if dipole transition are forbiden then life time will increas by a factor of $\left(\frac{\lambda}{a}\right)^2$ which may be estimated to give

$$\tau \sim 10^{-2} s.$$

---

§. Variational method.

this method is based on the following statement for E_0 of a certaine $\hat{H}$ (not possible to solve)

$$E_0 \leqslant \langle \psi | \hat{H} | \psi \rangle$$, for any ψ such that

$$\langle \psi | \psi \rangle = 1 .$$

Proof : assum $\hat{H} | \psi_n \rangle = E_n | \psi_n \rangle$

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n | n \rangle \quad , \quad \sum_n |C_n|^2 = 1$$

$$\langle \psi | \hat{H} | \psi \rangle = \sum_{n'=0}^{\infty} C_{n'}^* \langle n' | \hat{H} \sum_n C_n | n \rangle =$$

$$= \sum_{n'n=0}^{\infty} C_{n'}^* C_n E_n \langle n' | n \rangle =$$

$$\sum_{n',n=0}^{\infty} C_{n'}^* C_n E_n \delta_{nn'} = \sum_{n=0} E_n |C_n|^2 \geqslant E_0 \sum_{n=0}^{\infty} |C_n|^2 \geqslant$$

$$\geqslant E_0 \; !$$

Select certain trial function

$$\psi(q; \underbrace{a_1, \ldots, a_n}_{\text{variational parameters}})$$

$$\int |\psi_0|^2 \, dq = 1$$

$$E_0(a_1, \dots, a_n) = \int \psi_0^* \hat{H} \psi_0 \, dq$$

E_0 from the minimal condition

$$\frac{\partial E_0}{\partial a_i} = 0 \quad , i = 1, \dots, n. \quad \Rightarrow \bar{a}_i \quad \text{and}$$

Ground state estimate: $E_0 = E_0(\bar{a}_1, \dots, \bar{a}_2)$.

Similery we may calculate next energy level.

$\psi_1(q; a_1, \dots, a_n)$; $\langle \psi_1 | \psi_1 \rangle = 0$ and

$\langle \psi_1 | \psi_0 \rangle = 0$

$$\begin{cases} E_1 = \min\limits_{a_i} \int \psi_1^* \hat{H} \psi_1 \, dq \\ \langle \psi_1 | \psi_1 \rangle = 1 \\ \langle \psi_1 | \psi_0 \rangle = 0 \end{cases}$$

$$\psi_1 = \sum_{n=1} b_n | n \rangle$$

$$\langle \psi_1 | \psi_1 \rangle = \sum_{n=1}^{\infty} |b_n|^2 E_n \leq \left[\sum_{n=1}^{\infty} |b_n|^2 \right] \cdot E_1 \leq E_1$$

and so on.

Example: " Anharmonic oscillator".

$$\hat{H} = \frac{\hat{p}^2}{2m} + \alpha x^4$$

find the energy of a ground state using Variational method:

$x \to -x$ symetry therefore we select the trial function even:

$$\psi = C e^{-\frac{m\omega x^2}{2\hbar}}$$

, where ω is the variational parameter.

$$1 = \int |\psi|^2 \to C = \left[\frac{m\omega}{\pi\hbar}\right]^{1/4}$$

$$E_0(\omega) = \langle \hat{H} \rangle = \left\langle \frac{\hat{p}^2}{2m} \right\rangle + \alpha \langle x^4 \rangle$$

it is easy to calculate

$$\left\langle \frac{\hat{p}}{2m} \right\rangle = \frac{\hbar\omega}{4}$$

$$\langle \hat{x}^4 \rangle = 3\left(\frac{\hbar}{2m\omega}\right)^2$$

$$E_0(\omega) = \frac{\hbar\omega}{4} + 3\alpha\left(\frac{\hbar}{2m\omega}\right)^2$$

$$\frac{dE_0}{d\omega} = 0 \Rightarrow \frac{\hbar}{4} - \frac{3}{4}\alpha\frac{\hbar^2}{m^2}\frac{2}{\omega^3} = 0$$

or $$\omega = \left(6\alpha\frac{\hbar}{m^2}\right)^{1/3}$$

which renders.

$$E_0^{V.P} = \min_{\omega} E(\omega) = \frac{3}{8}\left(6\alpha \frac{\hbar^4}{m^2}\right)^{1/3} . \rightarrow \text{upper bound.}$$

E.g. perturbation theory gives.

$$E_{pert.} \frac{1}{2}\left(3\alpha \frac{\hbar}{m^2}\right)^{1/3}$$

from Heisenberg uncertainty principle

we may estimate

$$E_0^{H.P.} \geq \frac{3}{8}\left(2\alpha \frac{\hbar^4}{m^2}\right)^{1/3}$$

Such that. the true value is

$$E_0^{H.P} \leq E_0 \leq E_0^{V.P}$$

$$\frac{3}{8}\left(2\alpha \frac{\hbar^4}{m^2}\right)^{1/3} \leq E_0 \leq \frac{3}{8}\left(6\alpha \frac{\hbar^4}{m^2}\right)^{1/3}$$
