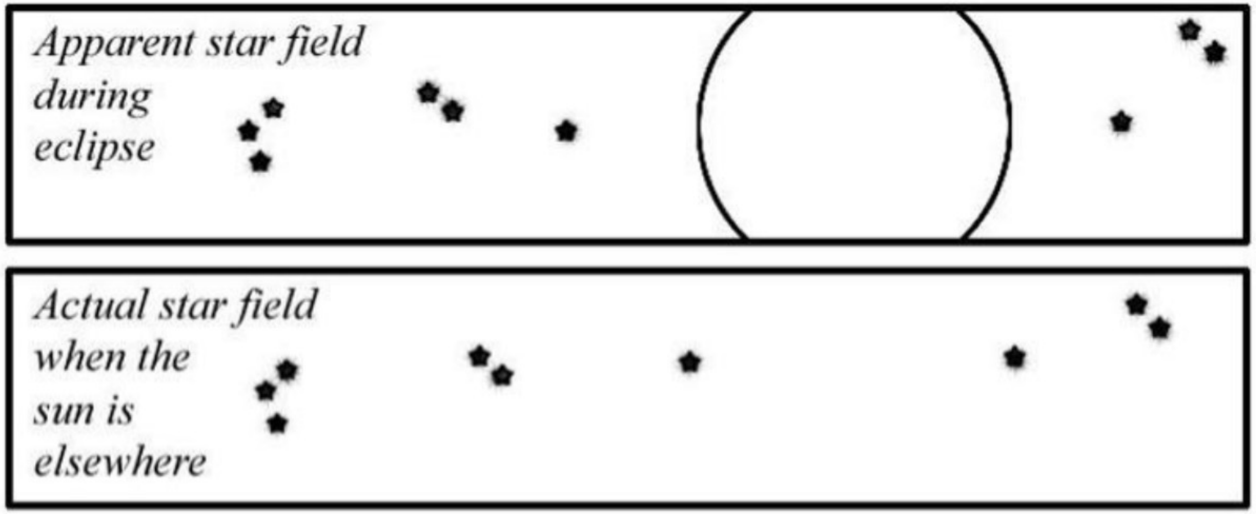
From the x and y components of the deflection angle vector, we compute its norm, which is the well-known **result**:

$$|\hat{\vec{\alpha}}| = \frac{4GM}{c^2 b}$$

Application 1: Deflection angle

Given the mass of the lens M, and an impact parameter b
→ determine the deflection angle

This was used to test GR in 1919 (Eddington eclipse expedition)



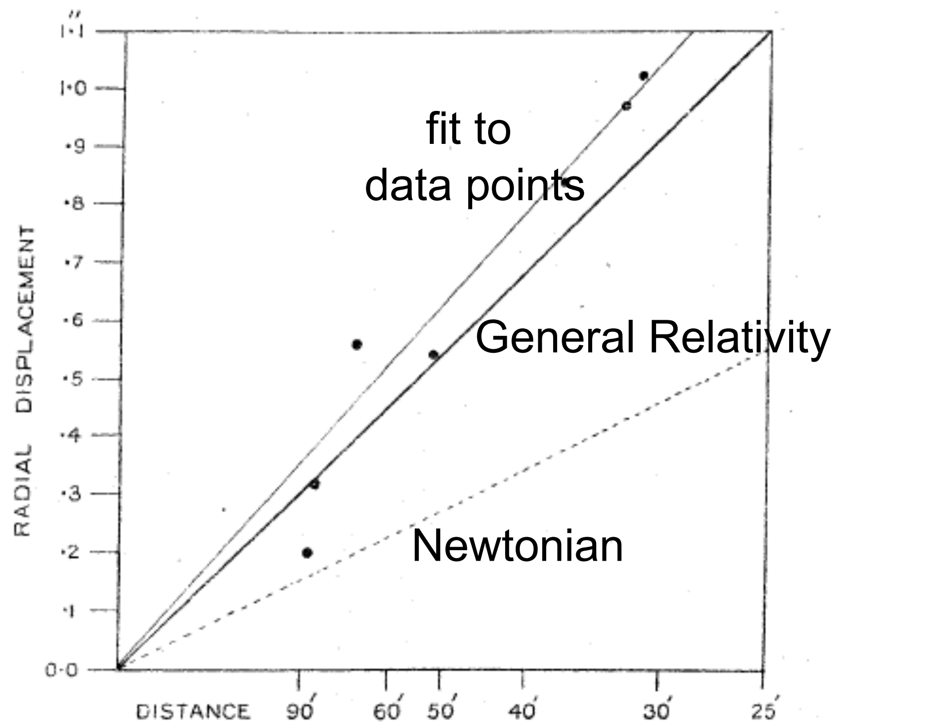
For b such that the star is at the sun's limb → the deflection is 1.75 arcsec (GR)

For b such that the star is at the sun's limb → the deflection is 0.87 arcsec (Newton)

Why this factor of 2?

because **acceleration** (which is equivalent to gravity in Newtonian thory) **induces only a change in time**

while **gravity** (as described by GR) **induces a change in time and in space** (it is the curvature of space-time)



data points are stars at various values of b

LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

Application 2: Mass of the lens

The source emits light in all directions, and various light paths reach the lens plane. But only one is deflected towards the observer.

From the lens equation (from the source-lens-observer diagram), we can see it is the one that passes at $\mathbf{b = D_d\, D_{ds} / D_s}$

D_d = distance from observer to lens (deflector)
D_{ds} = distance from lens to source
D_s = distance from observer to source

For this reason, **all lensing systems have a fundamental degeneracy between distances and lens properties**.

We can only compute the mass of the lens if we know the distances involved in the system.

Conversely, lensing can be used as a geometric probe of the Universe (i.e., it can be used to measure cosmological distance and use them to infer the density parameters) if the mass of the lens is known.

Let us consider that the lens is not a point mass, but it is an extended object (extended lens)

Since the deflection angle depends linearly on the mass M, the effect from a finite lens in a plane is just the sum of the deflection angles created from all points in the lens. If we discretize the lens as a set of N point lenses of masses M_i at positions ξ_i on the lens plane, then the deflection angle of a light ray crossing the plane at ξ will be:

$$\hat{\vec{\alpha}}(\vec{\xi}) = \sum_i \hat{\vec{\alpha}}_i(\vec{\xi} - \vec{\xi}_i) = \frac{4G}{c^2} \sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2}$$

We can also consider a lens in 3D with mass density ρ. The z extension of the lens is always just a small segment of the full source-observer light path, and it can be considered that it is in a plane - the thin-screen approximation. In this approximation, the lensing matter distribution is completely described by its **surface mass density**:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\, \mathrm{d}z$$

and the total deflection is given by:

$$\vec{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int \frac{(\vec{\xi} - \vec{\xi'})\Sigma(\vec{\xi'})}{|\vec{\xi} - \vec{\xi'}|^2}\, \mathrm{d}^2\xi'$$

Gravitational Lensing

Gravitational lensing, in a strict sense, refers to the case of **extended sources**, which give rise to differential effects.

Indeed, neighbouring points in the source suffer slightly different deflections in the lens plane: it is a differential effect that makes **the image of an extended source** (i.e. non point-like) **to become distorted**.

This is easily seen if we Taylor-expand the lens equation. Remember the lens equation is a mapping from image positions to source positions (it is usually written in that order, and not as a mapping from source to image). So a given point θ in the image plane corresponds to an original position β(θ) in the source plane, related by the deflection angle:

$$\vec{\beta}(\theta) = \vec{\theta} - \vec{\alpha}$$

(here the vectors have absorbed the distance factors present in the original lens equation) $\vec{\alpha}(\vec{\theta}) \equiv \frac{D_{\rm LS}}{D_{\rm S}} \hat{\vec{\alpha}}(\vec{\theta})$

The Taylor expansion of β(θ) to linear order is $\beta(\theta) = \beta(\theta_0) + A(\theta_0).(\theta - \theta_0)$

where A is the amplification matrix (the Jacobian) and describes the **lensing transformation between source and image planes** to first order:

$$A_{ij}(\theta) = \frac{\partial \beta_i}{\partial \theta_j} = \left(\delta_{ij} - \frac{\partial \alpha_i}{\partial \theta_j} \right)$$

it is a 2D matrix, since β (position in the source plane and θ (position in the lens plane) are 2D vectors.

Now, remember that a general matrix can be decomposed in 3 parts:

(traceless) symmetric + (traceless) antisymmetric + diagonal

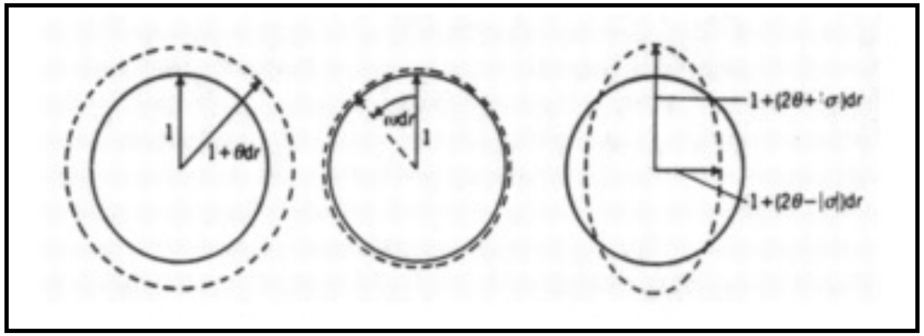
$$\begin{bmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

Applying a diagonal matrix to an image will expand it (or contract it) radially in an isotropic way → k is called **convergence.**

Applying an antisymmetric matrix to an image will rotate it → ω is called **rotation.**

Applying an symmetric matrix to an image will distort it in an anisotropic way, contracting in one dimension and expanding in the other → γ is called **shear.**

This means that any linear distortion of an image is a combination of convergence/expansion, rotation and shear



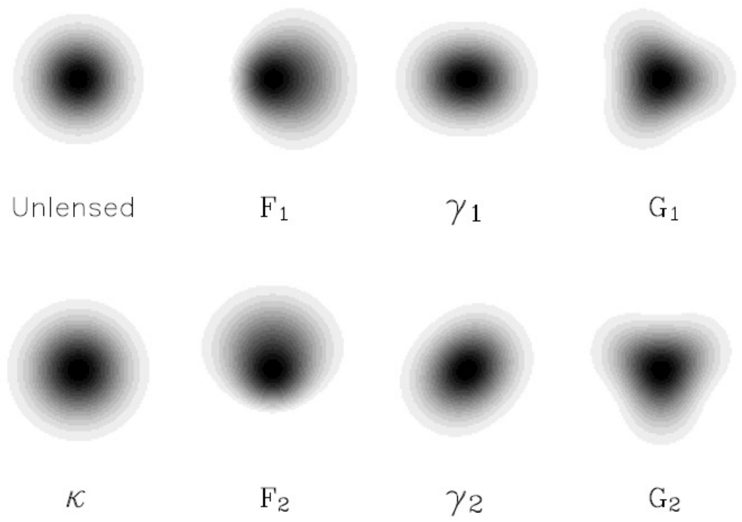
The amplification matrix is then written as

$$A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

Note that usually actual lensing distortions do not include rotation because the gravitational field is a gradient field (completely defined by a potential), and so, its rotational is zero (it is a so-called E field) and the deflection vector field does not produce rotations.

The presence of rotations in a lensed image (due to so-called B-modes) is an indication of systematic effects, i.e., distortion effects with non-lensing origin.

The distortions applied to a circular image result in:



isotropic distortion (k, convergence) → a circle expands/contracts (full rotational symmetry)

anisotropic distortion (γ, shear) → a circle transforms into a π-rotational symmetric shape (an ellipse)

second-order distortions (by continuing the Taylor expansion) (F, G, flexion) → a circle transforms into a 120º-rotational symmetric shape (a banana-shape F or a “Mercedes logo” G)

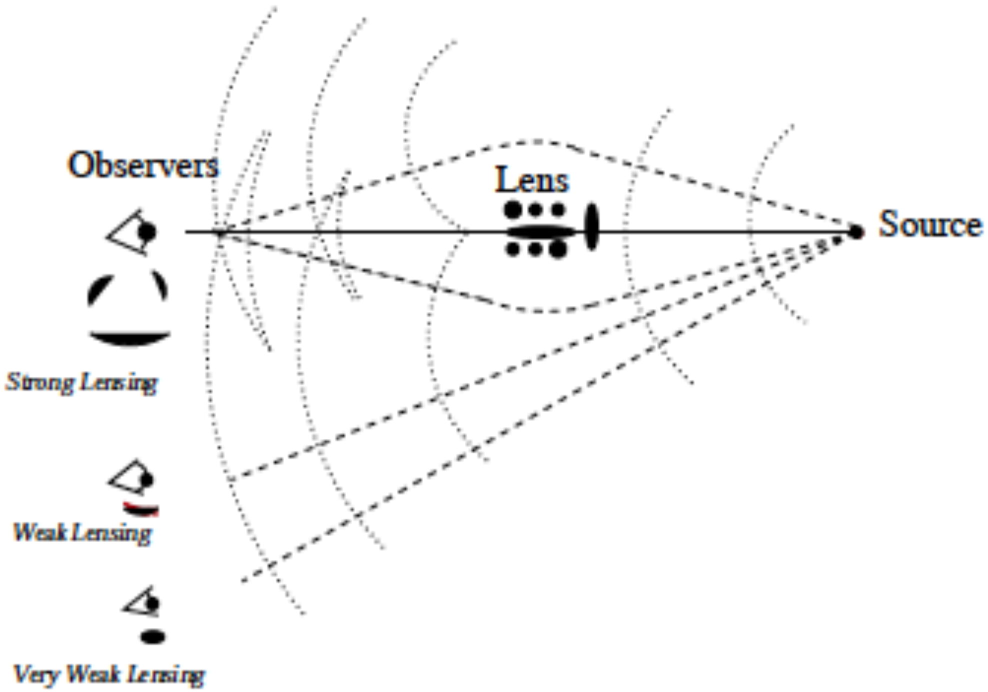
These are the fundamental distortions (also called the optical scalars) and contain the dependence on α → which contains the information on gravity

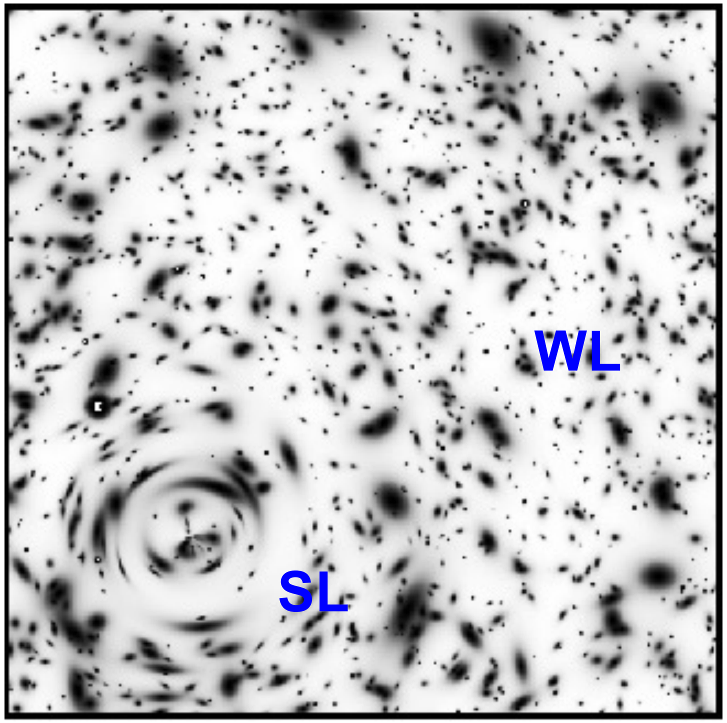
The determinant of the amplification matrix defines the magnification:

$$\mu = \frac{1}{det\, A} = \frac{1}{(1-\kappa)^2 - \gamma^2}$$

The magnification, and the amplitude of the optical scalars - which are fields in the 2D sky - define the **gravitational lensing regime** that occur in the positions of the sky.

There are two regimes - **weak lensing** and **strong lensing** - that occur in regions of the image plane where the values of the $k(\theta)$ and γ (θ) fields are small (<<1) (weak lensing) or large (strong lensing).



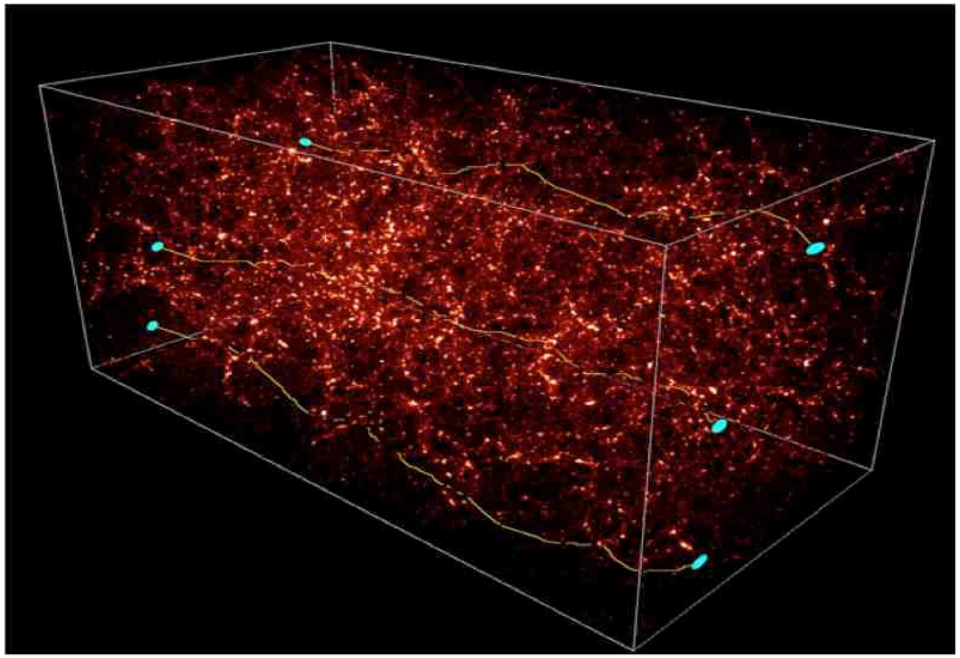


The observable effects are very different in the two regimes.

Weak Lensing occurs at larger separations from the source-lens-observer line (the **line-of-sight**), or with lenses of low density contrast.

The effects are: small increase of ellipticity of the source galaxy (shear), alignment of images.

Weak lensing is a very useful probe in a cosmological system where the lens is the large-scale structure of dark matter distribution. In this case the shear is so small that it cannot be detected in individual galaxies. What can be detected is a correlation of those ellipticities because their orientations get some degree of alignment and cease being randomly oriented → this effect is used to probe the structure formation of the Universe.



Increased ellipticities: weak lensing of galaxies by the large-scale structure of the Universe

Strong Lensing occurs near the line of sight, with lenses of high density contrast.

The effects are: very strong distortions (giant arcs), multiple images, flux magnification. They occur near lines where det A = 0 (infinite magnification), which are called critical lines of the image plane (the observed sky), and map back to the source plane to lines known as caustic lines.

Example:
Spherical lens

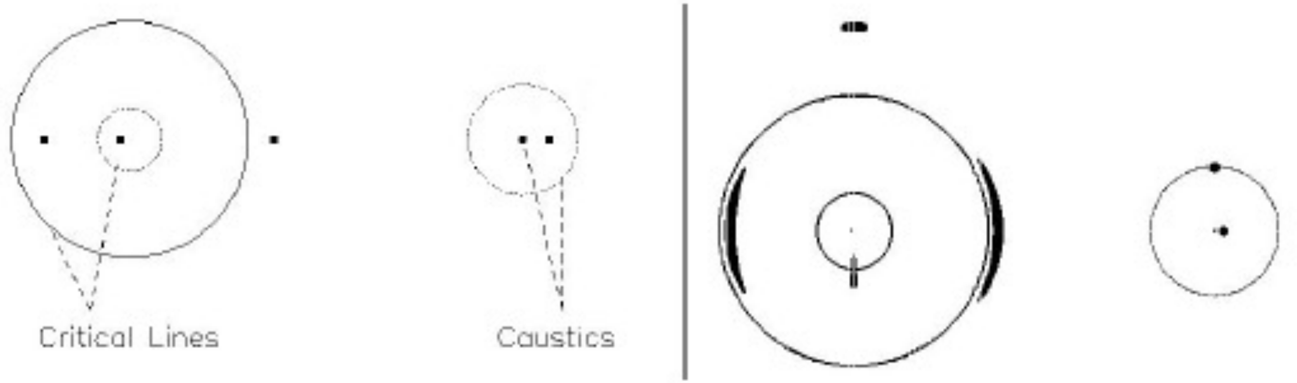


image — point source — image — extended source

Example:
Elliptical lens

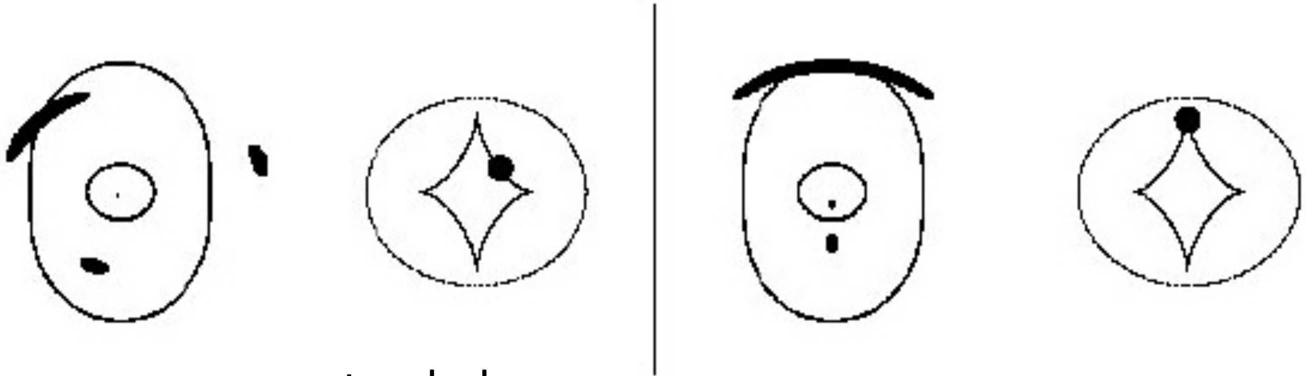
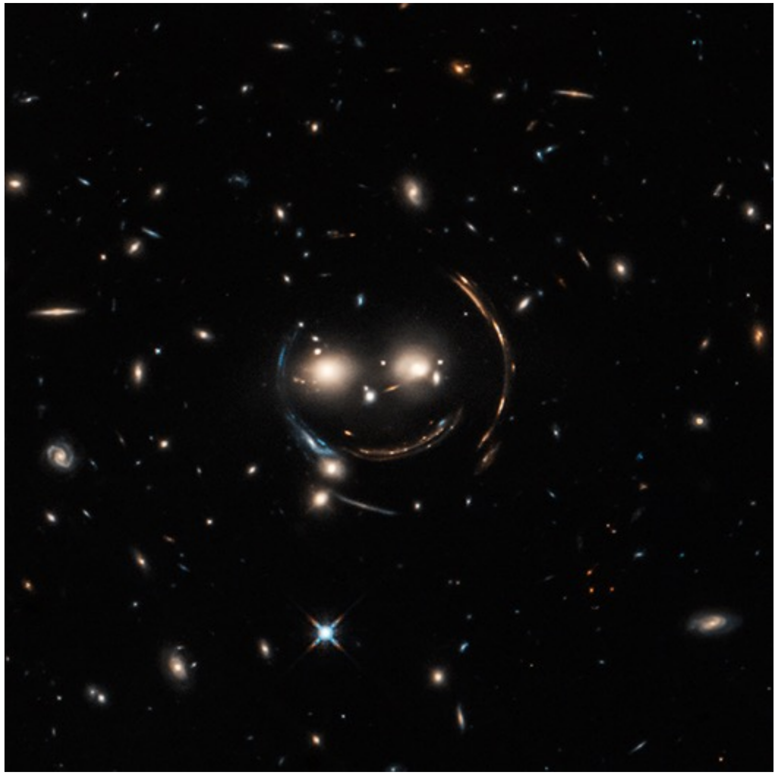


image — extended source — image — extended source

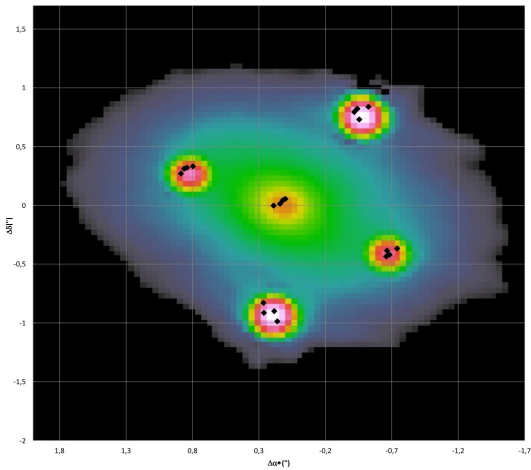
Actual observations of strong lensing:

Giant Arcs: Strong lensing of galaxies by a cluster



Giant Arcs: Strong lensing and Einstein ring of galaxies by a group that includes two massive ellipticals (The Cheshire Cat)

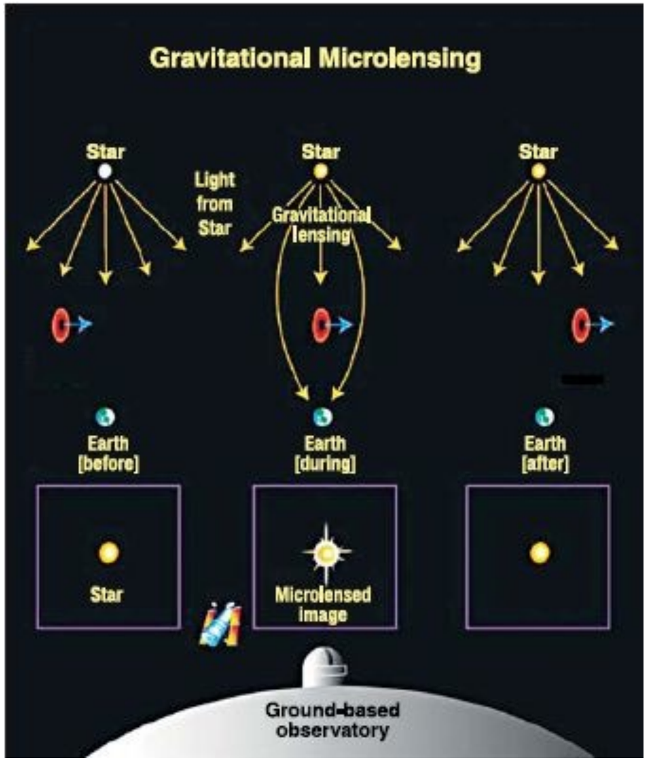
Einstein ring: Strong lensing of a galaxy by a galaxy, an infinite number of multiple images forms on a circle

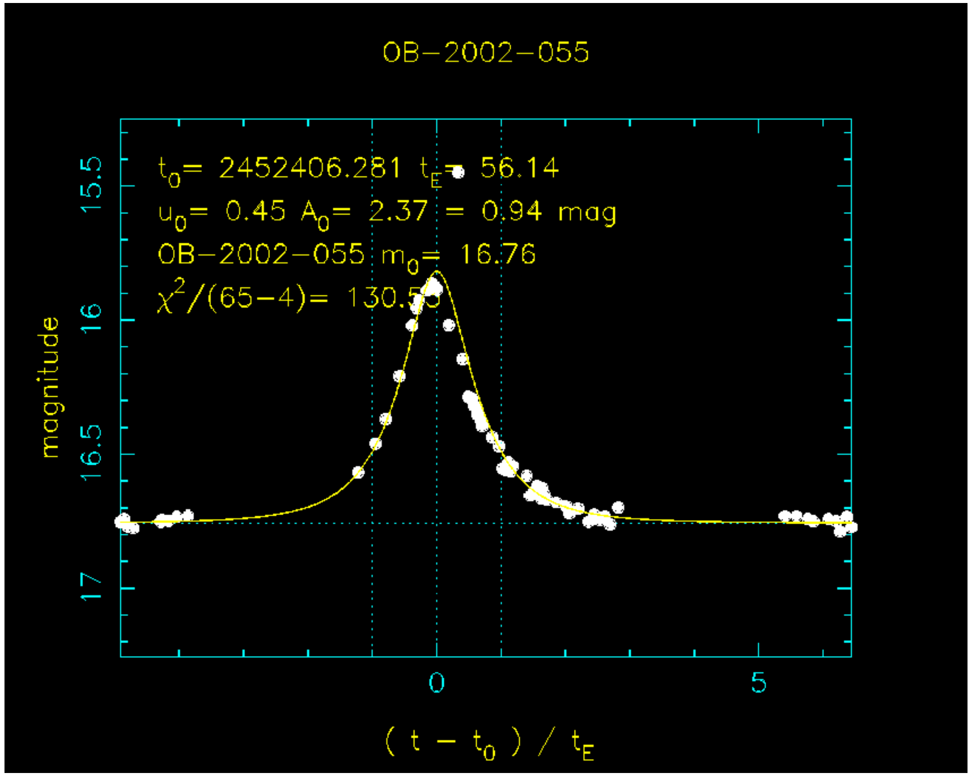


Einstein cross: Strong lensing of a quasar by a galaxy, forming a quadruple image of the quasar

When the angular scale of the strong lensing effects is small (ex: multiple images have small angular separation and are not resolved):

the strong lensing is called **microlensing**.





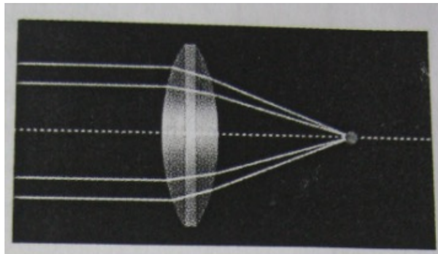
Increase of flux: → Microlensing of a star by a planet (used to detect exoplanets).

In summary, gravitational lensing has a number of fundamental properties:

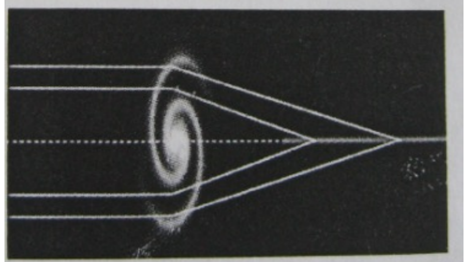
- it depends on the projected 2d mass density distribution of the lens
- it is independent of the luminosity of the lens
- it does not have a focal point
- it is achromatic, there is no frequency shift from source to image
- it involves no emission or absorption of photons
- it conserves the surface brightness

that lead to a number of observable features:

- change of apparent **positions**
- magnification (increase of size), which combined to the conservation of brightness implies an increase of **flux** → natural telescope
- **distortion** of extended sources (ellipticities, tangential giant arcs, radial arclets)
- **multiple images**
- **time-delay** between multiple images



CONVEX GLASS LENS
Light near the edge of a glass lens is deflected more than light near the optical axis. Thus, the lens focuses parallel light rays onto a point.



GRAVITATIONAL LENS
Light near the edge of a gravitational lens is deflected less than light near the center. Thus, the lens focuses light onto a line rather than a point.

These observables (positions, fluxes, distortions) can be used **to estimate the total mass and mass distribution of the lens**. For example:

- in (strong or weak) cluster lensing → mass distribution of the cluster
- in LSS weak lensing (cosmic shear) → dark matter power spectrum

In all systems, the general recipe to estimate the physical properties (or cosmological parameters) is:

i) (theoretical) ***define a lens model and derive its gravitational potential.***

For example the potential of a mass distribution, or the potential of a cosmological model

ii) (theoretical) ***derive the deflection and optical scalar fields from the gravitational potential***

From the definitions in the amplification matrix, we see that shear and convergence are derivatives of the deflection field, and second-order derivatives of the potential:

shear $\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \quad , \quad \gamma_2 = \psi_{,12} \, .$

convergence $\kappa = \frac{1}{2}(\psi_{,11} + \psi_{,22}) .$

where ψ is the gravitational potential projected on the lens plane (i.e. integrated along z) and dimensionless (with the distance factors included), i.e.,

$$\Psi = \frac{D_{\mathsf{L}}^2}{\xi_0^2} \frac{D_{\mathsf{LS}}}{D_{\mathsf{L}} D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(D_{\mathsf{L}}\vec{\theta}, z) \mathrm{d}z$$

this is called the **lensing potential**.

Note that indeed:

$$\begin{aligned} \vec{\nabla}_x \Psi(\vec{x}) &= \xi_0 \vec{\nabla}_\perp \left(\frac{D_{\mathsf{LS}} D_{\mathsf{L}}}{\xi_0^2 D_{\mathsf{S}}} \frac{2}{c^2} \int \Phi(\vec{x}, z) \mathrm{d}z \right) \\ &= \frac{D_{\mathsf{LS}} D_{\mathsf{L}}}{\xi_0 D_{\mathsf{S}}} \frac{2}{c^2} \int \vec{\nabla}_\perp \Phi(\vec{x}, z) \mathrm{d}z \\ &= \vec{\alpha}(\vec{x}) \end{aligned}$$

Note also that the convergence is the Laplacian of the lensing potential. This implies, from Poisson equation, that the **convergence is a (projected) mas**s. In particular, it is the (dimensionless) surface density:

$$\kappa(\vec{x}) \equiv \frac{\Sigma(\vec{x})}{\Sigma_{\rm cr}} \quad \text{with} \quad \Sigma_{\rm cr} = \frac{c^2}{4\pi G} \frac{D_{\rm S}}{D_{\rm L} D_{\rm LS}}$$

iii) (theoretical) ***predict the observables from the optical scalars fields*** (shear, image positions, fluxes)

iv) (observational) ***measure the observables in astrophysical images***

v) (statistical***) estimate the lens model parameters by fitting the theoretical predictions to the data***

Example: estimate the mass of a galaxy cluster (lens)

We need to build a complex model that takes into account different components of mass distribution: dark matter halo, gas, galaxy distribution,

and need to define a spatial distribution of background galaxies (sources)

and then predict the distortions, positions and fluxes on the image plane of source background galaxies.

Let us consider that the cluster only has one matter component: the dark matter halo (a NFW density profile):

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}$$ (with 2 free parameters)

The 2D **surface mass density** can be computed from the 3D density profile, and it is:

$$\Sigma(x) = \frac{2\rho_s r_s}{x^2 - 1} f(x) \ ,$$

with

$$f(x) = \begin{cases} 1 - \frac{2}{\sqrt{x^2-1}} \, \mathrm{arctan}\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ 1 - \frac{2}{\sqrt{1-x^2}} \, \mathrm{arctanh}\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

and so, the **convergence** is

$$\kappa(x) = \frac{\Sigma(\xi_0 x)}{\Sigma_{cr}} = 2\kappa_s \frac{f(x)}{x^2 - 1} \quad \text{with} \quad \kappa_s \equiv \rho_s r_s \Sigma_{\mathrm{cr}}^{-1}$$

from which we can obtain the **mass**,

$$m(x) = 2\int_0^x \kappa(x')x'dx' = 4k_s h(x)$$

with

$$h(x) = \ln\frac{x}{2} + \begin{cases} \frac{2}{\sqrt{x^2-1}} \operatorname{arctan}\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ \frac{2}{\sqrt{1-x^2}} \operatorname{arctanh}\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 1 & (x = 1) \end{cases}$$

We can also compute the **lensing potential**, which is,

$$\Psi(x) = 4\kappa_s g(x)\,,$$

where

$$g(x) = \frac{1}{2}\ln^2\frac{x}{2} + \begin{cases} 2\operatorname{arctan}^2\sqrt{\frac{x-1}{x+1}} & (x > 1) \\ -2\operatorname{arctanh}^2\sqrt{\frac{1-x}{1+x}} & (x < 1) \\ 0 & (x = 1) \end{cases}$$

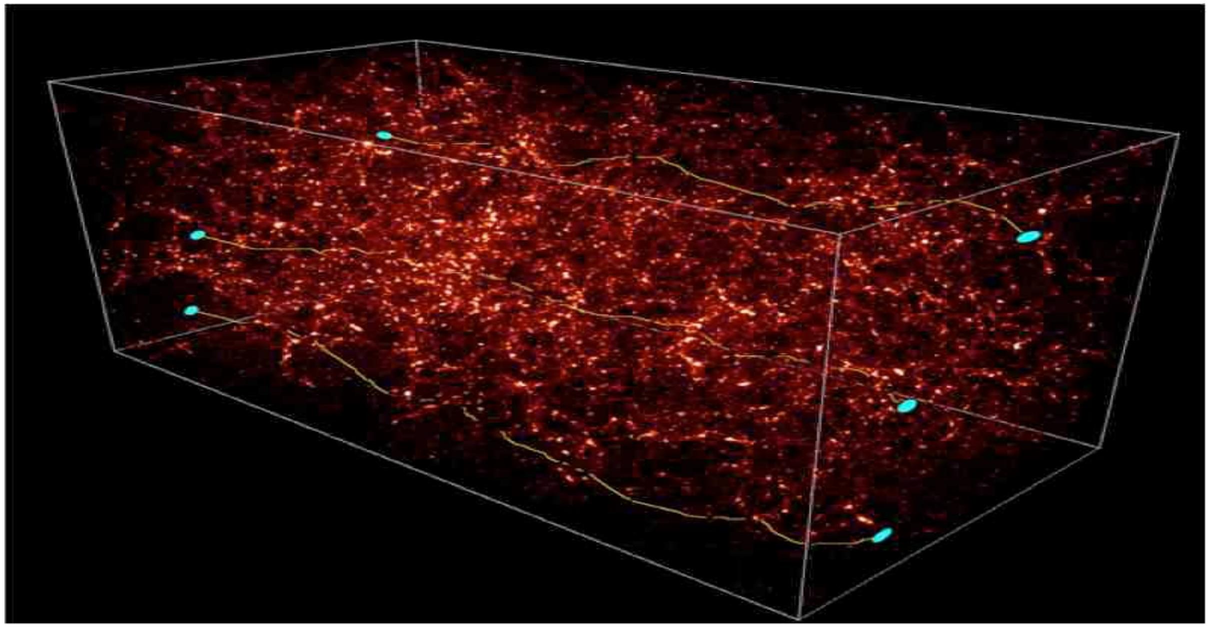
and the **deflection angle**, which is $\alpha(x) = \frac{4\kappa_s}{x}h(x)$

From this, we can predict the image positions of source galaxies, fit to the observed positions and constrain the two parameters r_s and ρ_s needed to determine the value of the cluster mass.

Cosmological weak lensing: theory

We are interested in a specific gravitational lensing system: cosmological weak lensing, i.e.,

the weak gravitational lensing produced by the large-scale structure of dark matter (the lens) in the light emitted by distant galaxies (the sources)



It is a direct tracer of the dark matter distribution

Light propagation in the inhomogeneous universe

We want to derive the lens equation for this system.

For this we need to consider **propagation of light in the inhomogeneous universe**

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \left[dx_1^2 + dx_2^2 + dx_3^2\right]$$

We assume that a bundle of light rays emitted from a galaxy travels through the homogeneous spacetime and is deflected on a series of lens planes where the LSS are placed.

We already derived the deflection using the **principle of Fermat,** and found that **the deflection is the gradient of the lens potential in the plane orthogonal to the tangent to the path.**

$$\vec{\alpha} = -\frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \dot{\vec{u}}_x \, d\lambda = \frac{2}{c^2}\int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \, d\lambda,$$

On the other hand, when travelling through the homogeneous universe there is no deflection → the separation between two light rays of the bundle is just the trivial separation x between two light rays:

$$x_i = \theta_i \, f_K(w) \qquad f_K(w) \text{ is the comoving diameter angular distance,}$$

It is useful to write this simple result as the solution of a differential equation for the evolution of the comoving transverse separation:

$$\frac{d^2 x_i}{dw^2} + K x_i = 0.$$

We can now add the **local deflection solution** (caused by the gravitational potentials) to this equation, to get the complete equation for the **evolution of the comoving transverse separation**
(defined with respect to a reference light ray at x = 0):

$$\frac{d^2 \vec{x}}{dw^2} + K\vec{x} = -\frac{2}{c^2}\left[\vec{\nabla}_\perp \Phi(\vec{x}(\vec{\theta}, w), w) - \vec{\nabla}_\perp \Phi(0, w)\right]$$

note it has a homogenous and an inhomogenous term

Lens equation and the optical scalars

The **lens equation** is the solution of the **differential equation of the evolution of the comoving transverse separation.**

The general solution of an inhomogeneous differential equation is a linear combination of the homogeneous solution and the convolution of the equation Green's function with the inhomogeneous term.

So:

$$f(x) = f^{(0)}(x) + \int dx'\, g(x')\, G(x, x')$$

where the Green's function of our differential equation is $G(w, w') = f_K(w - w')$.

The solution is thus:

$$\vec{x}(\vec{\theta}, w) = f_K(w)\vec{\theta} - \frac{2}{c^2}\int_0^w dw'\, f_K(w - w')\left[\vec{\nabla}_\perp \Phi(\vec{x}(\vec{\theta}, w'), w') - \vec{\nabla}_\perp \Phi(0, w')\right] \Leftrightarrow$$

$$\Leftrightarrow \beta_i(\vec{\theta}, w) = \theta_i - \frac{2}{c^2}\int_0^w dw'\, \frac{f_K(w - w')}{f_K(w)} f_K(w')\left[\Phi_{,i}\left(\vec{x}(\vec{\theta}, w'), w'\right) - \Phi_{,i}\left(0, w'\right)\right].$$

Note that this is essentially a deviation to the usual triangle $x = D_A\,\theta$ (or $x = f_K\,\theta$) (valid for the homogeneous spacetime).

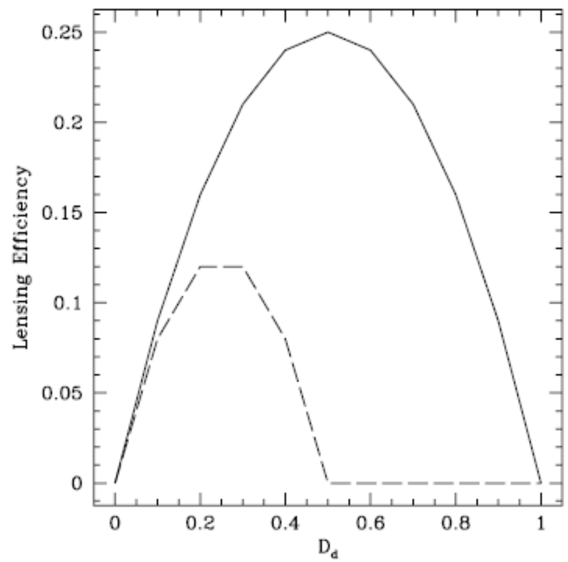
In the language of the lens equation this triangle would be $\beta = \theta$, i.e. a case with zero deflection.

When there are perturbations there is deflection and the 'triangle' changes to $\beta = \theta + \alpha$

The second term of the equation is thus the solution for the deflection as function of the potential.

Note that the total deflection is the integral over all local deflections (the gradients of the various local potentials), each one multiplied by the 'weight' that appeared naturally in the solution (the ratio of distances i.e. of f_K functions) → this is known as the lensing efficiency factor:

$$\frac{D_{ds}D_d}{D_s}$$



This factor shows that **the lens at halfway of the trajectory is the one that contributes the most for the total deflection.**

Also note that the solution is recursive, because the separation x depends on the potential at the position x.

To get rid of this difficulty, we can Taylor-expand the solution around the unperturbed trajectory (the one that lies on the positions $x = f_K \theta$).

$$\Phi_{,i}(x) = \Phi_{,i}(f_K\theta - f_K\alpha(x)) = \Phi_{,i}(f_K\theta) - f_K\alpha(x)\, f_K\Phi_{,ik}(f_K\theta) + \mathcal{O}(\alpha^2)$$

This results in:

$$\beta_i(\vec{\theta}, w) = \theta_i - \frac{2}{c^2}\int_0^w dw'\, \frac{f_K(w-w')}{f_K(w)} f_K(w')\Phi_{,i}(f_K\theta, w') +$$
$$+ \frac{2}{c^2}\int_0^w dw'\, \frac{f_K(w-w')}{f_K(w)} f_K^2(w')\alpha(\vec{x})\Phi_{,ik}(f_K\theta, w') + f(w) + \mathcal{O}(\alpha^2),$$

Born approximation (the same that is done in quantum mechanics scattering)
+
higher-order terms

Keeping only the solution in the Born approximation, we can insert the amplification matrix definition $A_{ij}(\theta) = \frac{\partial \beta_i}{\partial \theta_j}$ to get the optical scalars:

$$A_{ij}(\vec{\theta}, w) = \delta_{ij} - \frac{2}{c^2} \int_0^w dw' \, \frac{f_K(w-w')}{f_K(w)} f_K(w') \Phi_{,ij} \, (f_K \theta, w').$$

$$A_{ij}(\vec{\theta}, w) = \delta_{ij} - \psi_{,ij} \, (\vec{\theta}, w).$$

where we defined the
effective lensing potential:

$$\psi(\vec{\theta}, w) = \frac{2}{c^2} \int_0^w dw' \, \frac{f_K(w-w')}{f_K(w)} f_K(w') \Phi(f_K \theta, w')$$

We recover the result that **the optical scalar fields are second-order derivatives of the potential:**

convergence

$$\kappa = \frac{1}{2}(\psi_{,11} + \psi_{,22}).$$

shear

$$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \quad , \quad \gamma_2 = \psi_{,12} \, .$$

rotation

$$\omega = 0.$$

Lensing produces no **Rotation**. This is a consequence of the fact that a gravitational field is a gradient field (of a potential) → its rotational is zero.

Shear γ has two components, two terms in the optical matrix → it is a polar vector

Convergence k is a scalar and is the Laplacian of the potential → it is related with the mass of the lens through the Poisson equation:

$$\nabla_p^2\Phi = 4\pi G\rho = 4\pi G\bar{\rho}\,\delta \Leftrightarrow \nabla^2\Phi = a^2\,4\pi G\,\Omega_m\rho_c a^{-3}\,\delta = \frac{3H_0^2\,\Omega_m\,\delta}{2a},$$

$$\boxed{\kappa(\vec{\theta},w) = \frac{3}{2}\left(\frac{H_0}{c}\right)^2\Omega_m\int_0^w dw'\,\frac{f_K(w-w')f_K(w')}{f_K(w)\,a(w')}\,\delta(f_K(w')\vec{\theta},w')}$$

We see that the convergence field is a weighted integral of the density contrast field.

This also means that **the power spectrum of the convergence can be related to the power spectrum of dark matter.**

Lensing signal

The convergence and shear amplitudes (i.e. the **lensing signal**) from the cosmological lensing effect over one galaxy are very small.

For example, consider a source galaxy at z_s = 0.8 and a lens at z_l = 0.4 with comoving size 8 Mpc (a cluster). For this system:

$$\kappa \approx \frac{3}{2}\Omega_{\rm m}\left(\frac{H_0}{c}\right)^2 \frac{D_{\rm LS}D_{\rm L}}{D_{\rm S}} \frac{R}{a^2(z_{\rm L})} \frac{\delta\rho}{\rho}$$

Inserting the distances D_L=1120 Mpc, D_S = 1500 Mpc, D_{LS} = 400 Mpc and r_H = 3000 Mpc/h, we get:

k ~ 0.0001

With these redshifts (which are typical of current surveys), a number N = D_S / R of lens planes fit along the trajectory. If a light ray typically crosses D_S / R ~100 planes, the signal increases to

k ~ 0.01

This is a small number, well inside the weak lensing regime.

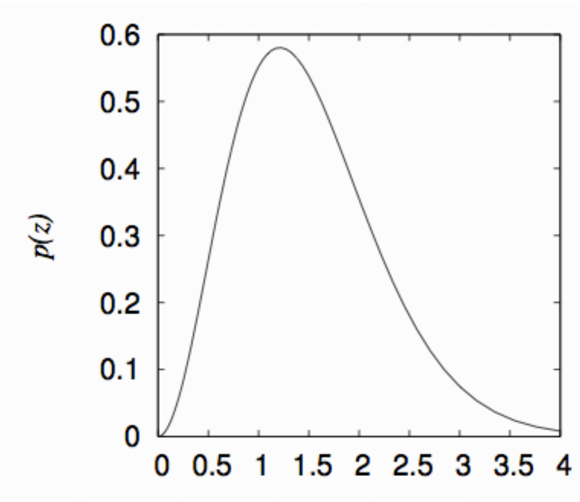
Note that a shear of 0.01 corresponds to the difference in ellipticity between the ellipticities of Uranus and the Moon.

This shows that the cosmological weak lensing signal can only be detected **statistically**, measuring it over a large number of source galaxies.

For this, we **need to consider the convergence from a distribution of sources at various redshifts.**

The signal is integrated over the distribution:

$$\kappa(\vec{\theta}) = \int_0^{w_H} dw\, p(w)\, \kappa(\vec{\theta}, w),$$



with for example,

$$p(z) \propto \left(\frac{z}{z_0}\right)^{\alpha} \exp\left[-\left(\frac{z}{z_0}\right)^{\beta}\right]$$

For a distribution of sources, the convergence can be rewritten as

$$\kappa(\vec{\theta}) = \frac{3}{2}\left(\frac{H_0}{c}\right)^2 \Omega_m \int_0^{w_H} dw' \, \frac{f_K(w')}{a(w')} \, \delta(f_K(w')\vec{\theta}, w') \, g(w').$$

(integral along the line of sight, over the lenses at w')

where

$$g(w') = \int_w^{w_H} dw \, p(w) \, \frac{f_K(w - w')}{f_K(w)}.$$

(integral over the sources at w, for each lens at w')

Note that the optical scalars are perturbed quantities (as we say they do not arise in the homogeneous space-time).

They have zero mean, $< \kappa > = < \gamma > = 0.$

and **the cosmological information is on the moments, i.e., in the correlation function and power spectrum**.

The **power spectrum of the convergence** field is of course related with the power spectrum of dark matter:

$$P_\kappa(\ell) = \frac{9}{4}\left(\frac{H_0}{c}\right)^4 \Omega_m^2 \int_0^{w_H} dw \, \frac{g^2(w)}{a^2(w)} P_\delta\left(\frac{\ell}{f_K(w)}, w\right)$$

The convergence power spectrum is a weighted line-of-sight integral of the matter power spectrum

matter/gravity relation

diameter angular distances

redshift of sources

$$P(k,a) = A\, k^n T_k^2 D_+^2(a)$$

primordial power spectrum (inflation)

transfer function

linear growth

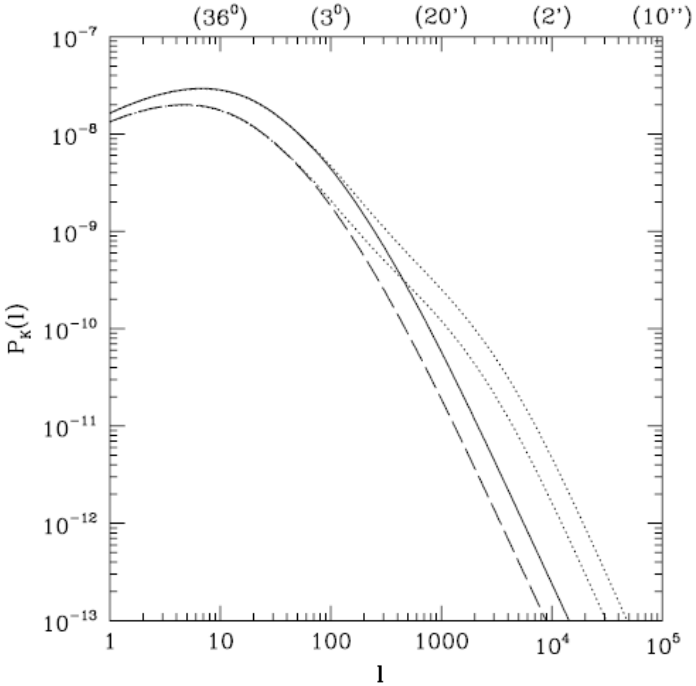
non-linear corrections

cosmological parameters

The convergence power spectrum is a projected power spectrum.

At each angular scale $l = 2\pi/\theta$, its amplitude has contributions from various k scales from the matter power spectrum at different redshifts:

$$\mathbf{k = l / f_K (w (z_{lens}))}$$



Weak lensing cosmic shear surveys measure a lensing signal in the scale range from few arcmin to few degrees. A **typical scale of cosmic shear measurements** is:

θ=30 arcmin → l=1000
if z_s=1 → k=0.8 h/Mpc → r = 8 Mpc/h
(mildly non-linear scales)

Linear and non-linear convergence power spectrum for two different source redshift distributions (higher z_s has higher amplitude) → there is a **strong degeneracy between z_s and σ_8** → this shows it is crucial to know the redshifts in cosmic shear surveys.

We can also derive the **power spectrum of the shear**.

Since shear and convergence are both second-order derivatives of the cosmological lensing potential, their power spectra are related.

The Fourier transform of a function of the form $f(\vec{\theta}) = \psi_{,ij}$ is:

$$\tilde{f}(\vec{\ell}) = \int d^2\theta \, e^{i\vec{\ell}.\vec{\theta}} \frac{d^2}{d\theta_i d\theta_j} \int \frac{d^2\ell'}{(2\pi)^2} \tilde{\psi}(\vec{\ell'}) e^{-i\vec{\ell'}.\vec{\theta}} = \int \frac{d^2\ell'}{(2\pi)^2} (-\ell'_i \ell'_j) \tilde{\psi}(\vec{\ell'}) (2\pi)^2 \delta_D(\vec{\ell} - \vec{\ell'}) = -\ell_i \ell_j \, \tilde{\psi}(\vec{\ell}).$$

and so:

$$\tilde{\kappa} = -\frac{1}{2}\left(\ell_1^2 + \ell_2^2\right)\tilde{\psi} \quad , \quad \tilde{\gamma} = \left[-\frac{1}{2}\left(\ell_1^2 - \ell_2^2\right) - i\ell_1\ell_2\right]\tilde{\psi}.$$

Computing the shear power spectrum:

$$(2\pi)^2 \delta_D(\vec{\ell} - \vec{\ell'}) P_\gamma(\ell) = \langle \tilde{\gamma}(\ell) \tilde{\gamma}^*(\ell') \rangle$$

we get

$$\langle \tilde{\gamma}(\ell) \tilde{\gamma}^*(\ell') \rangle = \frac{(\ell_1^2 - \ell_2^2 + 2i\ell_1\ell_2)(\ell_1^2 - \ell_2^2 - 2i\ell_1\ell_2)}{\ell^4} \langle \tilde{\kappa}(\ell) \tilde{\kappa}^*(\ell') \rangle = \langle \tilde{\kappa}(\ell) \tilde{\kappa}^*(\ell') \rangle$$

i.e., **the shear and the convergence power spectra are identical.**

$$\boxed{P_\gamma(\ell) = P_\kappa(\ell)}$$

We can also derive the **correlation function of the shear**

$$\xi_\gamma(\vartheta) = \int \frac{d^2\ell}{(2\pi)^2} e^{-i\vec{\theta}.\vec{\ell}} \int \frac{d^2\ell'}{(2\pi)^2} e^{i\vec{\theta}'.\vec{\ell}'} \langle \tilde{\gamma}(\ell)\tilde{\gamma}^*(\ell') \rangle = \int \frac{d^2\ell}{(2\pi)^2} e^{-i\ell\,\vartheta\cos(\varphi)} P_\gamma(\ell)$$

Writing the scale vector (l_1, l_2) in polar coordinates, $\vec{\ell} = (\ell_1, \ell_2) = \ell(\cos\varphi, \sin\varphi)$.

d^2 l = dl l dφ, and the angular part dφ can be integrated out, since from isotropy the power spectrum only depends on the modulus of l.

The integral of the angular part of the plane wave is given by a Bessel function:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi\, e^{i(n\varphi - x\sin(\varphi))}$$

Bessel function of the first kind, with order n

After integrating out the angular part, the correlation function is the following radial integral of the power spectrum:

$$\xi_\gamma(\vartheta) = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\ell\vartheta\cos(\varphi)} = 2\pi \int d\ell \, \ell P_\kappa(\ell) \, \frac{J_0(\ell\vartheta)}{(2\pi)^2}.$$

This shows that, as usual, the correlation function is a filtered version of the power spectrum, mixing the power of its scales, depending on the filter function.

We can also define a **power spectrum and correlation function for individual components of shear**:

$$\xi_{11}(\vartheta) = \left\langle \gamma_1(\vec{\theta}) \gamma_1^*(\vec{\theta}') \right\rangle$$

$$\langle \tilde{\gamma}_1(\ell) \tilde{\gamma}_1^*(\ell) \rangle = \left(\frac{\ell_1^2 - \ell_2^2}{\ell^2} \right)^2 \langle \tilde{\kappa}(\ell) \tilde{\kappa}^*(\ell) \rangle = (\cos^2 \varphi - \sin^2 \varphi)^2 \langle \tilde{\kappa}(\ell) \tilde{\kappa}^*(\ell) \rangle$$

and so it relates with the convergence power spectrum as,

$$P_{11} = \cos^2(2\varphi)\, P_\kappa = \frac{1}{2}(1 + \cos 4\varphi)\, P_\kappa$$

The corresponding correlation function is the Fourier transform of P_{11}.

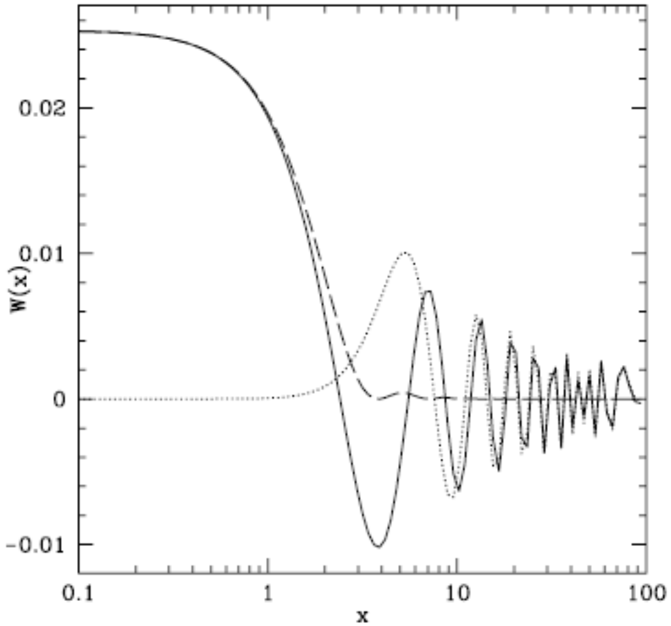
After integrating out the angular part, the 11 correlation function is:

$$\xi_{11}(\vartheta) = \frac{1}{2} \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \; [J_0(\ell\vartheta) + J_4(\ell\vartheta)]$$

Similarly for the other components:

$$\xi_{22}(\vartheta) = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\ell\,\vartheta\cos(\varphi)} \sin^2(2\varphi) = \frac{1}{2} \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \left[J_0(\ell\vartheta) - J_4(\ell\vartheta)\right]$$

$$\xi_{12}(\vartheta) = \frac{1}{2} \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\ell\,\vartheta\cos(\varphi)} \sin(4\varphi) = 0.$$



Solid: filter ξ+ (low-pass band)
Dotted: filter ξ- (narrow-band)

Usually the following linear combinations of shear correlation functions are defined:

$$\xi_+ = \xi_{11} + \xi_{22} = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell)\, J_0(\ell\theta) \;,\; \xi_- = \xi_{11} - \xi_{22} = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell)\, J_4(\ell\theta) \;,\; \xi_\times = \xi_{12} = 0$$

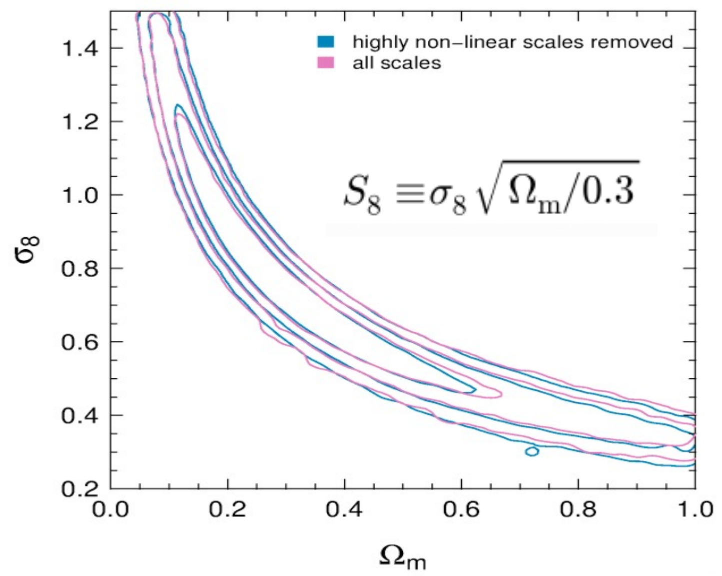
Cosmological content

The cosmological weak lensing power spectra define various filtered versions of the matter power spectrum P_δ

The cosmological weak lensing deflections are produced by LSS gravitational potentials → by the total mass in the structure (which is mostly dark matter) → lensing is sensitive to the total mass, **It is independent of the nature of matter (baryonic or dark) and of its dynamical state (relaxed or merging)**

It is sensitive to the cosmological parameters:

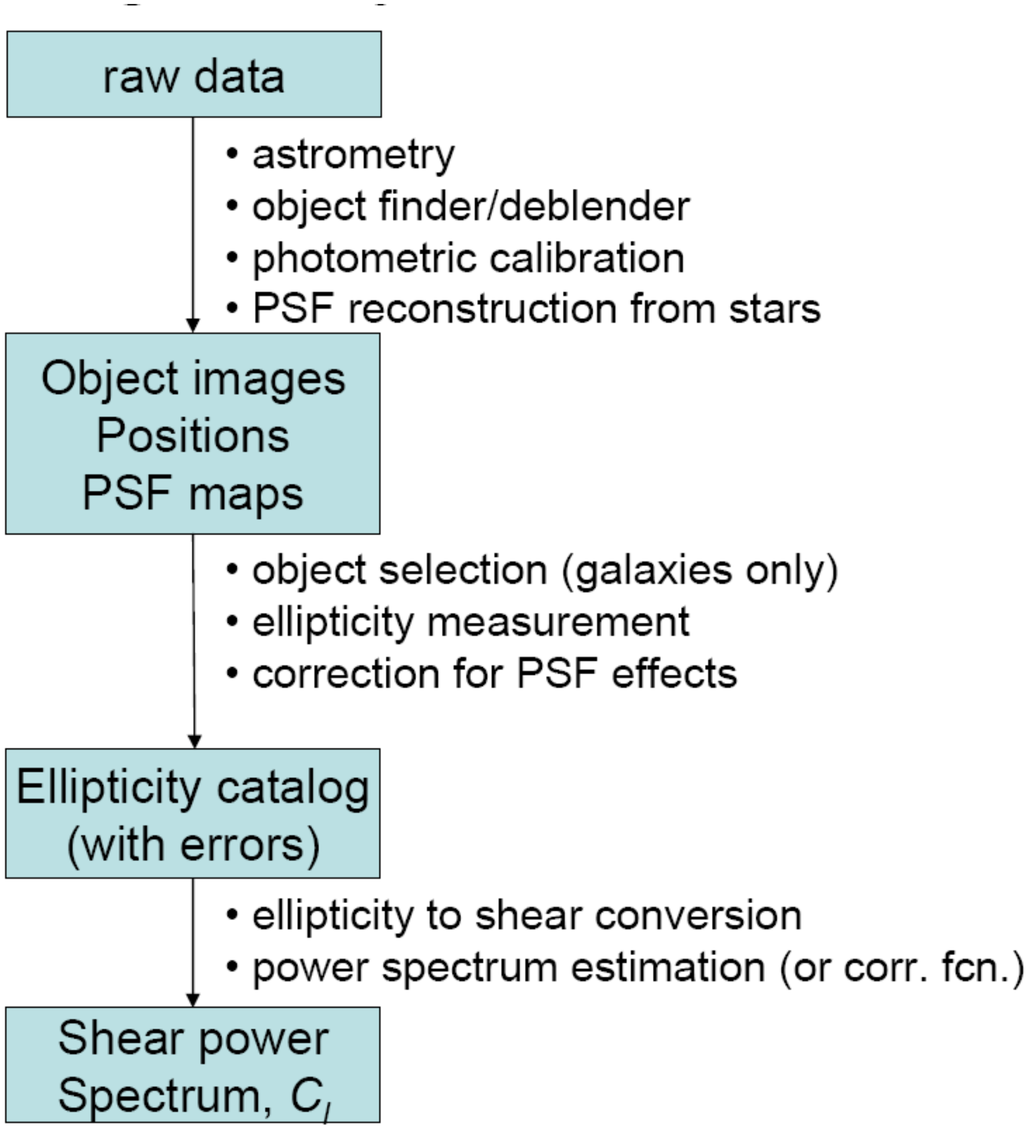
- through structure formation (P_δ)
- through direct dependences on H_0, Ω_m
- through the background evolution ($D_A(z)$ in the function g(w))

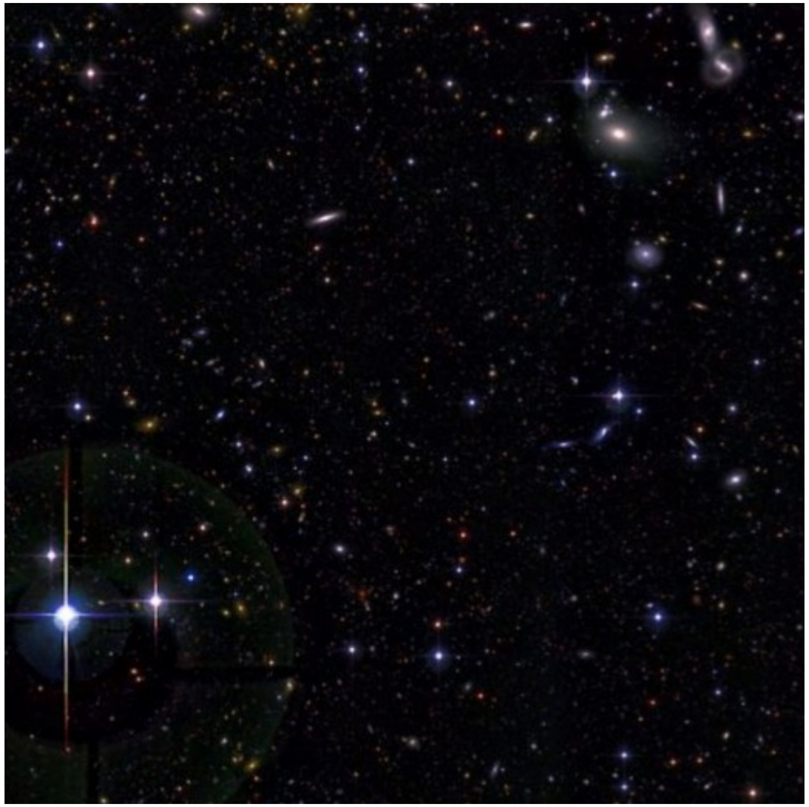


It is mainly sensitive to Ω_m and to the amplitude of P_δ (i.e., to σ_8) with a well defined degeneracy direction, and to the sources redshift distribution

Cosmological weak lensing: estimator

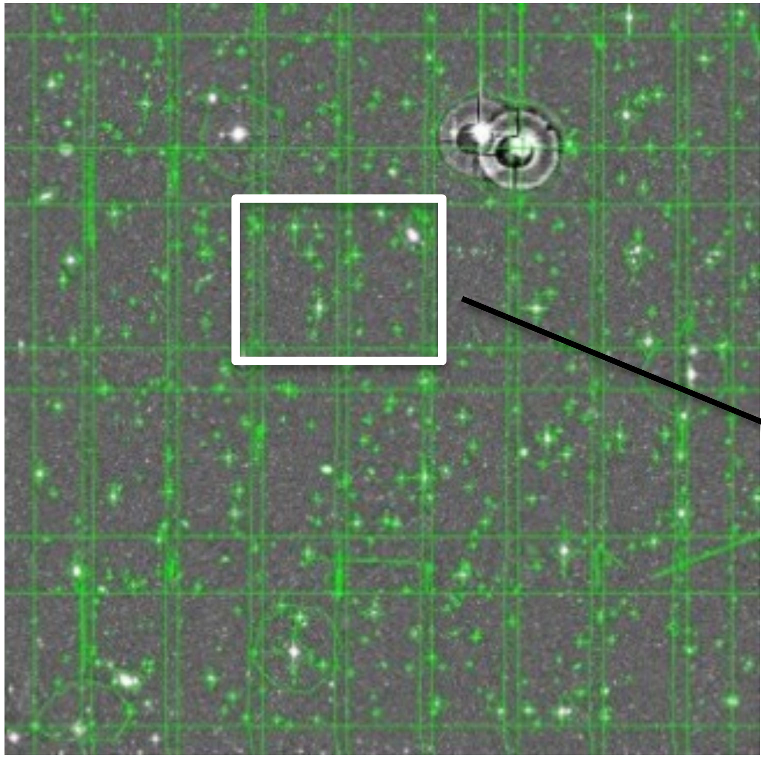
Lensing Analysis Pipeline



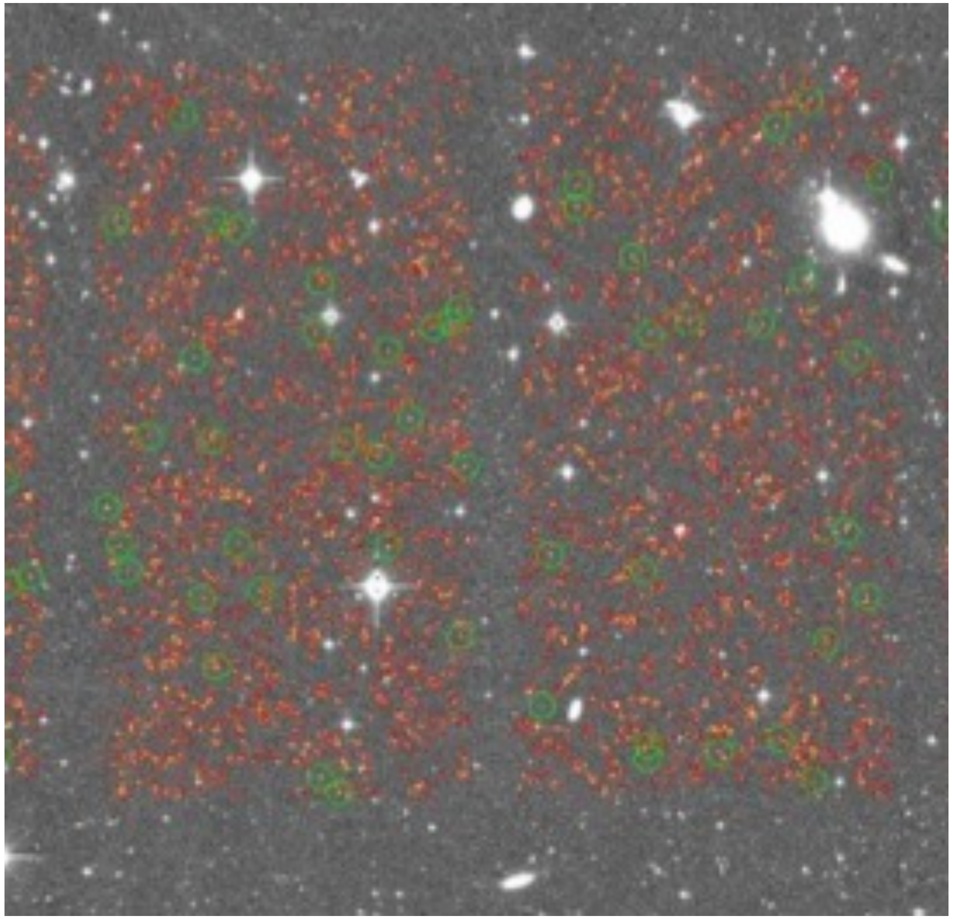


only stars and foreground galaxies
are visible in this image

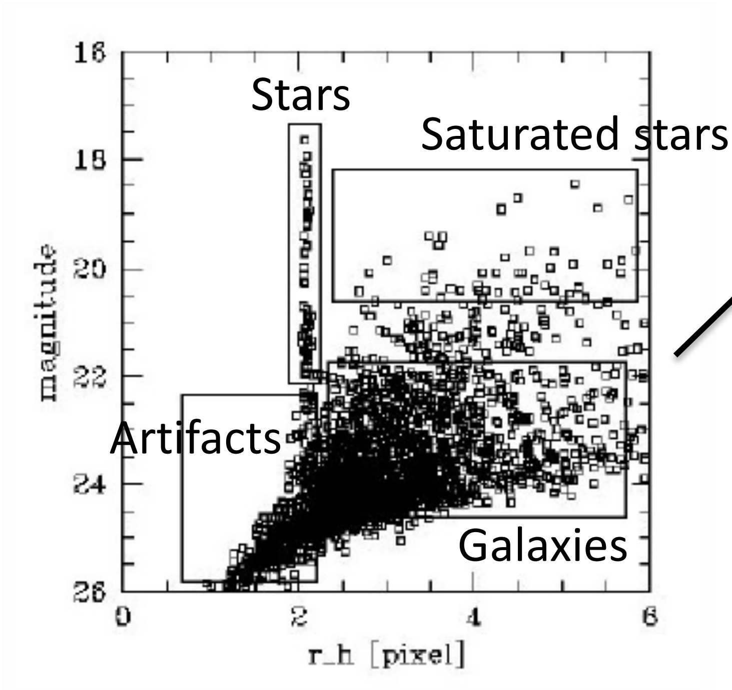
background galaxies are visible now,
and also the “ghosts” from a
saturated star



Masking



Stars (green) / Galaxies (red) separation



Shear estimator

The estimator of shear is the **ellipticity**.

The shapes of distant galaxies in a 2D image are approximately ellipses (valid for both elliptical and spiral galaxies). They can be described by 2 parameters: eccentricity |e| (deviation from a circle) and orientation φ. These 2 parameters define the ellipticity, which is a traceless symmetric tensor.

Note that under a rotation of α, a traceless symmetric tensor transforms in the same way as a vector under a rotation of 2α.

$$\begin{bmatrix} e_1' & e_2' \\ e_2' & -e_1' \end{bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} e_1 & e_2 \\ e_2 & -e_1 \end{bmatrix} \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix},$$

which is equivalent to,

$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} cos2\alpha & sin2\alpha \\ -sin2\alpha & cos2\alpha \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

For this reason, traceless symmetric tensors are also called **pseudo-vectors**, which have π symmetry, instead of 2 π.
They are also called **spin-2** quantities and its components can be written in vector form:

$$e = |e| \exp(2i\varphi) = e_+ + i e_\times$$

The ellipticity of an object is computed from the second-order moments of brightness (with respect to the centroid of the image),

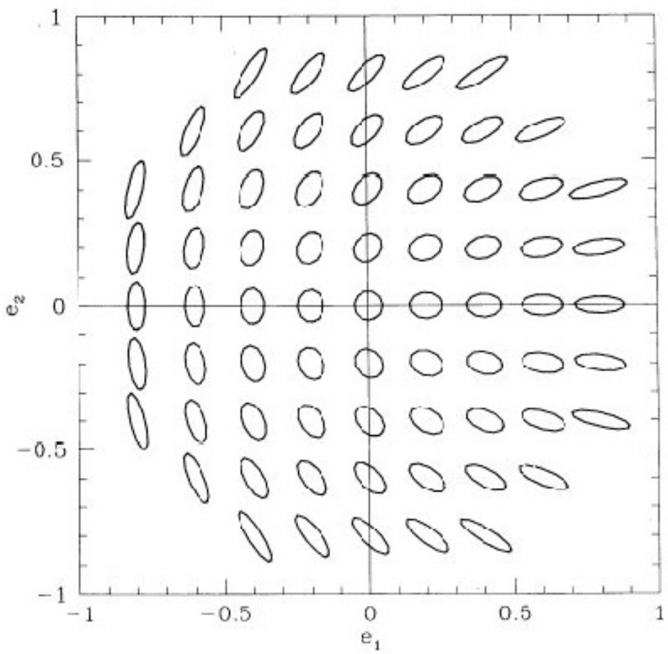
$$Q_{ij} = \int d^2\theta \, I(\theta)(\theta_i - \theta_i^0)(\theta_j - \theta_j^0),$$

as,

$$e = \left(\frac{Q_{xx} - Q_{yy}}{Q_{xx} + Q_{yy}}, \frac{2Q_{xy}}{Q_{xx} + Q_{yy}} \right)$$

So, component e_+ measures the normalized excess of flux along the x-axis with respect to the flux along the y-axis

and component e_X measures the normalized excess of flux along the y = x line with respect to the flux along the y = −x line



The ellipticity ranges from 0 → the ellipticity of a circular object, to 1 → the limiting case of an extremely elliptical object that becomes one-dimensional.

It is dimensionless, not containing information about the size of the object, which is encoded in the trace $Q_{xx} + Q_{yy}$

To understand why the ellipticity is an estimator of the shear, let us consider a 2D image of a galaxy (the source shape) that is subject to weak gravitational lensing and will be transformed into a slightly different 2D (the image shape).

The moments of the source are transformed by the lens equation (the lensing transformation) into the moments of the image:

$$Q^s = A(\theta)\, Q\, A^t(\theta)$$

For example, for the trace of the moments we get,

$$Q^s_{11} + Q^s_{22} = Q_{11}\,[(1-\kappa-\gamma_1)^2 + \gamma_2^2] + Q_{22}\,[(1-\kappa+\gamma_1)^2 + \gamma_2^2] + Q_{12}\,4\gamma_2(1-\kappa)$$

Computing the transformation for all moments, and combining them to form the ellipticities, we get an expression for the transformation of the ellipticities.

We may neglect quadratic terms in the transformation, because we are in the weak lensing regime:

$$\kappa << 1\ ,\ |\gamma| << 1\ ,\ g \approx \gamma + \gamma\kappa.$$

where g is the reduced shear

$$g_i = \gamma_i/(1-\kappa)$$

In the weak lensing approximation the resulting transformation is:

$$e_i^s = e_i - 2g_i$$

and this is the **shear estimator.**

So the reduced shear produced by the lensing effect (which is $g \sim \gamma$) **adds linearly to the intrinsic (source) ellipticity of the galaxy to produce the image galaxy ellipticity.**

The estimator cannot give us the exact value of the shear acting on a galaxy because we do not know the source ellipticity e^s of a galaxy.

But it can be used to estimate the shear from the measured ellipticity, if we know the properties of the intrinsic ellipticity distribution.

$$2\gamma = e_{\mathrm{obs}} - < e_s > \pm \frac{\sigma_s^2}{N}$$

If the galaxies have intrinsically random ellipticities, which implies random orientations → $<e_s> = 0$ → the estimator is unbiased.

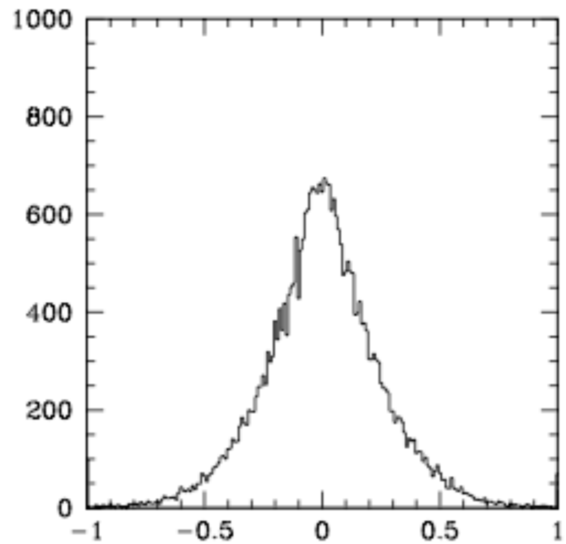
If the galaxies eccentricities and orientations are intrinsically correlated (for example for having been formed together in the same DM halo) → $<e_s> \neq 0$ → the estimator is biased.

In general it is always possible to find a sample of uncorrelated galaxies in the same 2D area of the sky, and have an **unbiased estimator**.

We already saw that the typical convergence (and shear) signal is 0.01.

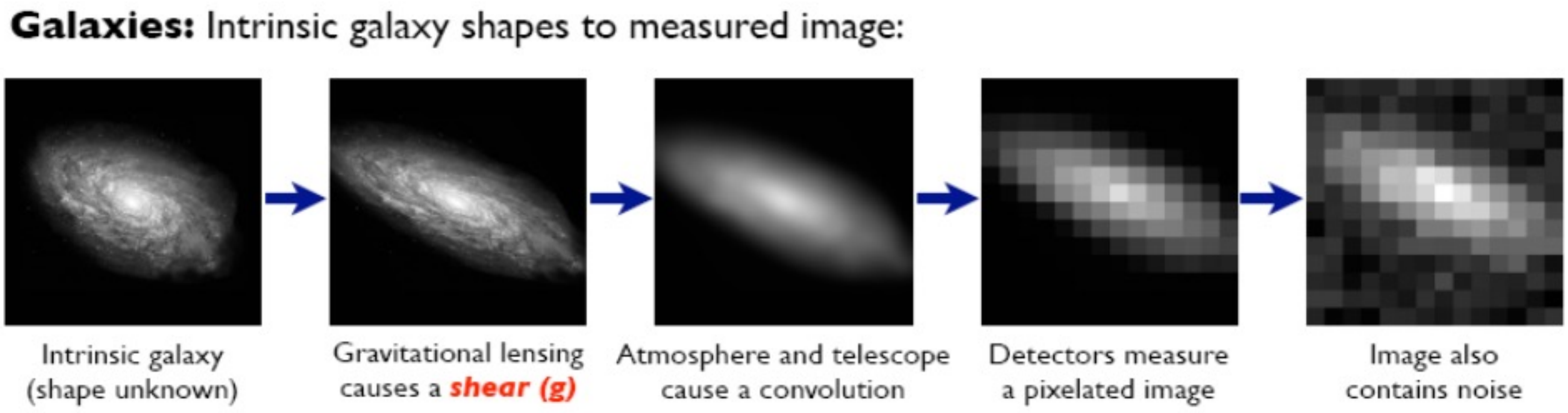
The measured rms of ellipticity distributions is ~0.3 → it is much larger that the cosmic shear signal → it is due to the intrinsic ellipticities dispersion.

This means that the shear estimator is very noisy → a large number of galaxies is needed to be able to detect the cosmological lensing signal.



intrinsic ellipticities distribution

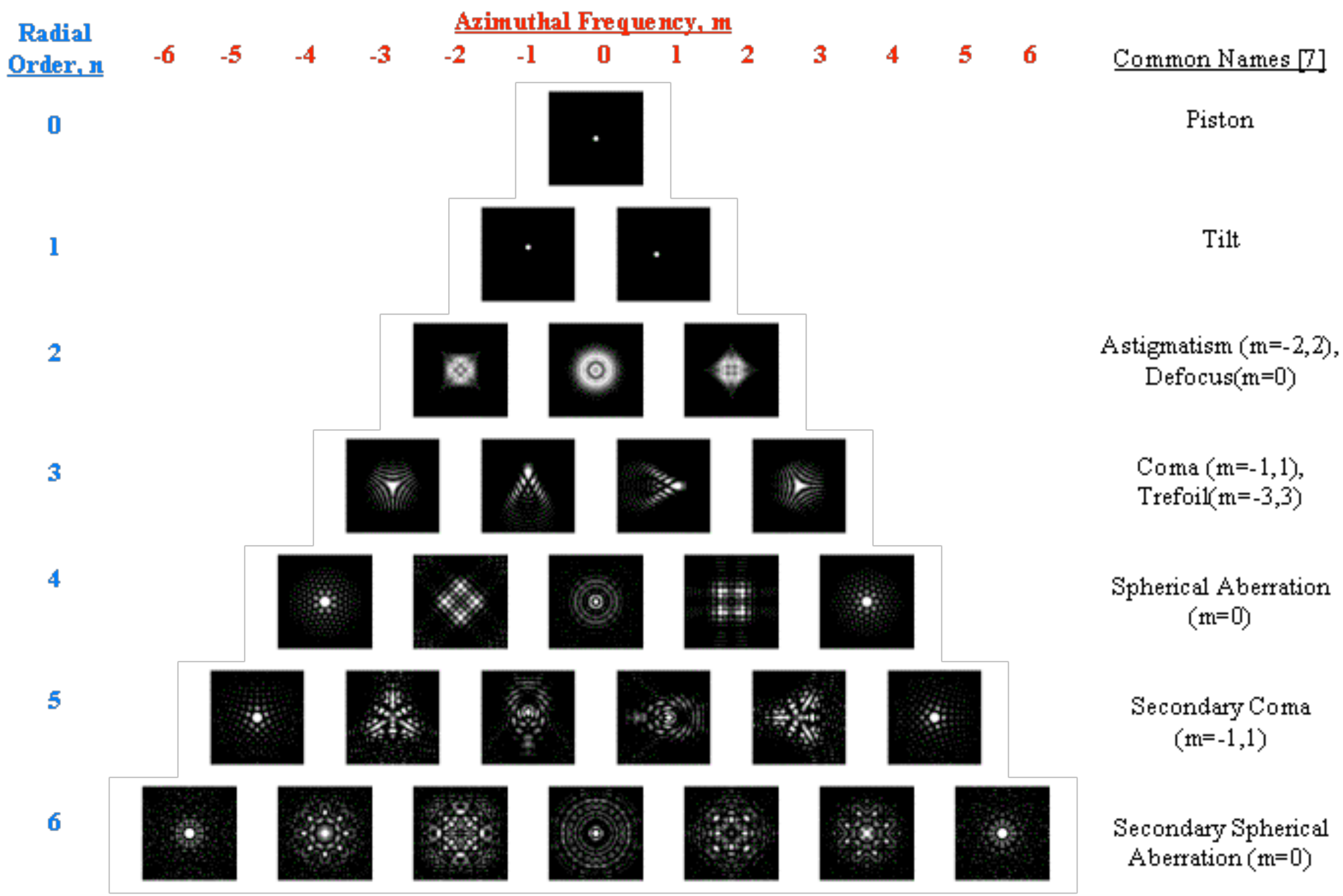
But the ellipticity of a galaxy image is not only induced by gravitational lensing → there are several other effects



So in reality **the estimator is biased**, and the non-cosmological distortions need to be corrected.

The dominating effect is the Point Spread Function (PSF) produced by the atmosphere and by the optical system of the telescope.

The PSF model convolves the image.
The amplitude of the PSF effect is much larger than the cosmological effect.



The types of PSF present in the optical system are characterized when building the telescope by simulating its wavefront.

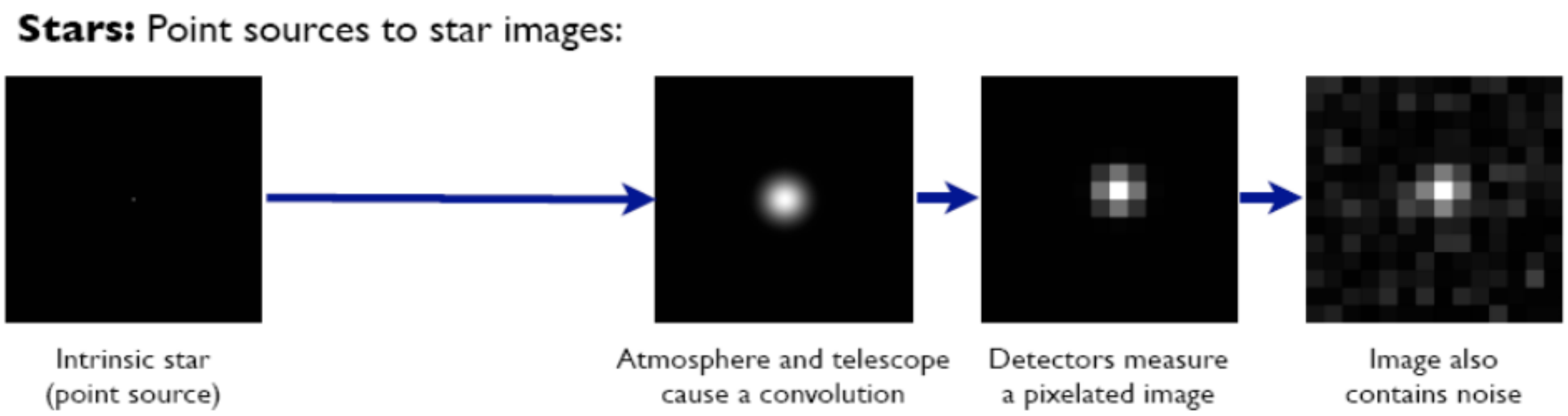
The biased shear estimator can be written as:

$$e_i^{s'} = e_i^{obs} - P_{ij}^{\gamma} g_j - P_{ij}^{sm} q_j$$

It includes the PSF anisotropy q → modeled as an **additive bias**
and the PSF isotropy, which decreases the response of the galaxies to shear, producing a change in the factor 2 in the original unbiased estimator → modeled as a **multiplicative bias**

The bias can be corrected because the PSF can be measured using stars. Stars are not affected by cosmological lensing → **any ellipticity detected in the stars in the image is produced by the PSF.**

In fact, stars are point-like and would not even be seen in an image if there was no isotropic PSF (like the seeing produced by the atmosphere).

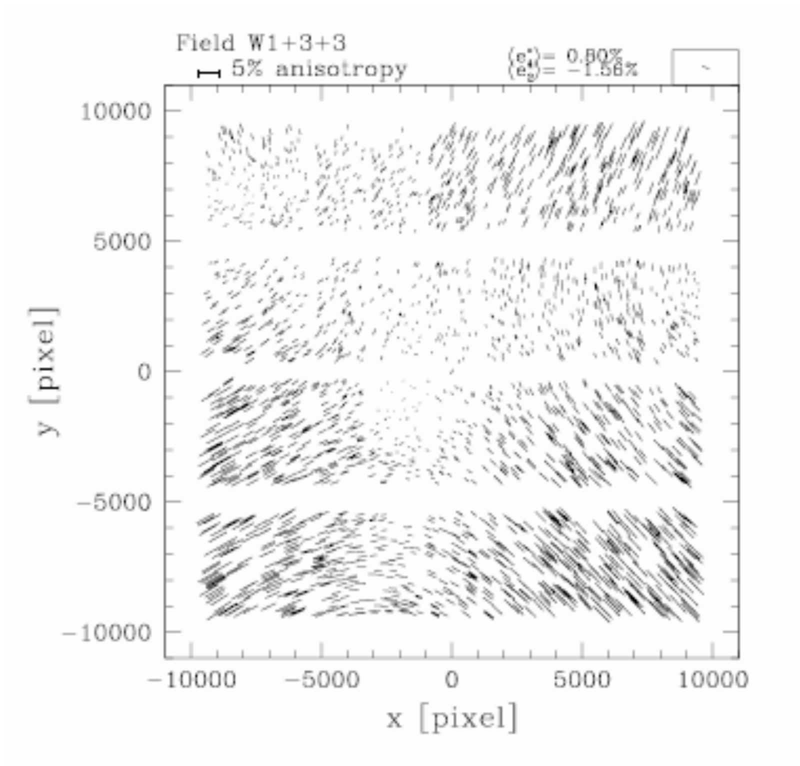


PSF is measured at stars positions → It is then interpolated across the FoV to find its values at the galaxies positions

PSF deconvolution:

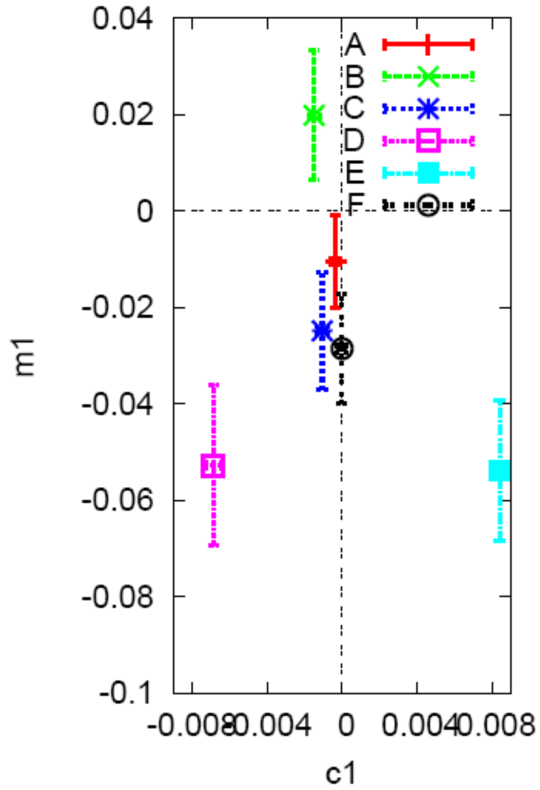
The PSF can then be subtracted (deconvolved) from the image.

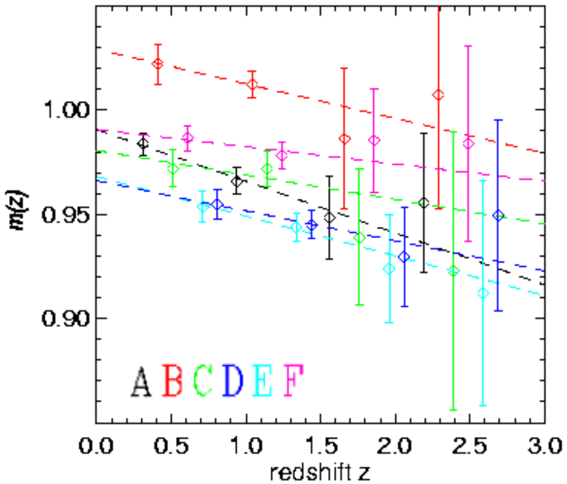
Simulations with known cosmic shear and PSF models may be used to check for residuals of the correction procedure → to calibrate the result:

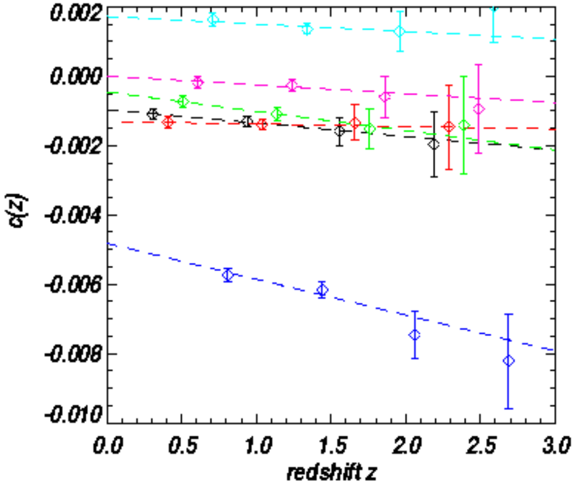


Multiplicative and additive residuals
for 6 PSF simulations:

$$\langle \gamma \rangle - \gamma^{\mathrm{input}} = m\, \gamma^{\mathrm{input}} + c$$



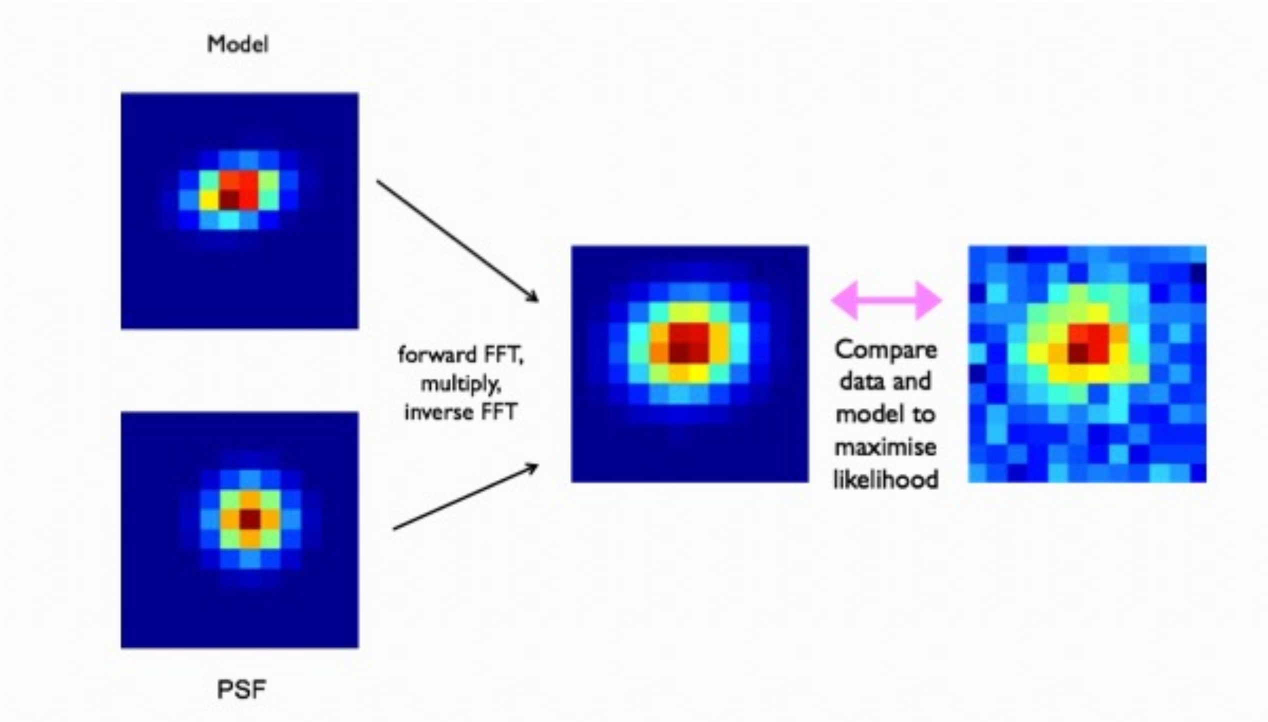




m and c evolution with redshift

If shear simulations are not used, the values of the residuals are not known. In that case they may be included in the estimator as nuisance parameters → **PSF calibration**

Alternatively, PSF may be corrected with **forward model fitting**:



The PSF (measured from stars) is convolved (multiplied in Fourier space) with models for the galaxy image.

Compare the results with the observed image → Bayesian analysis to find the best model.

In both cases calibration nuisance parameters are introduced, to ensure greater accuracy.

Shear correlation function estimator

The goal of weak lensing measurements is to go from ellipticity measurements → to 2D correlation function of shear → to 2D metric (potential) power spectrum or dark matter power spectrum → to compare with theoretical predictions

We are interested in the statistical properties of the ellipticity field and not on finding the individual shear of each galaxy → **we may estimate directly the shear correlation function instead of the shear.**

Roughly speaking, we saw that the shear is estimated from

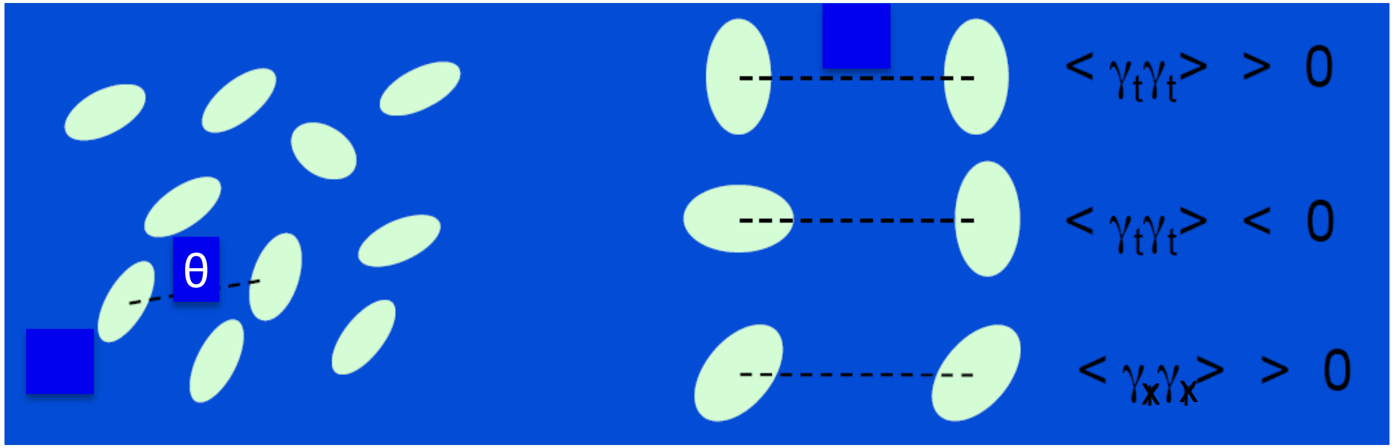
$$e = e^s + \gamma \text{ (neglecting calibration factors)}$$

So the ellipticity correlation function is an estimator of the shear correlation function:

$$\hat{\xi}_{ee} = \xi_{\gamma\gamma} + \xi_{e^s e^s} + \xi_{\gamma e^s}$$

The ellipticity correlation function of a discrete galaxy field is measured from the correlation of ellipticity pairs as function of separation:

$$\hat{\xi}_{tt}(\theta) = \frac{\sum_{i,j} w_i w_j \, e_t(\mathbf{x}_i) e_t(\mathbf{x}_j)}{\sum_{i,j} w_i w_j} \qquad \hat{\xi}_{\times\times}(\theta) = \frac{\sum_{i,j} w_i w_j \, e_\times(\mathbf{x}_i) e_\times(\mathbf{x}_j)}{\sum_{i,j} w_i w_j}.$$



Bias of the estimator

However, the ellipticity correlation function does not give us directly the shear correlation function. It is a biased estimator of it, due to the two additional effects that also contribute to the ellipticity correlation function:

- $\xi_{e^s e^s}$ **correlation function of the source ellipticities** (i.e., the intrinsic distribution of ellipticities, before the lensing effect).

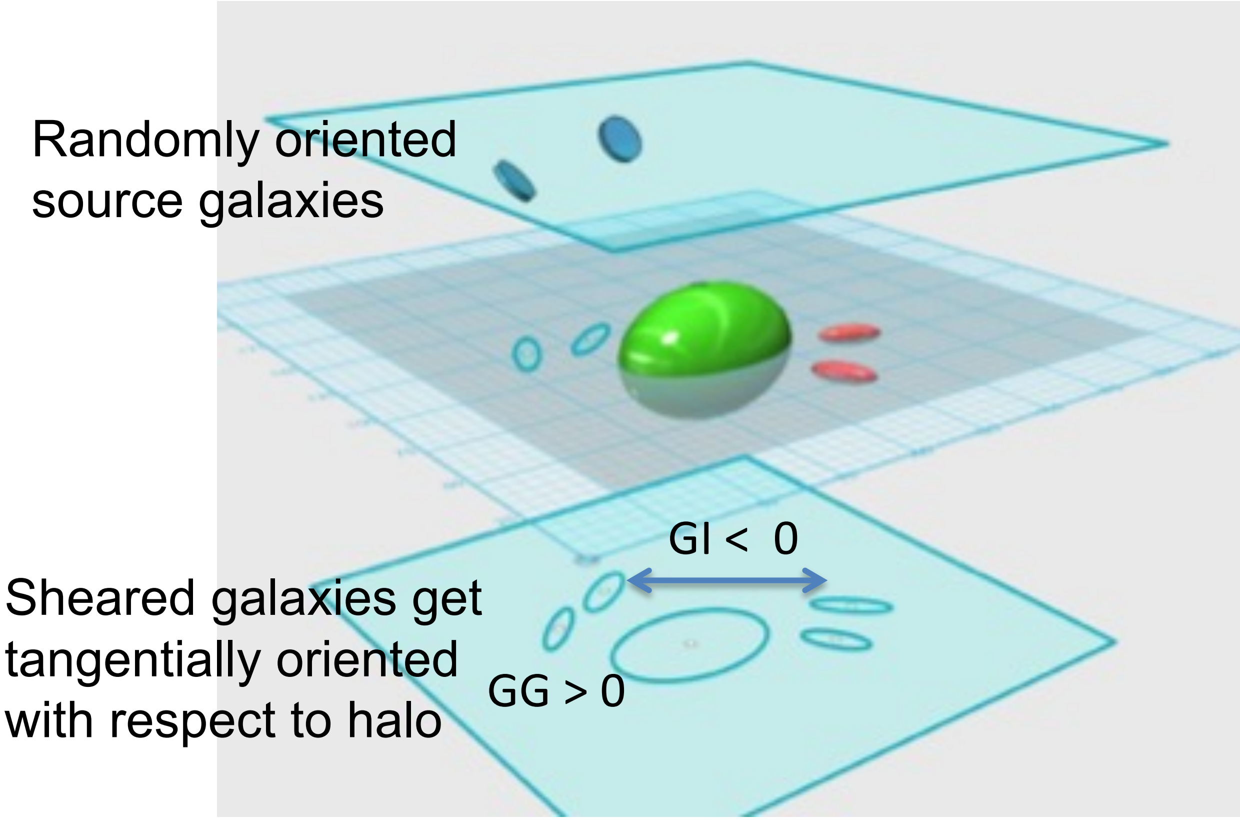
  It depends on the type of pairs involved:

  - for i=j it is a monopole constant term → a shot noise $\frac{\sigma_{e^s}^2}{n}$

  - for i ≠j it is the correlation of the intrinsic ellipticities between different galaxies → an intrinsic alignment (II)

- $\xi_{\gamma e^s}$ **shear-ellipticity cross-correlation**

  It is the correlation between the intrinsic shape of a galaxy and the shear produced in a second galaxy (its i=j contribution is zero, but i ≠j is not zero)→ another type of intrinsic alignment (GI)



The contamination from $\xi_{e^s e^s}$ (II) is zero if we do not consider galaxies at the same redshift bin

The contamination from $\xi_{\gamma e^s}$ (GI) depends on galaxy formation. It can be measured with <eδ> (galaxy-galaxy lensing)

Origin of the intrinsic alignments

Elliptical galaxies near halos are tidally streched → creates II

$$\varepsilon_+ \propto (\partial_y^2 - \partial_x^2)\phi$$
$$\varepsilon_X \propto 2\partial_x \partial_y \phi$$

Spiral galaxies orientation near halos determined by angular momentum L → do not correlate with halo orientation → no GI

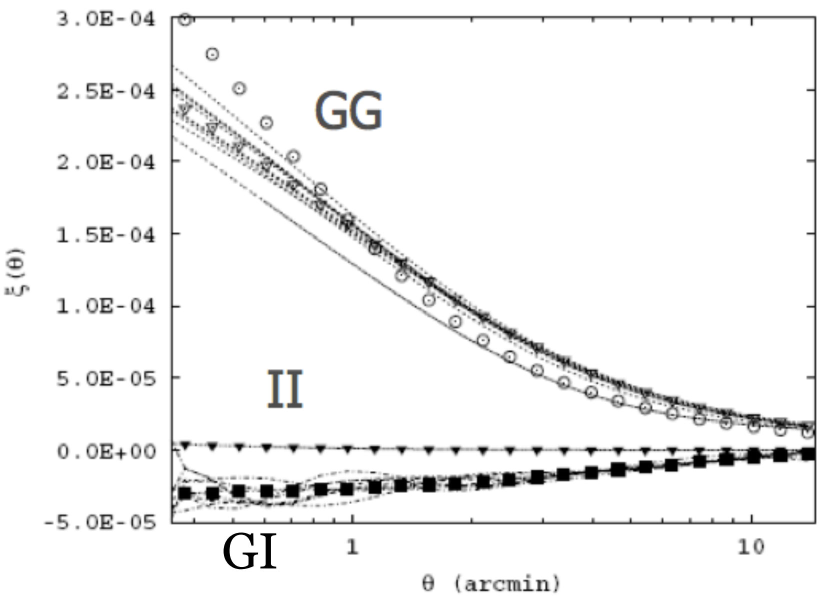
$$\varepsilon_+ \propto (L_y^2 - L_x^2)$$
$$\varepsilon_x \propto 2L_x L_y$$
$$L_\alpha \propto \varepsilon_{\alpha\beta\gamma} J_{\beta\delta} \partial_\delta \partial_\gamma \phi$$

So **the shear correlation function estimator is biased** by construction, due to the presence of intrinsic alignments.

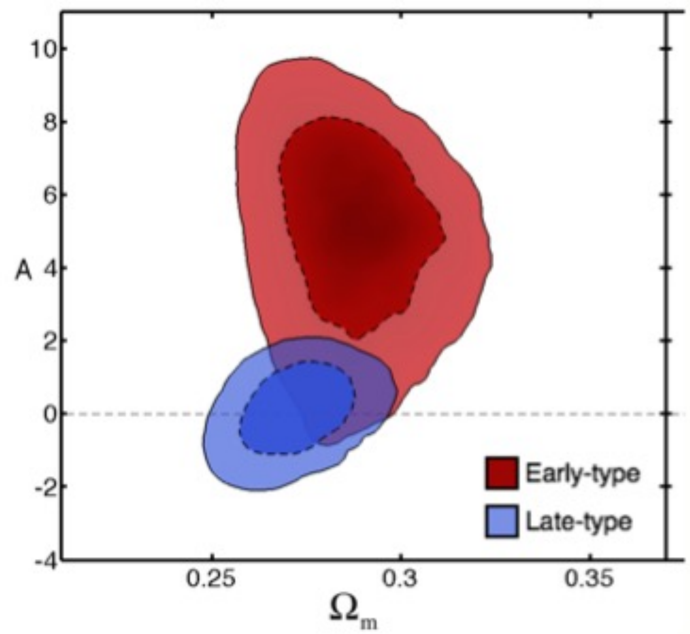
However, if the galaxy pairs in the correlation are at different redshifts, the dominant contribution for the ellipticity correlation is the shear correlation (GG):

II is zero (because the two galaxies are distant in redshift)

GI < 0 and ~ 10% GG



GI can be estimated from galaxy-galaxy lensing measurements using early-type (ellipticals) and late-type (spirals) galaxies



Besides the fundamental intrinsic alignment biases, there are **3 other main classes of systematics** that affect the measurement of the shear signal and impact the estimation of cosmological parameters.

They come from the **measurement of the ellipticities**, from the **determination of the source redshift distribution**, and from uncertainties on the **shear theoretical power spectrum.**

i) **Bias in the shear measurement :** there are many sources of bias in the measurement of shear, besides PSF residuals, that propagate into the correlation measurement. For example:

- Light-profile model bias: due to noise, the brightness moments need to be computed using a filter. This needs to correctly model the light profile, otherwise it will introduce a bias. It is easy to use a non-appropriate filter in cases of non-elliptical isophotes, or when there are color gradients (different profiles in different filters → bias broad-band measurements)

- Noise bias: in general, ellipticity is non-linear in pixel data → the simple fact that the flux values in the image pixels are noisy changes the shear-to-ellipticity linear relation → if we use it, we introduce a bias

- PSF residuals

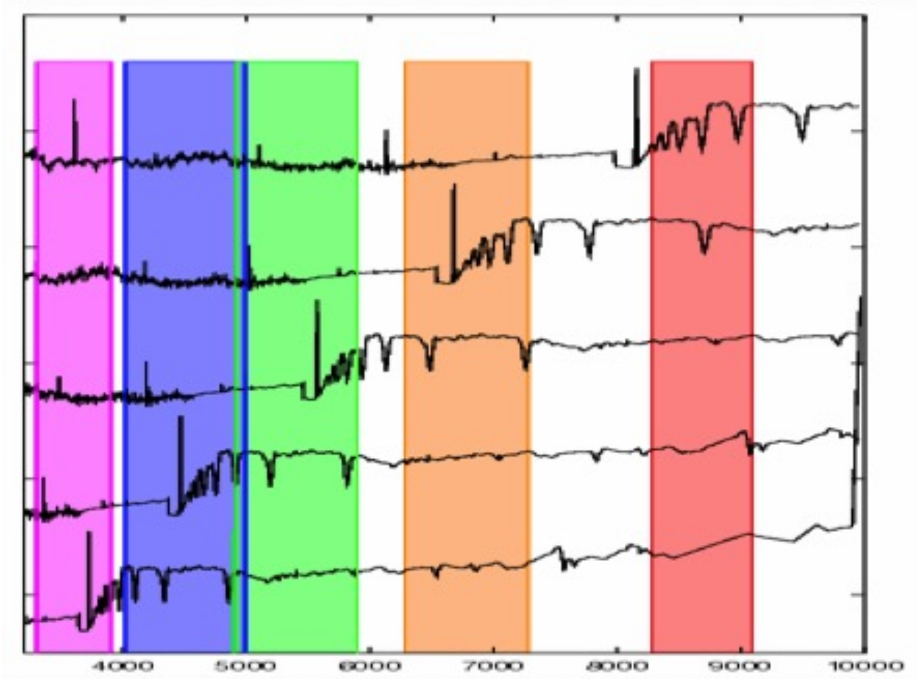
- Detector effects: charge transfer inefficiency (CTI)

ii) **Bias in the redshift distribution**:

- Wrongly identified photometric redshifts

Typical filters u ; g r i y z ; I J K

used to detect the strongest features, like the 4000 Angstrom-break for galaxies at various redshifts



Some **properties** of photometric redshift estimation:

In the redshift desert, $z \sim 1.5 - 2.5$ → neither 4000 A-break or Ly-break in visible range → very hard to access from ground.

Confusion between low-z dwarf ellipticals and high-z galaxies and confusion between Balmer and Lyman break → catastrophic outliers

UV band and IR needed for high redshifts → but UV is very inefficient and IR is absorbed by atmosphere → need space observations.

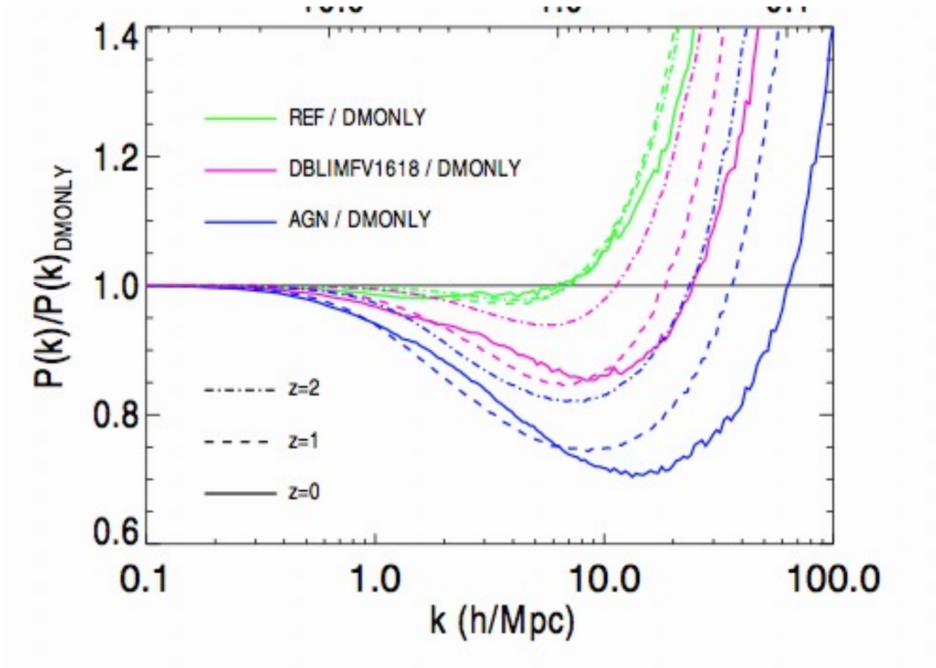
Need spectroscopic galaxy sample for comparison and calibration, or also for cross-correlation.

The typical **accuracy of photo-z** determination is: $\sigma \sim 0.05\,(1 + z)$

- Selection effects: for example, blended galaxy images are discarded → under-representation of galaxies in crowded fields, which are high-density regions and have typically lower redshifts → biased $n(z)$.

iii) **Bias in the shear power spectrum from baryonic effects**:

- **on small scales** 1 < k < 10 h/Mpc
gas pressure is important (baryonic matter is no longer dust) → suppression of structure formation, gas distribution is more diffuse than DM → less power in the total matter power spectrum



- **on very small scales** k > 10 Mpc
(~ R < 0.1 Mpc) there is baryonic cooling and AGN+SN feedback →

increase condensation of baryons → formation of stars and galaxies → increase of power spectrum amplitude

The shear correlation estimator can then be written with all the biases terms by including N = 4 + n_zbins **nuisance parameters**:

$$\xi_{ee}(\theta) = m\,\xi_{\gamma\gamma}(\theta) + c + A_{\rm I}\,f(\theta) + A_{\rm b}\,f(\theta, z_s) + A_{\rm phz}(z_s)$$

where the calibration parameters (m,c) account for all shear measurement biases.

Variance of the estimator

The full measurement of a cosmological quantity of interest (power spectrum, correlation function, etc) must include not only the estimate of the quantity but we also need to quantify the precision of the measurement (compute the error bars).

$\hat{\xi}$ is the estimator of the correlation function → it is the measurement.

The measurement $\hat{\xi}$ is interpreted as one possible realization of the true value of ξ.

$< \xi >$ is the true value of the correlation function → it is the theoretical computation of ξ, computed from the model (structure formation).

Even for direct measurements in the real space,
$\hat{\xi}$ and $< \xi >$ are different because of noise (variance of the estimator, and also intrinsic 'cosmological noise') and bias (the estimator may have systematic errors that need to be corrected or taken into account in nuisance parameters).

The **variance of the estimator is**:

$$C_{ij} = \left\langle (\hat{\xi} - < \xi >)_i \, (\hat{\xi} - < \xi >)_j \right\rangle$$

In the case of the cosmic shear correlation functions, we can already see that, since ξ depends on <e e> → <YY> → <δδ>, its variance will depend on 4-pt functions <δδδδ> → **the full computation of the error bars of a power spectrum requires the theoretical computation of the trispectrum**. (It is the variance of a variance)

Let us consider the estimator for ξ+

$$\hat{\xi}_{+}(\vartheta) = \frac{\sum_{ij} w_i w_j (\epsilon_{it}\epsilon_{jt} + \epsilon_{i\times}\epsilon_{j\times}) \Delta_{\vartheta}\left(\left|\boldsymbol{\theta}_i - \boldsymbol{\theta}_j\right|\right)}{N_{\mathrm{p}}(\vartheta)}$$

(assuming all external biases are accounted for)

ξ+ combines the two components (t,X) of the ellipticity.

The weights are needed to distinguish the quality of the measurements of different galaxies.

So the correlation for each separation ϑ is the sum of all contributions (e_i e_j) from the N_p galaxy pairs in the bin ϑ :

$$\Delta_{\vartheta}(\phi) = 1 \text{ for } \vartheta - \Delta\vartheta/2 < \phi \leq \vartheta + \Delta\vartheta/2,$$

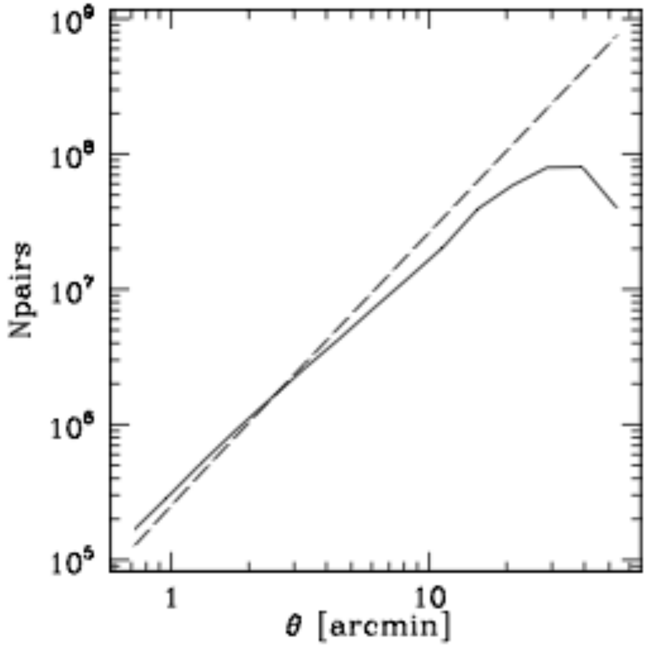
$$= 0 \text{ otherwise}$$

and is divided by the number of contributing pairs (i.e., it is an average).

The number of pairs **increases with separation** and depends on the survey area and density:

$$N_{\rm p}(\vartheta) = A\, n\, 2\pi\vartheta\, \Delta\vartheta\, n.$$

This approximate expression comes from considering that the survey is a single connected field of area A with N galaxies (density n) → the galaxies on a circular shell of radius ϑ around a central galaxy, form pairs of separation ϑ. Then, consider shells around all galaxies to get the total number of pairs for that separation.



Npairs

dashed : formula

solid: measured

The measured number of pairs on large scales is smaller due to the edge of the field-of-view.

Assuming no intrinsic alignments, this estimator is unbiased:

$$\left\langle \hat{\xi}_+(\vartheta) \right\rangle = \xi_+(\vartheta),$$

using e = $e^s + \gamma$ we can write:

$$\left\langle \epsilon_{it}\epsilon_{jt} + \epsilon_{i\times}\epsilon_{j\times} \right\rangle = \sigma_\epsilon^2 \delta_{ij} + \xi_+ \left(\left| \boldsymbol{\theta}_i - \boldsymbol{\theta}_j \right| \right)$$

(where σ_e^2 is the shot noise term, i.e., the auto-correlation term)

Now, let us **compute the variance** of the unbiased estimator:

$$\mathrm{Cov}(\hat{\xi}_\pm, \theta_1; \hat{\xi}_\pm, \theta_2) = \left\langle \left(\hat{\xi}_\pm(\theta_1) - \xi_\pm(\theta_1) \right) \left(\hat{\xi}_\pm(\theta_2) - \xi_\pm(\theta_2) \right) \right\rangle$$

To compute it, **we need to compute the cross-correlation between the correlation function at two separations**:

$$\left\langle \hat{\xi}_+(\theta_1)\hat{\xi}_+(\theta_2)\right\rangle = \frac{1}{N_p(\theta_1)N_p(\theta_2)} \sum_{ijkl} w_i w_j w_k w_l \left\langle (e_{i1}e_{j1} + e_{i2}e_{j2})(e_{k1}e_{l1} + e_{k2}e_{l2})\right\rangle$$

Notice that, since the correlation function separations are not independent (contrary to linear power spectrum scales), we have to consider all cases e_i e_j and cannot simplify them to e_i^2

The calculation is involved because of this, and also due to the presence of the extra term of e_s, and also because the ellipticity and shear fields have two components.

Inserting $e = e^s + \gamma$, the quantities $<(e_i\ e_j)\ (e_k\ e_l)>$ become,

$$\langle e_{i\alpha}e_{j\beta}e_{k\mu}e_{l\nu}\rangle = \frac{\sigma_e^2}{2}\left(\delta_{jl}\delta_{\beta\nu}\langle\gamma_{i\alpha}\gamma_{k\mu}\rangle + \delta_{jk}\delta_{\beta\mu}\langle\gamma_{i\alpha}\gamma_{l\nu}\rangle + \delta_{il}\delta_{\alpha\nu}\langle\gamma_{j\beta}\gamma_{k\mu}\rangle + \delta_{ik}\delta_{\alpha\mu}\langle\gamma_{j\beta}\gamma_{l\nu}\rangle\right) +$$

$$+ \langle\gamma_{i\alpha}\gamma_{j\beta}\gamma_{k\mu}\gamma_{l\nu}\rangle + \left\langle e^s_{i\alpha}e^s_{j\beta}e^s_{k\mu}e^s_{l\nu}\right\rangle$$

(greek indexes account for the 2 components 1,2)

Notice that even though none of the correlation functions $\xi(\theta_1)$ and $\xi(\theta_2)$ are computed at separation zero, their variance depends on the shot noise, because it includes terms $\theta_1 = \theta_2$ → **the covariance of a quantity that is itself a pure covariance, also depends on the variance of the covariance (and not just on the covariance of the covariance).**

(in other words, a 2-pt signal at non-zero separations is not affected by shot noise, but its covariance is).

Now, using Wick's theorem and **assuming Gaussianity** (no connected 4-pt), we can write all 4-pt quantities as products of 2-pt quantities. In this Gaussian approximation, we get:

$$
\begin{aligned}
\left\langle \epsilon_{i\alpha}\epsilon_{j\beta}\epsilon_{k\mu}\epsilon_{l\nu} \right\rangle = & \frac{\sigma_\epsilon^2}{2}\left(\delta_{jl}\delta_{\beta\nu}\left\langle \gamma_{i\alpha}\gamma_{k\mu}\right\rangle + \delta_{jk}\delta_{\beta\mu}\left\langle \gamma_{i\alpha}\gamma_{l\nu}\right\rangle + \delta_{il}\delta_{\alpha\nu}\left\langle \gamma_{j\beta}\gamma_{k\mu}\right\rangle + \delta_{ik}\delta_{\alpha\mu}\left\langle \gamma_{j\beta}\gamma_{l\nu}\right\rangle\right) \\
& + \left\langle \gamma_{i\alpha}\gamma_{j\beta}\right\rangle\left\langle \gamma_{k\mu}\gamma_{l\nu}\right\rangle + \left\langle \gamma_{i\alpha}\gamma_{k\mu}\right\rangle\left\langle \gamma_{j\beta}\gamma_{l\nu}\right\rangle + \left\langle \gamma_{i\alpha}\gamma_{l\nu}\right\rangle\left\langle \gamma_{j\beta}\gamma_{k\mu}\right\rangle \\
& + \left(\frac{\sigma_\epsilon^2}{2}\right)^2\left(\delta_{ik}\delta_{jl}\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{il}\delta_{jk}\delta_{\alpha\nu}\delta_{\beta\mu}\right)
\end{aligned}
$$

Inserting this in the variance of the estimator, we obtain 3 different contributions for the **error budget**.

The third term is diagonal, it only affects the diagonal of the covariance matrix.

$$\mathrm{Cov}(\hat{\xi}_+, \vartheta_1; \hat{\xi}_+, \vartheta_2) = \frac{1}{N_\mathrm{p}(\vartheta_1) N_\mathrm{p}(\vartheta_2)} \left[\sigma_\epsilon^4 \bar{\delta}(\vartheta_1 - \vartheta_2) \sum_{ij} w_i^2 w_j^2 \Delta_{\vartheta_1}(ij) \right.$$

It is the shot noise contribution to the error budget.
It depends only on the intrinsic ellipticity dispersion, i.e., on the shape noise.

The second term contains only shear correlations. It is a purely cosmological term, coming from the shear 4-pt function.

$$\mathrm{Cov}(\hat{\xi}_+, \vartheta_1; \hat{\xi}_+, \vartheta_2) = \frac{1}{N_\mathrm{p}(\vartheta_1) N_\mathrm{p}(\vartheta_2)} \times$$

$$\times \sum_{ijkl} w_i w_j w_k w_l \Delta_{\vartheta_1}(ij) \Delta_{\vartheta_2}(kl) \Big(\xi_+(il) \xi_+(jk) + \cos\left[4 \left(\varphi_{il} - \varphi_{jk} \right) \right] \xi_-(il) \xi_-(jk) \Big) \Big]$$

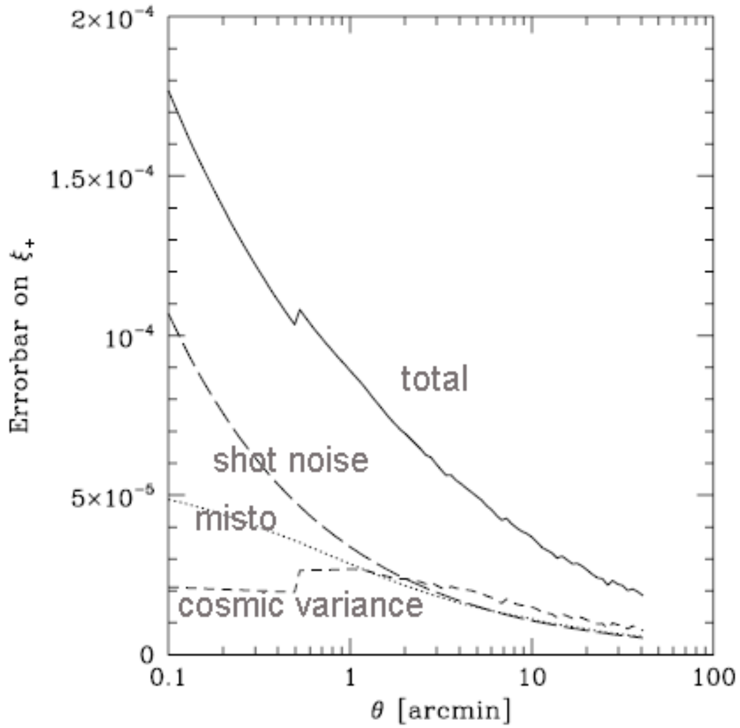
It is the only source of noise remaining in the absence of shape noise.
It is the cosmic variance contribution to the error budget.

The first term correlates shot noise with cosmic variance:

$$\mathrm{Cov}(\hat{\xi}_+, \vartheta_1; \hat{\xi}_+, \vartheta_2) = \frac{1}{N_\mathrm{p}(\vartheta_1) N_\mathrm{p}(\vartheta_2)} 2\sigma_\epsilon^2 \sum_{ijk} w_i^2 w_j w_k \Delta_{\vartheta_1}(ij) \Delta_{\vartheta_2}(ik) \xi_+(jk)$$

It is a mixed term.

The error bars are the square root of the diagonal of the covariance matrix (or noise matrix). Their relative contribution to the error budget is:



The variance is larger on small scales and:

- Shot noise dominates on small scales
- Cosmic variance dominates on large scales

Notice that the amplitude of the error bars depends essentially on the number of pairs, (divides all error terms) i.e., the uncertainty of cosmic shear surveys depends mainly on:

- Area of the survey
- Density of source galaxies

This analytical result is valid in the Gaussian fields approximation.

To compute the covariance matrix without this approximation we need to **consider the trispectrum** or measure the **dispersion of the correlation function on the data** or on **numerical simulations** of the shear field.

The **observed shear field** follows a non-Gaussian distribution, not only due to the non-linear regime of structure formation, but also because in practice a complex survey geometry introduces couplings in the measured modes and modifies the distribution → non-Gaussian covariance matrix is really needed.

Numerical simulations of the lensing field consist on N-body simulations + Ray-tracing. They are anyway needed in cosmic shear analysis for various reasons, besides computing the non-Gaussian covariance matrix:

- To compute the theoretical non-linear power spectrum (analytical extensions of the linear theory are only valid up to k ~ 0.5h/Mpc)

- To include baryonic physics, which further modify dark-matter halo properties → hydrodynamic simulations needed.

- To model systematic effects that correlate to astrophysics or the LSS, like intrinsic alignments that may also be included in the N-body simulation.

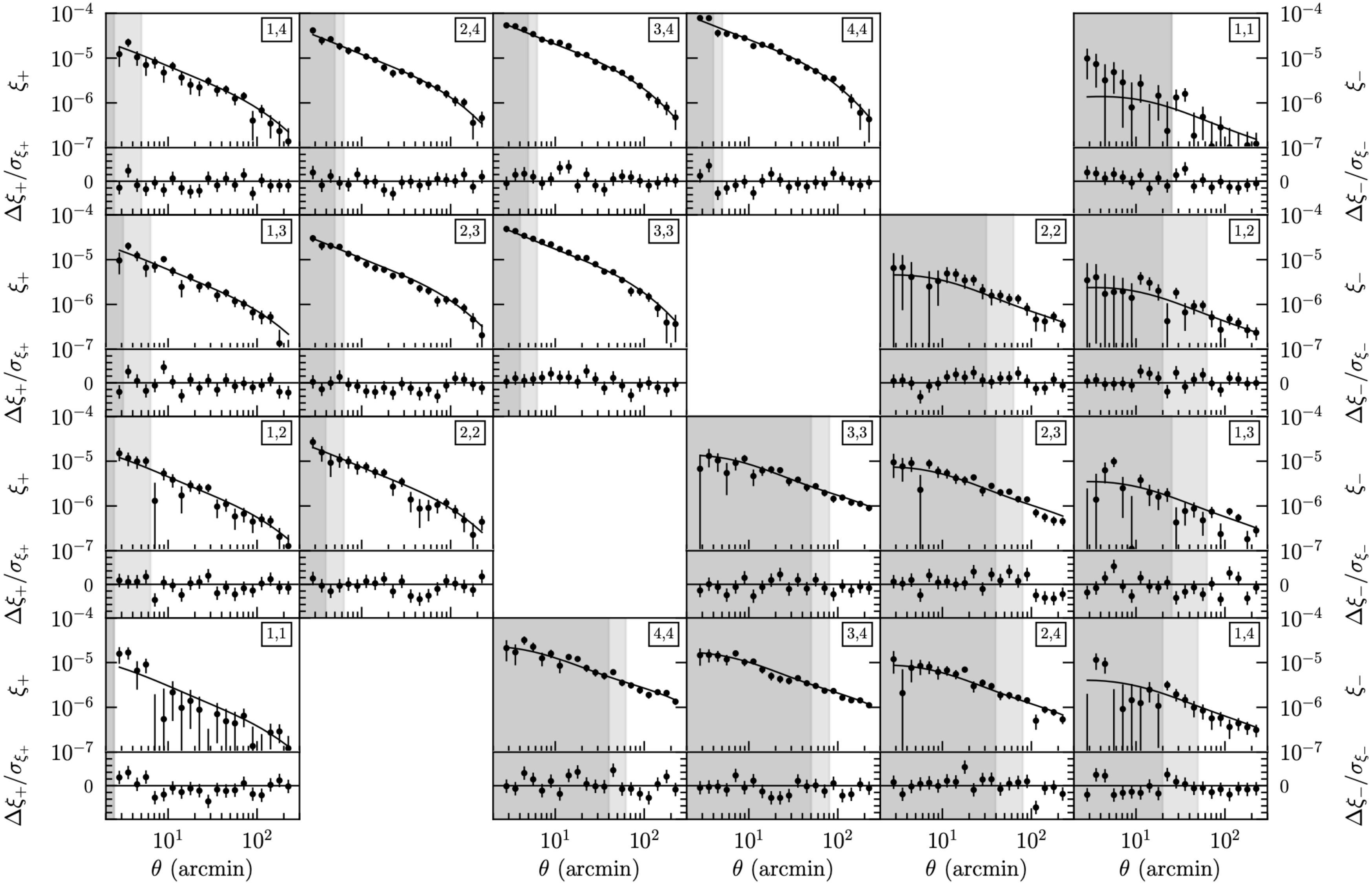
- To test the mathematical approximations made : Born approximation, neglecting of lens-lens coupling (second-order terms in the light-propagation equation), replacement of reduced shear by shear.

Numerical shear maps are produced by ray-tracing through N-body output snapshot boxes: light-rays are sent to every direction from the observer to a source at high redshift → they travel on straight lines between lens planes → N-body particles are projected onto lens planes and their surface mass density and gravitational potential computed → the induced deflection angle α is computed → the ray changes direction → this is repeated until reaching a source galaxy.

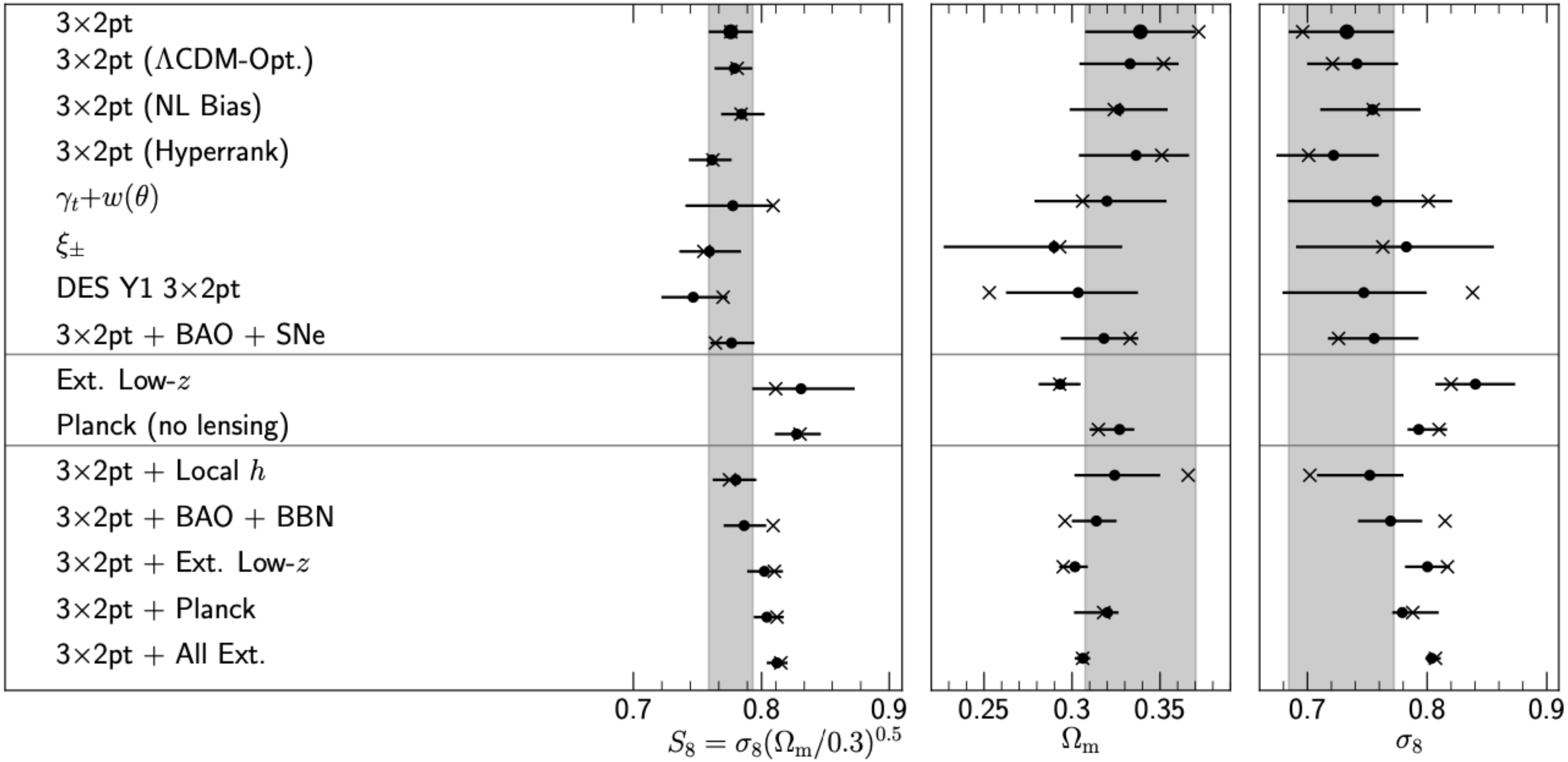
From multiple rays, the shear at each observing direction of the image is obtained.

Results

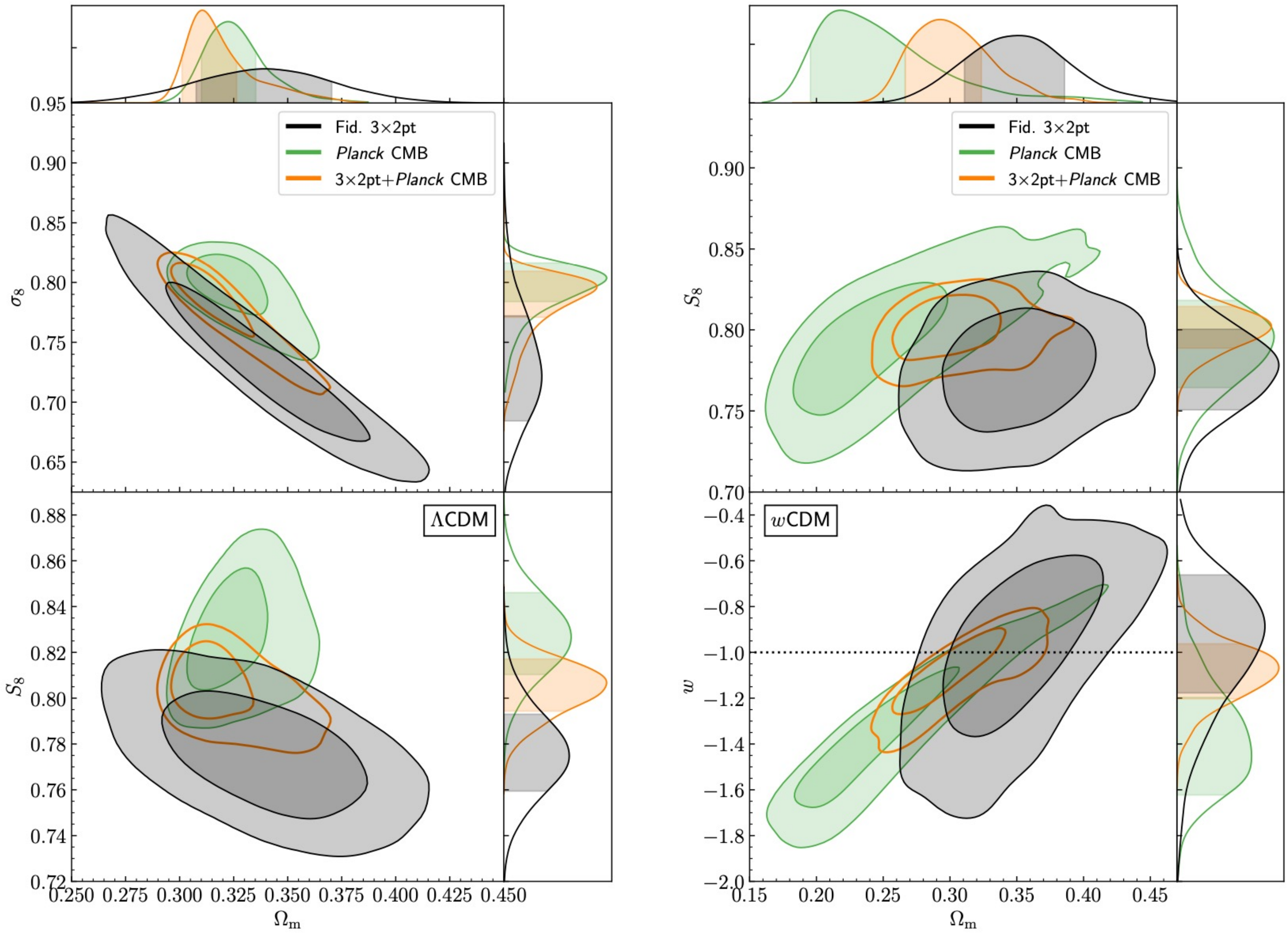
from the Dark Energy Survey (arXiv : 2105.13549)



Parameter	Prior	
Cosmology		
Ω_{m}	Flat	(0.1, 0.9)
$10^9 A_{\mathrm{s}}$	Flat	(0.5, 5.0)
n_{s}	Flat	(0.87, 1.07)
Ω_{b}	Flat	(0.03, 0.07)
h	Flat	(0.55, 0.91)
$10^3 \Omega_\nu h^2$	Flat	(0.60, 6.44)
w	Flat	(−2.0, −0.33)
Lens Galaxy Bias		
$b_i (i \in [1, 4])$	Flat	(0.8, 3.0)
Lens magnification		
C_{l}^1	Fixed	0.42
C_{l}^2	Fixed	0.30
C_{l}^3	Fixed	1.76
C_{l}^4	Fixed	1.94
Lens photo-z		
$\Delta z_{\mathrm{l}}^1 \times 10^2$	Gaussian	$(-0.9, 0.7)$
$\Delta z_{\mathrm{l}}^2 \times 10^2$	Gaussian	$(-3.5, 1.1)$
$\Delta z_{\mathrm{l}}^3 \times 10^2$	Gaussian	$(-0.5, 0.6)$
$\Delta z_{\mathrm{l}}^4 \times 10^2$	Gaussian	$(-0.7, 0.6)$
$\sigma_{z,\mathrm{l}}^1$	Gaussian	$(0.98, 0.06)$
$\sigma_{z,\mathrm{l}}^2$	Gaussian	$(1.31, 0.09)$
$\sigma_{z,\mathrm{l}}^3$	Gaussian	$(0.87, 0.05)$
$\sigma_{z,\mathrm{l}}^4$	Gaussian	$(0.92, 0.05)$
Intrinsic Alignment		
$a_i \ (i \in [1, 2])$	Flat	$(-5, 5)$
$\eta_i \ (i \in [1, 2])$	Flat	$(-5, 5)$
b_{TA}	Flat	$(0, 2)$
z_0	Fixed	0.62
Source photo-z		
$\Delta z_{\mathrm{s}}^1 \times 10^2$	Gaussian	$(0.0, 1.8)$
$\Delta z_{\mathrm{s}}^2 \times 10^2$	Gaussian	$(0.0, 1.5)$
$\Delta z_{\mathrm{s}}^3 \times 10^2$	Gaussian	$(0.0, 1.1)$
$\Delta z_{\mathrm{s}}^4 \times 10^2$	Gaussian	$(0.0, 1.7)$
Shear calibration		
$m^1 \times 10^2$	Gaussian	$(-0.6, 0.9)$
$m^2 \times 10^2$	Gaussian	$(-2.0, 0.8)$
$m^3 \times 10^2$	Gaussian	$(-2.4, 0.8)$
$m^4 \times 10^2$	Gaussian	$(-3.7, 0.8)$



3×2pt
3×2pt (ΛCDM-Opt.)
3×2pt (NL Bias)
3×2pt (Hyperrank)
$\gamma_t + w(\theta)$
$\xi_\pm$
DES Y1 3×2pt
3×2pt + BAO + SNe
Ext. Low-z
Planck (no lensing)
3×2pt + Local h
3×2pt + BAO + BBN
3×2pt + Ext. Low-z
3×2pt + Planck
3×2pt + All Ext.
0.7
0.8
0.9
$S_8 = \sigma_8(\Omega_m/0.3)^{0.5}$
0.25
0.3
0.35
Ω_m
0.7
0.8
0.9
σ_8



Fid. 3×2pt
Planck CMB
3×2pt+Planck CMB
ΛCDM
wCDM
σ_8
S_8
w
Ω_m