# Fluid Kinematics

Margarida Telo da Gama

Rodrigo Coelho

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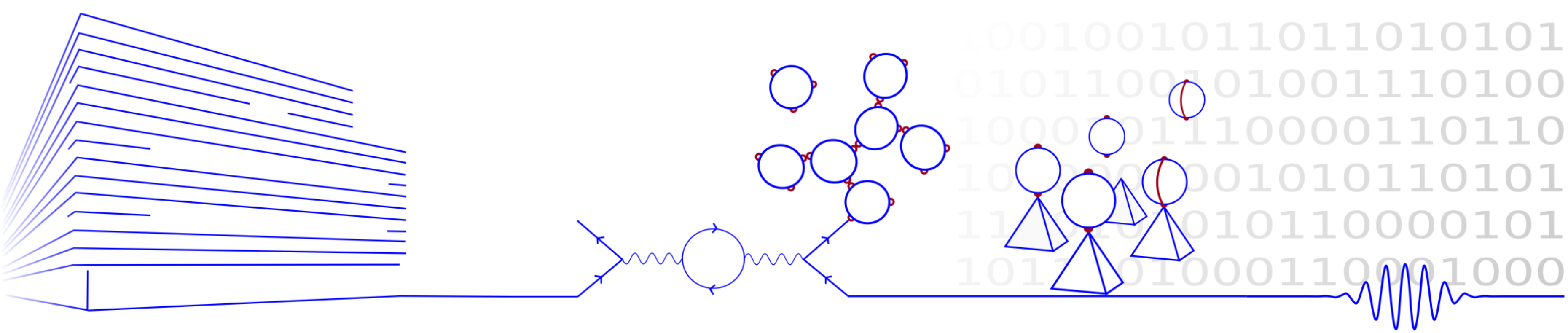


# Overview

- Fluid Kinematics deals with the motion of fluids without considering the forces and moments which create the motion.

- Reference: (Chap. 4) Fluid Mechanics: Fundamentals and Applications, by Çengel & Cimbala, McGraw-Hill series in mechanical engineering.

# What is a fluid ?

Tension (or stress): Force per unit area

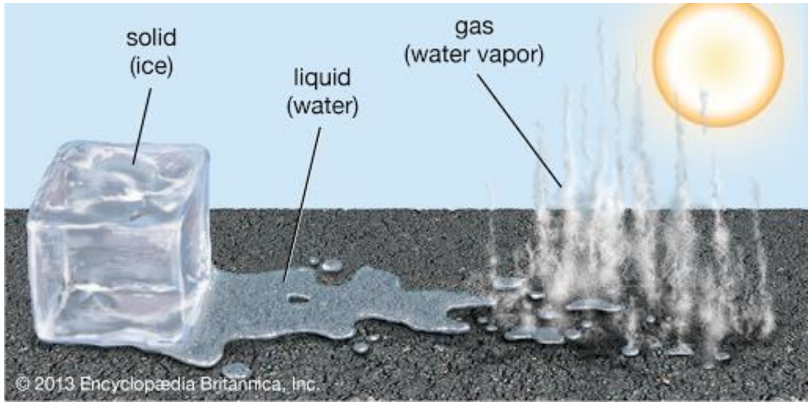
- Normal tension: perpendicular to the surface
- Shear tension: parallel to the surface

Materials respond differently to shear stresses:

- Solids deform non-permanently
- Plastics deform permanently
- Fluids do not resist: they flow

In a fluid at mechanical equilibrium the shear stresses are ZERO.

A fluid may be a gas or a liquid



# Lagrangian Description

- Lagrangian description of fluid flow tracks the position and velocity of individual particles.

- Based upon Newton's laws of motion.

- Difficult to use for practical flow analysis.
  - Fluids are composed of *billions* of molecules.
  - Interaction between molecules hard to describe/model.

- However, useful for specialized applications
  - Sprays, particles, bubble dynamics, rarefied gases.
  - Coupled Eulerian-Lagrangian methods.

- Named after Italian mathematician Joseph Louis Lagrange (1736-1813).

# Eulerian Description

- Eulerian description of fluid flow: a **flow domain** or **control volume** is defined by which fluid flows in and out.

- We define **field variables** which are functions of space and time.
  - Pressure field, P=P(x,y,z,t)
  - Velocity field, $\vec{V} = \vec{V}(x, y, z, t)$

$$\vec{V} = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$

  - Acceleration field,

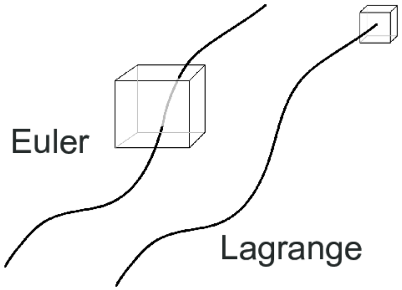
$$\vec{a} = \vec{a}(x, y, z, t)$$

$$\vec{a} = a_x(x, y, z, t)\vec{i} + a_y(x, y, z, t)\vec{j} + a_z(x, y, z, t)\vec{k}$$

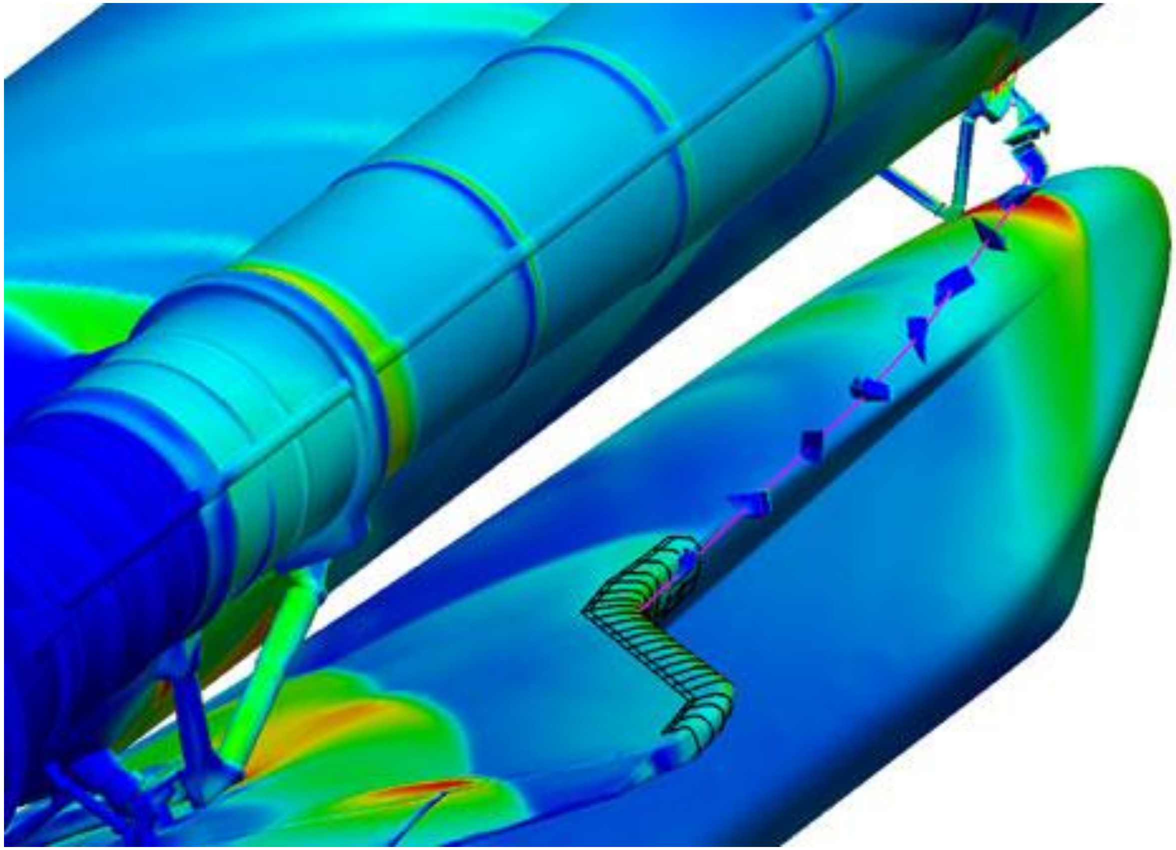
  - These (and other) field variables define the **flow field**.

- Well suited for formulation of initial boundary-value problems (PDE's).

- Named after Swiss mathematician Leonhard Euler (1707-1783).



# Example: Coupled Eulerian-Lagrangian Method



Forensic analysis of Columbia accident: simulation of shuttle debris trajectory using Eulerian CFD for flow field and Lagrangian method for the debris.

# Acceleration Field

- Consider a fluid particle and Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity.

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

- However, particle velocity at a point is the same as the fluid velocity,

$$\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

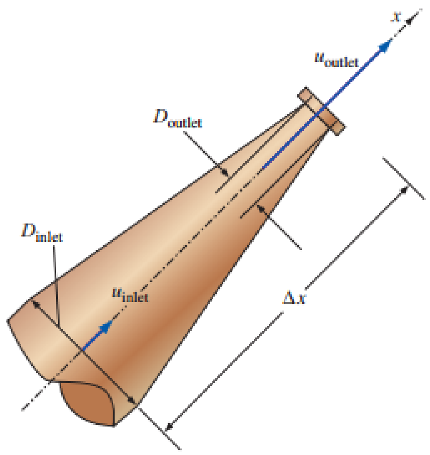
- To take the time derivative of $V_{particle}$ the chain rule must be used.

Obtain the expression for the acceleration and for the material derivative.

# Acceleration Field

**EXAMPLE 4–2 Acceleration of a Fluid Particle through a Nozzle**

Nadeen is washing her car, using a nozzle similar to the one sketched in Fig. 4–8. The nozzle is 3.90 in (0.325 ft) long, with an inlet diameter of 0.420 in (0.0350 ft) and an outlet diameter of 0.182 in (see Fig. 4–9). The volume flow rate through the garden hose (and through the nozzle) is $\dot{V}$ = 0.841 gal/min (0.00187 ft³/s), and the flow is steady. Estimate the magnitude of the acceleration of a fluid particle moving down the centerline of the nozzle.




Solve the problem and discuss the assumptions.

Answer: 160 ft/s^2

# Material Derivative

- The total derivative operator is called the **material derivative** and is often given special notation, D/Dt.

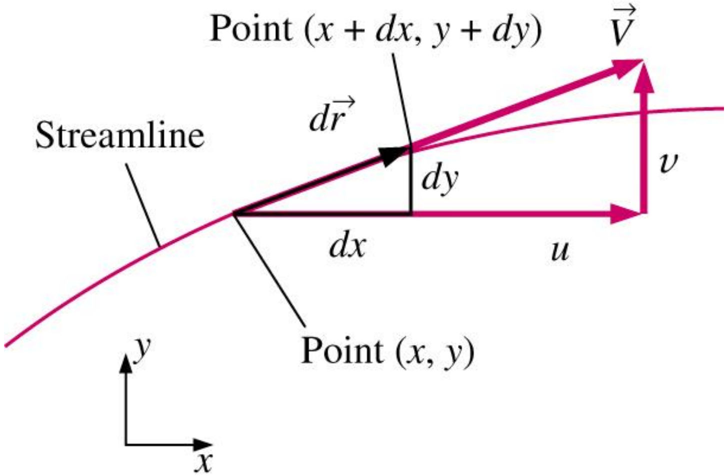
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

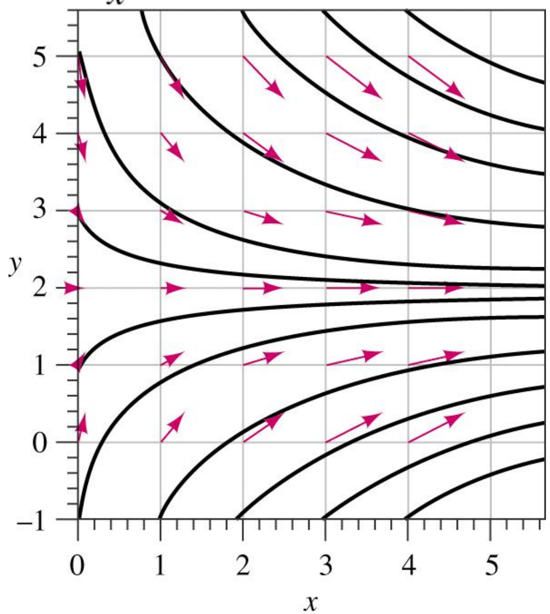
- Advective acceleration is nonlinear: source of many phenomena and primary challenge in solving fluid flow problems.
- Provides ``transformation'' between Lagrangian and Eulerian frames.
- Other names for the material derivative include: **total, particle,** and **substantial** derivative.

# Flow Visualization

- Flow visualization is the visual examination of flow-field features.
- Important for both physical experiments and numerical (CFD) solutions.
- Numerous methods
  - Streamlines and streamtubes
  - Pathlines
  - Streaklines
  - Timelines
  - Refractive techniques
  - Surface flow techniques

# Streamlines




- A **Streamline** is a curve that is everywhere tangent to the *instantaneous* local velocity vector.
- Consider an arc length

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

- $d\vec{r}$ must be parallel to the local velocity vector

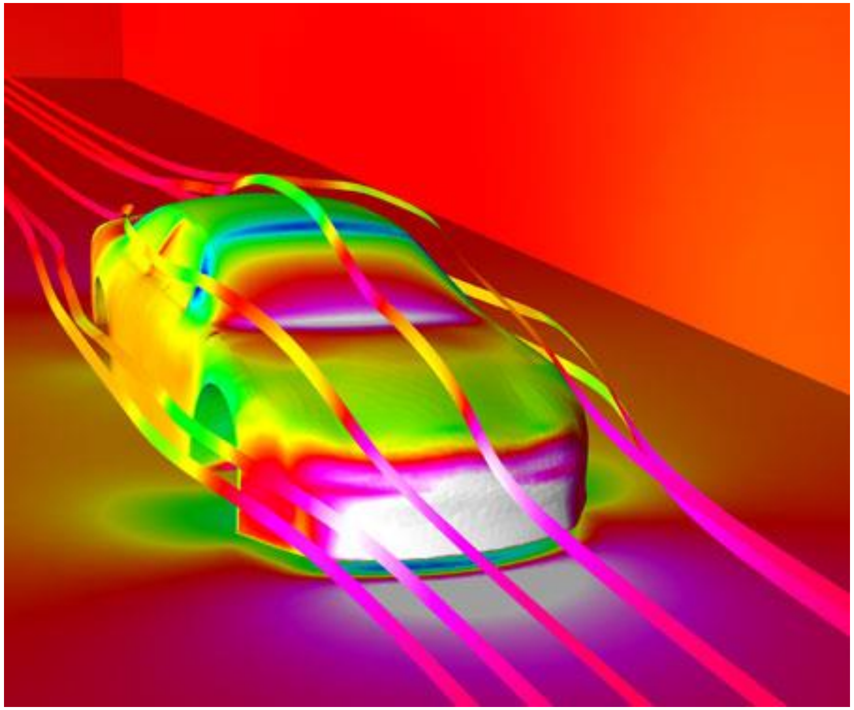
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- Geometric arguments results in the equation for a streamline

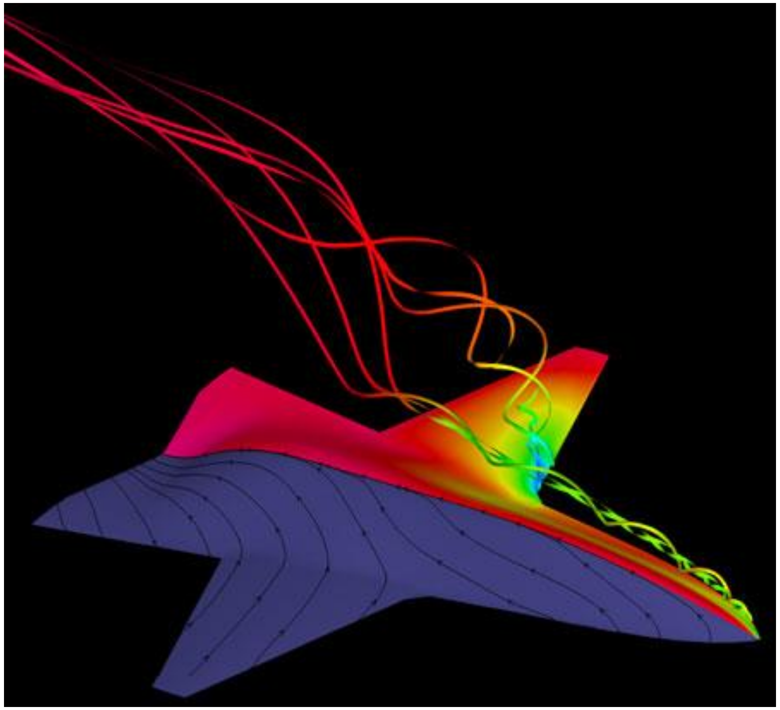
Expression

# Streamlines

NASCAR surface pressure contours and streamlines

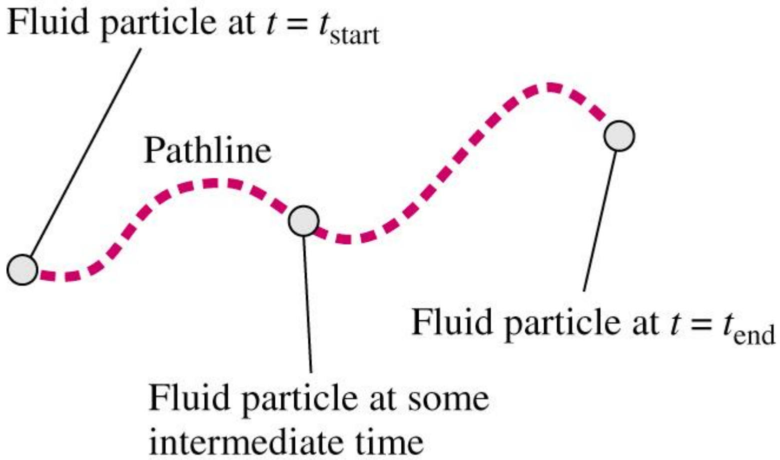


Airplane surface pressure contours, volume streamlines, and surface streamlines



Calculate the **streamlines** for the following velocity field: $v_x = sin(t) \text{ and } v_y = 1$

# Pathlines



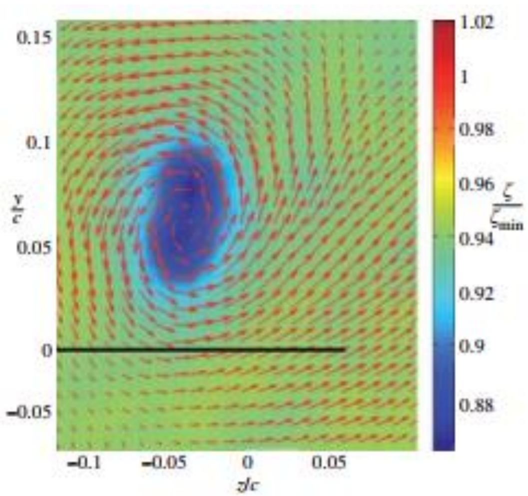




*Photo by Michael H. Krane, ARL-Penn State.*

- A **Pathline** is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

$$\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

- Particle location at time t:

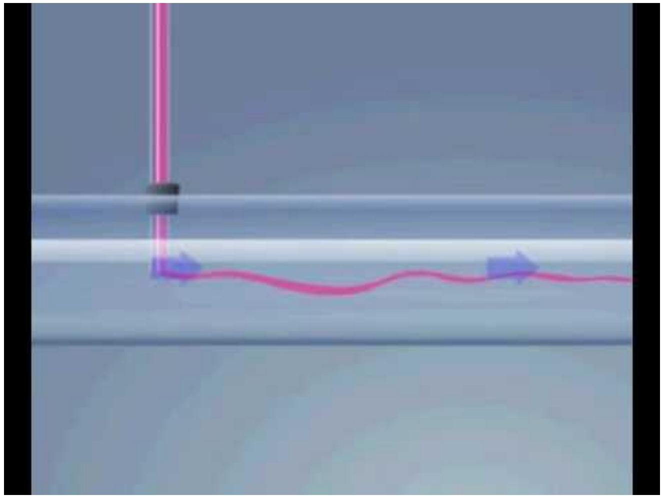
$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V} dt$$

- Particle Image Velocimetry (PIV) is a modern experimental technique to measure velocity field over a plane in the flow field.

Calculate the **pathlines** for the following velocity field: $v_x = sin(t)\ and\ v_y = 1$

# Streaklines

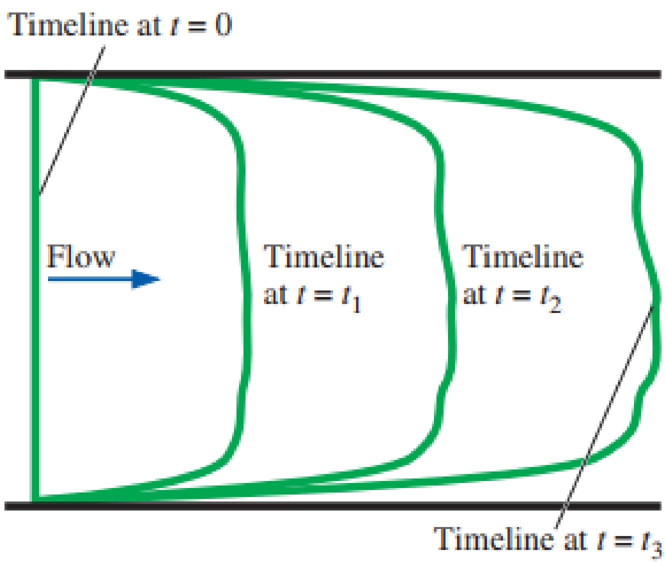
- A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.

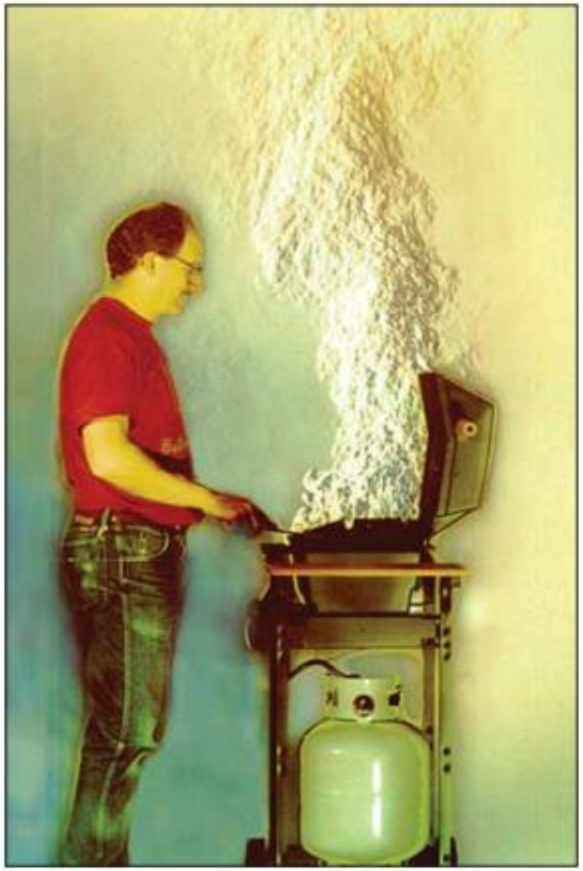


- Easy to generate in experiments:  dye in a water flow, or smoke in an airflow.

Refractive Flow Visualization Techniques

Timelines: A timeline is a set of adjacent fluid particles that were marked at the same (earlier) instant in time.



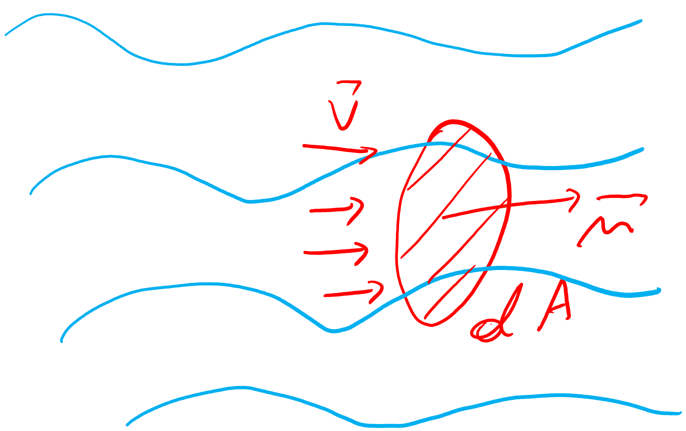
# Comparisons

- For steady flow, streamlines, pathlines, and streaklines are identical.
- For unsteady flow, they can be very different.
  - Streamlines are an instantaneous picture of the flow field.
  - Pathlines and Streaklines are flow patterns that have a time history associated with them.
  - Streakline:  instantaneous snapshot of a time-integrated flow pattern.
  - Pathline:  time-exposed flow path of an individual particle.

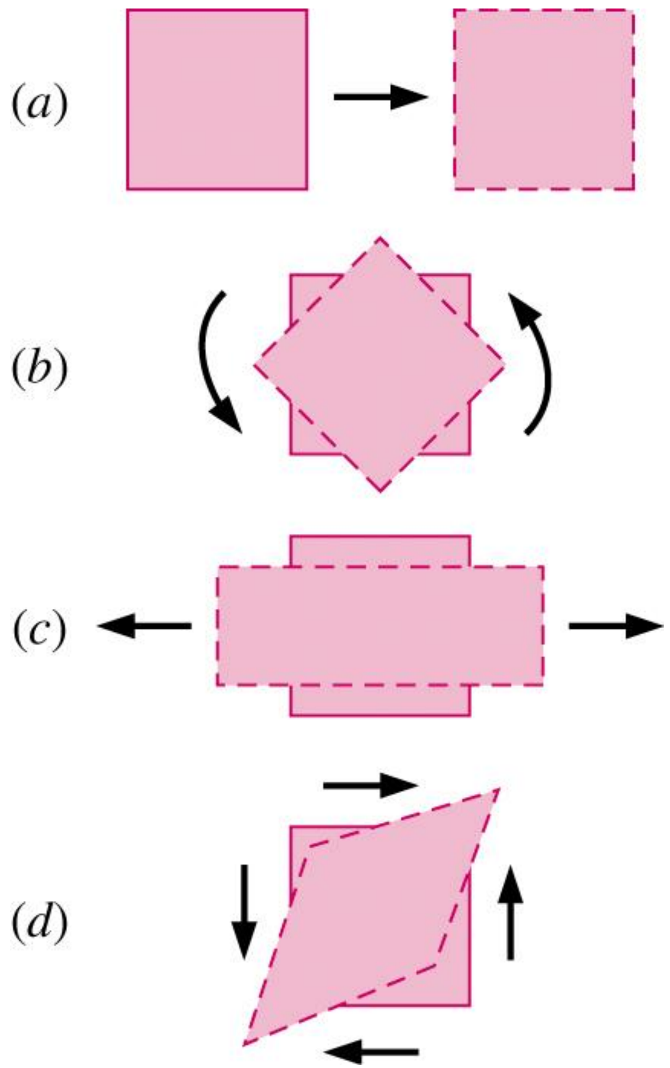
# Flow rate

- The **volumetric flow rate** is the volume of fluid which passes per unit time; usually it is represented by the symbol *Q*.

$$Q = \int \vec{V} \cdot \vec{n} dA$$




# Kinematic Description




- In fluid mechanics, an element may undergo four fundamental types of motion.
  - a) Translation
  - b) Rotation
  - c) Linear strain
  - d) Shear strain
- Because fluids are in constant motion and deformation, they are better described in terms of rates
  - a) velocity: rate of translation
  - b) angular velocity: rate of rotation
  - c) linear strain rate: rate of linear strain
  - d) shear strain rate: rate of shear strain

# Rate of Translation and Rotation

- To be useful, these rates must be expressed in terms of velocity and derivatives of velocity
- The **rate of translation vector** is described as the velocity vector. In Cartesian coordinates:

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- **Rate of rotation** at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point. The rate of rotation vector in Cartesian coordinates:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$= \boxed{\frac{1}{2}\nabla \times \vec{V}}$$

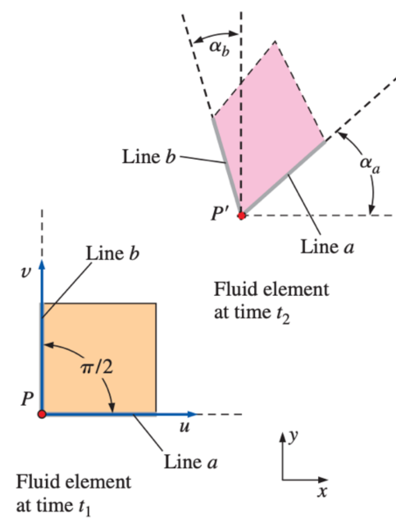


Tabela com o operador del em coordenadas cartesianas, cilíndricas e esféricas

| Operação | Coordenadas cartesianas $(x, y, z)$ | Coordenadas cilíndricas $(\rho, \varphi, z)$ | Coordenadas esféricas $(r, \theta, \varphi)$, onde $\varphi$ é o polar e $\theta$ é o ângulo azimutal [α] |
|---|---|---|---|
| campo vetorial $\mathbf{A}$ | $A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$ | $A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$ | $A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$ |
| Gradiente $\nabla f$ | $\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$ | $\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$ | $\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$ |
| Divergência $\nabla \cdot \mathbf{A}$ | $\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ | $\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$ | $\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$ |
| Rotacional $\nabla \times \mathbf{A}$ | $\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$ | $\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$ | $\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial \varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{\varphi}}$ |
| Operador de Laplace $\nabla^2 f \equiv \Delta f$ | $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ | $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$ | $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$ |
| Vetor de Laplace $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$ | $\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$ | $\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\rho}} + \left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}} + \nabla^2 A_z\hat{\mathbf{z}}$ | $\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial \theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\mathbf{r}} + \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\theta}}$ }} $+ \left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial \varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}}$ |

https://pt.wikipedia.org/wiki/Del_em_coordenadas_cil%C3%ADndricas_e_esf%C3%A9ricas

# Linear Strain Rate

- **Linear Strain Rate** is defined as the rate of increase in length per unit length.

  In Cartesian coordinates

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

- Volumetric strain rate in Cartesian coordinates

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Since the volume of a fluid element is constant for an incompressible flow, the volumetric strain rate must be zero.

Interpret the divergence

# Shear Strain Rate

- **Shear Strain Rate** at a point is defined as *half of the rate of decrease of the angle between two initially perpendicular lines that intersect at a point*.

- Shear strain rate can be expressed in Cartesian coordinates as:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

# Shear Strain Rate

We can combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain-rate tensor.**

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

Write in tensor form

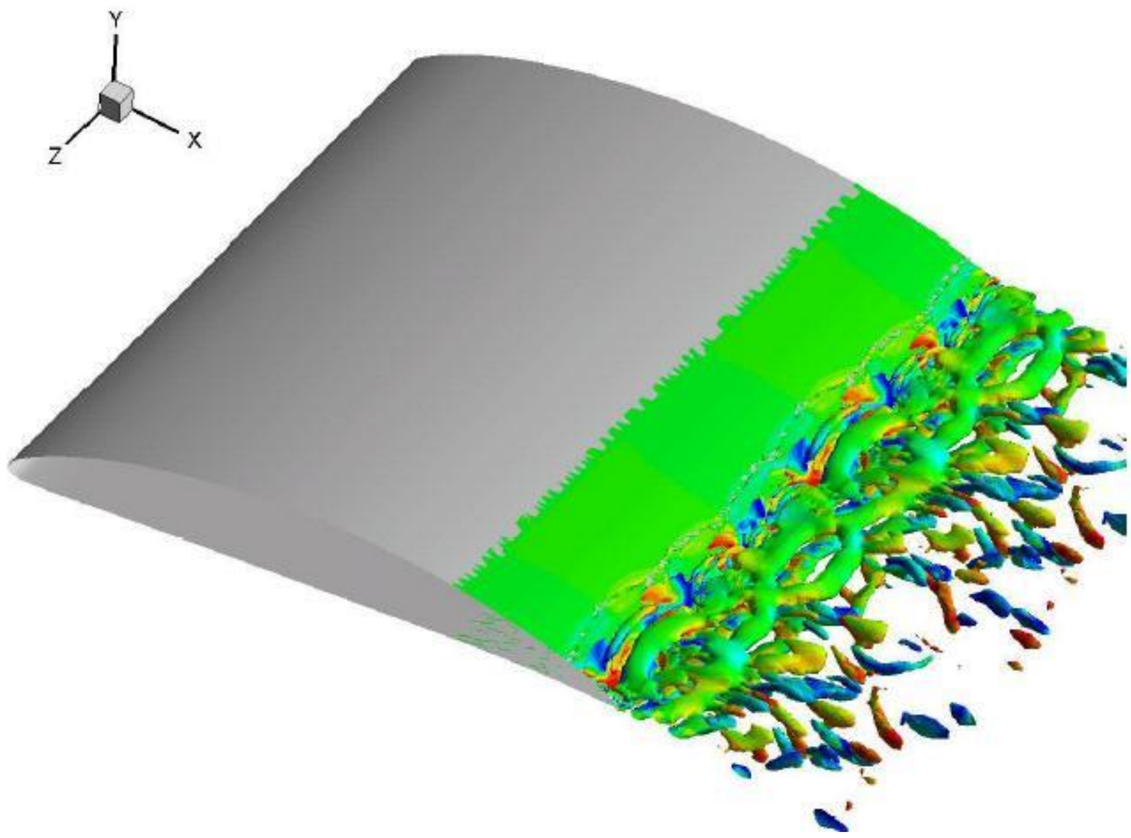
# Shear Strain Rate

- Purpose of our discussion of fluid element kinematics:
  - Better appreciation of the inherent complexity of fluid dynamics
  - Mathematical sophistication required to fully describe fluid motion

- Strain-rate tensor is important for numerous reasons. For example,
  - Develop relationships between fluid stress and strain rate.
  - Feature extraction and flow visualization in CFD simulations.

# Shear Strain Rate

Example: Visualization of trailing-edge turbulent eddies
for a hydrofoil with a beveled trailing edge




Feature extraction method is based upon eigen-analysis of the strain-rate tensor.

# Vorticity and Rotationality

- The **vorticity vector** is defined as the curl of the velocity vector

$$\vec{\zeta} = \vec{\nabla} \times \vec{V}$$

- Vorticity is equal to twice the angular velocity of a fluid particle.

$$\vec{\zeta} = 2\vec{\omega}$$

Cartesian coordinates

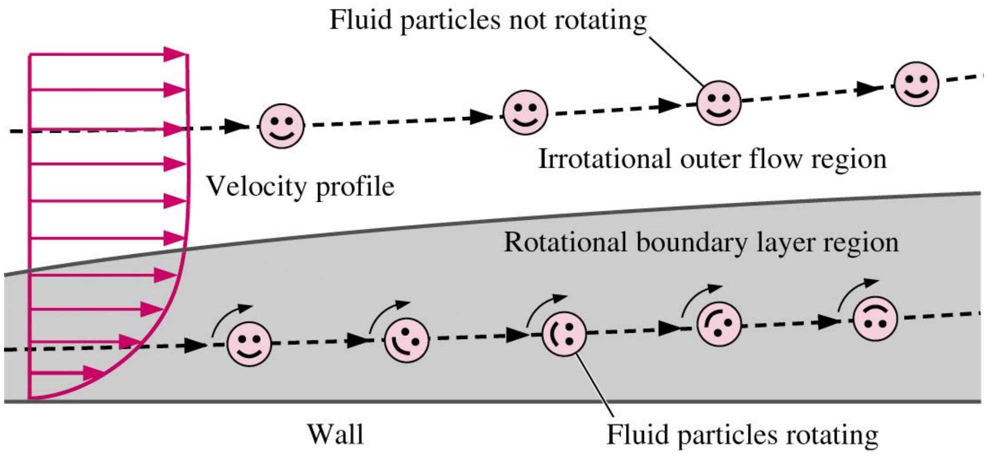
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

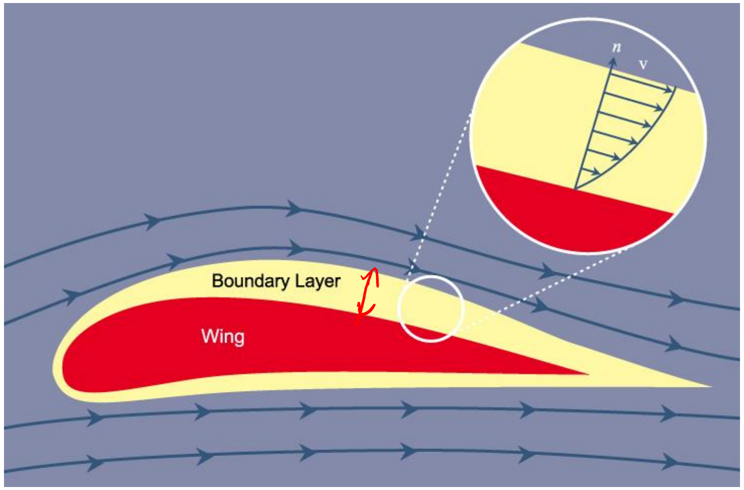
Cylindrical coordinates

$$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z \; \frac{1}{r}$$

- In regions where $\zeta = 0$, the flow is called **irrotational.**
- Elsewhere, the flow is called **rotational.**

# Vorticity and Rotationality



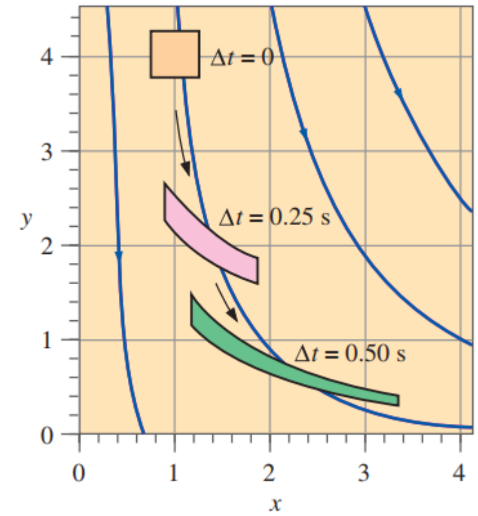



## EXAMPLE 4–8 Determination of Rotationality in a Two-Dimensional Flow

Consider the following steady, incompressible, two-dimensional velocity field:

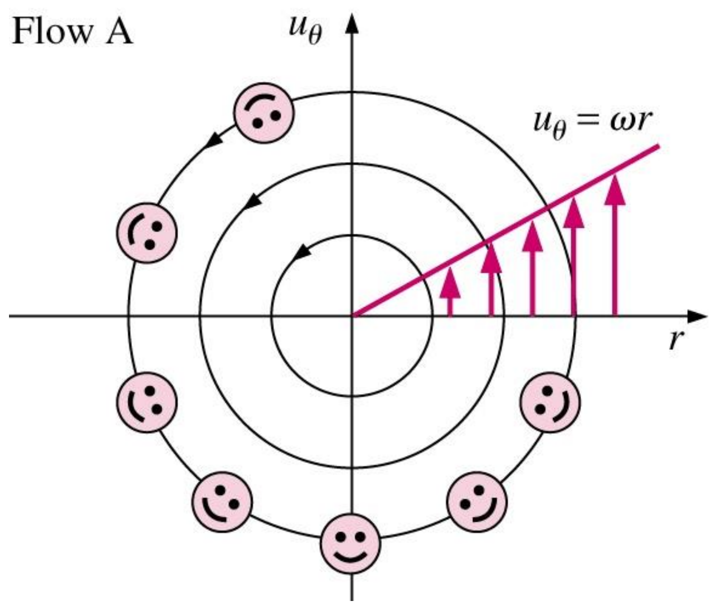
$$\vec{V} = (u, v) = x^2\vec{i} + (-2xy - 1)\vec{j} \tag{1}$$

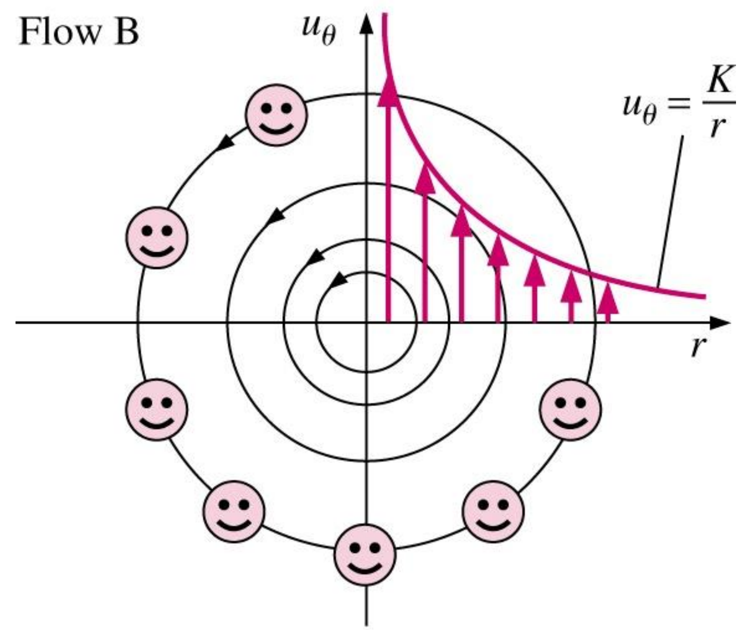
Is this flow rotational or irrotational? Sketch some streamlines in the first quadrant and discuss.



# Comparison of Two Circular Flows

Special case:  consider two flows with circular streamlines




(b)

$$u_r = 0, u_\theta = \omega r$$

$$u_r = 0, u_\theta = \frac{K}{r}$$

# Reynolds—Transport Theorem (RTT)

- A **system** is a quantity of matter of fixed identity. *No mass can cross a system boundary.*
- A **control volume** is a region in space chosen for study. Mass can cross a control surface.
- The fundamental conservation laws (conservation of mass, energy, and momentum) apply directly to systems.
- However, in most fluid mechanics problems, control volume analysis is preferred over system analysis (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to transform the conservation laws from a system to a control volume. This is accomplished with the Reynolds transport theorem (RTT).

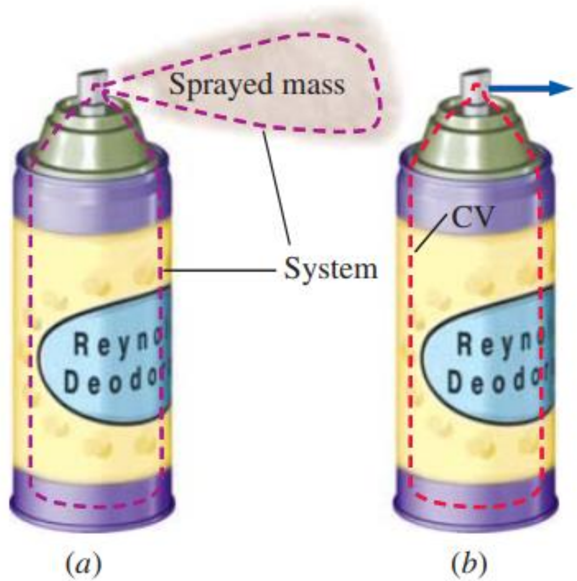
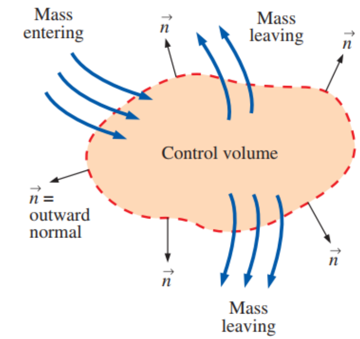



**FIGURE 4–53**

Two methods of analyzing the spraying of deodorant from a spray can: (*a*) We follow the fluid as it moves and deforms. This is the *system approach*—no mass crosses the boundary, and the total mass of the system remains fixed. (*b*) We consider a fixed interior volume of the can. This is the *control volume approach*—mass crosses the boundary.

# Reynolds—Transport Theorem (RTT)




$$\dot{B}_{net} = \dot{B}_{out} - \dot{B}_{in} = \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

- Material derivative (differential analysis):

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right) b$$

- General RTT (integral analysis):

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

control volume — control surface

$$b = \frac{B}{m}$$

- Interpretation of the RTT:
  - Time rate of change of the property B of the system is equal to (Term 1) + (Term 2)
  - Term 1: the time rate of change of B of the control volume
  - Term 2: the net flux of B out of the control volume by mass crossing the control surface
  - b=B/m (intensive property)

# Reynolds—Transport Theorem (RTT)

General

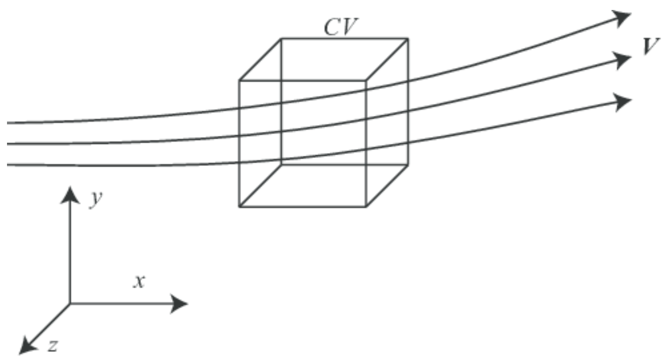
$$\frac{d}{dt}\int_{\Omega(t)} \mathbf{f}\, dV = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t}\, dV + \int_{\partial\Omega(t)} \left(\mathbf{v}^b \cdot \mathbf{n}\right) \mathbf{f}\, dA$$

Intensive property

$$f = \rho\, b$$

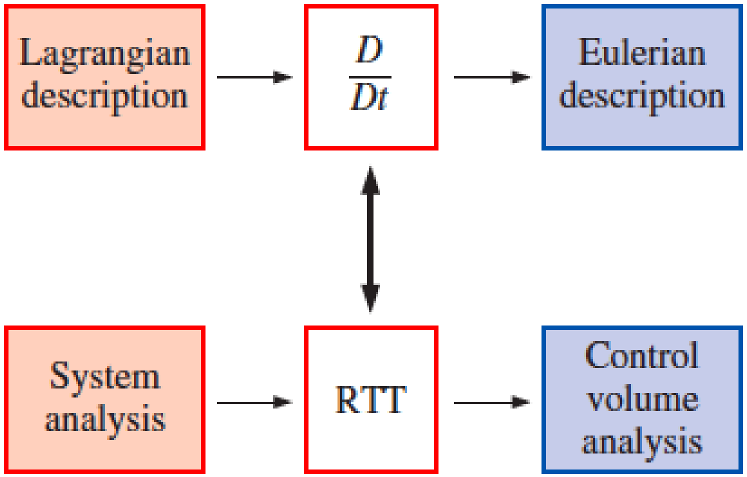
Material element

$$\mathbf{v}^b \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}.$$




https://en.wikipedia.org/wiki/Reynolds_transport_theorem

# Reynolds—Transport Theorem (RTT)




There is an analogy between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and that from systems to control volumes (for integral analysis using finite flow fields).

# Conservation of mass (continuity equation)

- Integral form

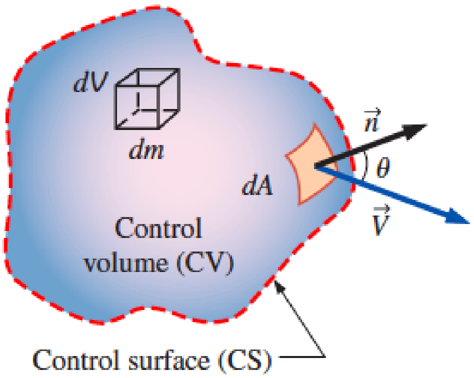
$b = 1$

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho\vec{V} \cdot d\vec{A} = 0$$

$\vec{J} = \rho \vec{V}$

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall \quad = \quad -\int_{CS} \rho\vec{V} \cdot d\vec{A}$$

Rate of increase of mass in CV = Net influx of mass




*General conservation of mass:*

$$\frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho(\vec{V}\cdot\vec{n}) \, dA = 0$$

$\int_{CV} \nabla\cdot(\rho\vec{V}) d\forall$

$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0$

# Conservation of mass (continuity equation)

- Differential form

- Use divergence theorem to transform the surface integral into a volume integral and equate the integrands,

$$\nabla \cdot (\rho \vec{V}) = -\frac{\partial \rho}{\partial t}$$

$\rho = C_{te} \Rightarrow \nabla \cdot \vec{V} = 0$

- For an incompressible fluid (constant density) the continuity equation reduces to

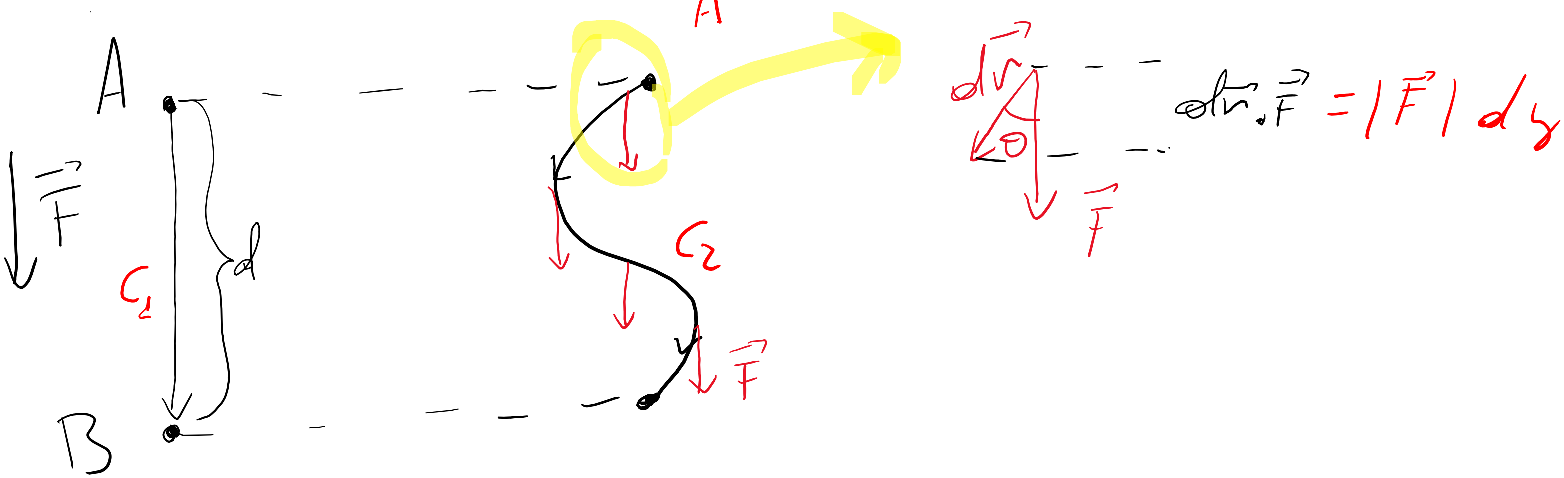
$$\nabla \cdot \vec{V} = 0$$

- The velocity field has ZERO divergence.

# Math review

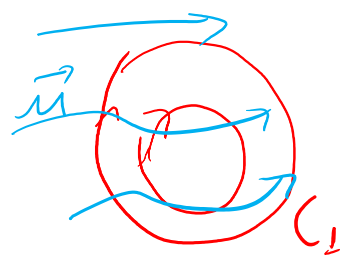
# Line integral

$$W = \int \vec{F} \cdot d\vec{r} = \int_A^B F\, dy = F \cdot d$$




Ex.: circulation

$$\Gamma = \oint \vec{u} \cdot d\vec{r}$$

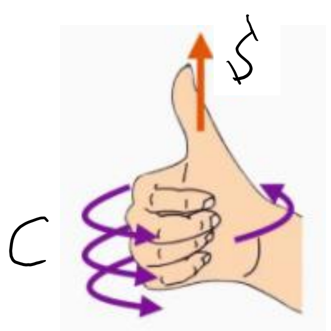


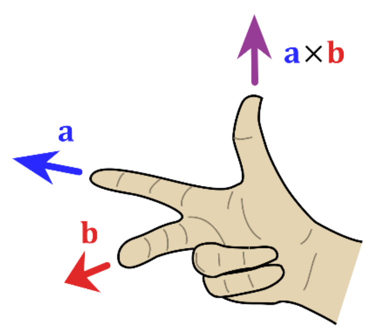
# Stokes theorem

Acheson (appendix)

$$\oint_C \vec{F} \cdot d\vec{l} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$$


The surface S is connected and limited by C. Use the right-hand rule to determine the direction of S according to C.



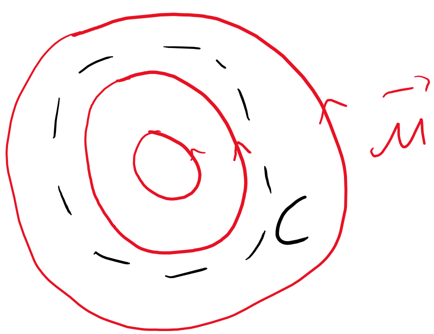




Other useful rule for cross product

$$\vec{c} = \vec{a} \times \vec{b}$$

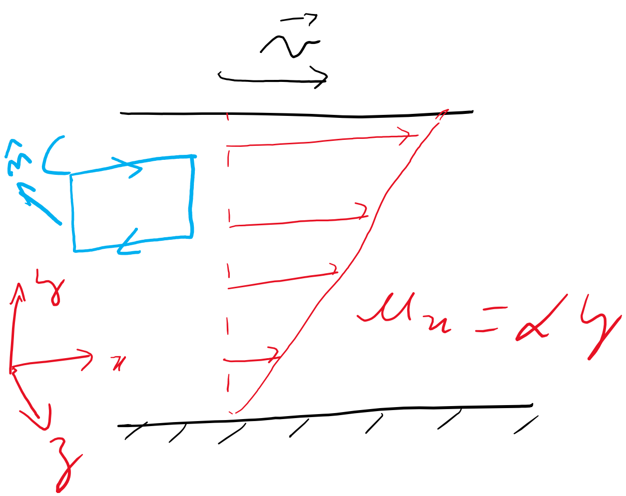
# Stokes theorem

Ex.: vorticity $\vec{\omega} = \nabla \times \vec{u}$



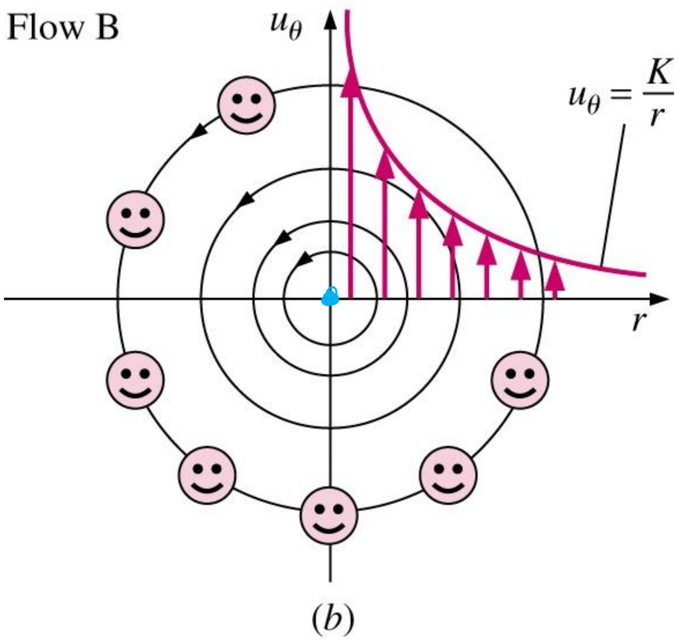

$$\int_S \nabla \times \vec{u} \cdot d\vec{S} = \oint_C \vec{u} \cdot d\vec{l} = \Gamma$$

Lid driven cavity




$$\vec{\omega} = \nabla \times \vec{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \alpha y & 0 & 0 \end{vmatrix}$$

$$= -\alpha \hat{z}$$




$\Gamma \neq 0 \quad , \quad u(r=0) \to \infty$

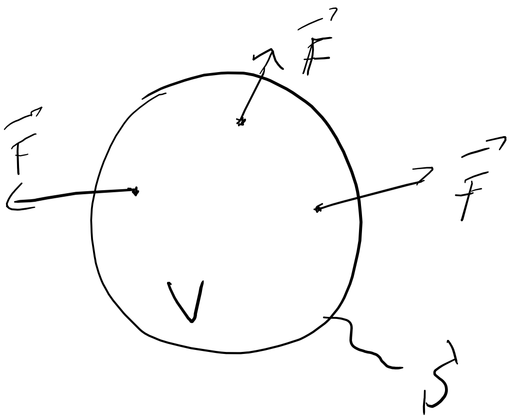
$$\int_S \nabla \times \vec{u} \cdot d\vec{S} \neq \int_C \vec{u} \cdot d\vec{l} = \Gamma$$

$$u_r = 0, u_\theta = \frac{K}{r}$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

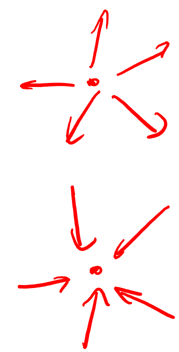
# Gauss theorem

$$\int_S \vec{F}\cdot d\vec{S} = \int_V \nabla\cdot\vec{F}\, dV$$




Ex.:

$$\begin{cases} \nabla\cdot\vec{u} > 0 \\ \nabla\cdot\vec{u} < 0 \end{cases}$$



# Kronecker Delta

$j = \{x, y, z\}$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Einstein notation

$$A_j B_j = \sum_{j=1}^{3} A_j B_j$$

$$= A_x B_x + A_y B_y + A_z B_z$$

$$= \vec{A} \cdot \vec{B}$$

Ex.:

$$B_j = A_i \delta_{ij} = A_x \delta_{xj} + A_y \delta_{yj} + A_z \delta_{zj} = A_j$$

$$C = A_i B_j \delta_{ij} = A_j B_j = \vec{A} \cdot \vec{B}$$

# Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$

ex.:

$\varepsilon_{xyz} = 1$

$\varepsilon_{xzy} = -1$

$\varepsilon_{zxy} = 1$

$\varepsilon_{zxx} = 0$

Properties

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij} = \dots$$

$$\varepsilon_{ijk} \cdot \varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

índices na ordem — Trocados

Ex.: $\vec{A} \times \vec{B} = \varepsilon_{ijk} A_j B_k$

índice livre

(Acheson)

## A.1. Vector identities

$$(\boldsymbol{a} \wedge \boldsymbol{b}) \wedge \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}, \tag{A.1}$$

$$\nabla \wedge \nabla\phi = 0, \qquad \nabla \cdot (\nabla \wedge \boldsymbol{F}) = 0, \qquad \text{(A.2, A.3)}$$

$$\nabla \cdot (\phi\boldsymbol{F}) = \phi\nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla\phi, \tag{A.4}$$

$$\nabla \wedge (\phi\boldsymbol{F}) = \phi\nabla \wedge \boldsymbol{F} + (\nabla\phi) \wedge \boldsymbol{F}, \tag{A.5}$$

$$\nabla \wedge (\boldsymbol{F} \wedge \boldsymbol{G}) = (\boldsymbol{G} \cdot \nabla)\boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + \boldsymbol{F}(\nabla \cdot \boldsymbol{G}) - \boldsymbol{G}(\nabla \cdot \boldsymbol{F}), \tag{A.6}$$

$$\nabla \cdot (\boldsymbol{F} \wedge \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \wedge \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \wedge \boldsymbol{G}), \tag{A.7}$$

$$\nabla(\boldsymbol{F} \cdot \boldsymbol{G}) = \boldsymbol{F} \wedge (\nabla \wedge \boldsymbol{G}) + \boldsymbol{G} \wedge (\nabla \wedge \boldsymbol{F}) + (\boldsymbol{F} \cdot \nabla)\boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\boldsymbol{F}, \tag{A.8}$$

$$(\boldsymbol{F} \cdot \nabla)\boldsymbol{F} = (\nabla \wedge \boldsymbol{F}) \wedge \boldsymbol{F} + \nabla(\tfrac{1}{2}\boldsymbol{F}^2), \tag{A.9}$$

$$\nabla^2\boldsymbol{F} = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla \wedge (\nabla \wedge \boldsymbol{F}). \tag{A.10}$$

Show that:

$$\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

$$= \varepsilon_{ijk}\,\partial_j\,(\nabla \times \vec{u})_k = \varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,\partial_l\,u_m$$

$$= \underbrace{\varepsilon_{ijk}\,\varepsilon_{klm}}\,\partial_j\,\partial_l\,u_m$$

$$-\varepsilon_{ikj}\,\varepsilon_{klm} = \varepsilon_{kij}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$$= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\,\partial_j\,\partial_l\,u_m$$

$$= \delta_{il}\,\delta_{jm}\,\partial_j\,\partial_{\cancel{l}\,i}\,u_{\cancel{m}\,j} - \delta_{im}\,\delta_{jl}\,\partial_j\,\partial_{\cancel{l}\,j}\,u_{\cancel{m}\,i}$$

$$= \partial_j\,\partial_i\,u_j - \partial_j\,\partial_j\,u_i = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u} \quad \square$$