The Dirac Equation

Rui Santos

FCUL & CFTC

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Non-relativistic energy and plane waves

The Schrödinger equation is based on the non-relativistic energy relation

$$E = \frac{p^2}{2m}.$$

Consider a plane wave solution

$$\psi(x,t) = e^{i(px-\omega t)}.$$

Then

$$\frac{\partial \psi}{\partial t} = -i\omega\psi, \qquad \nabla\psi = i\mathrm{p}\psi.$$

Using the relation

$$E = \hbar\omega,$$

we obtain the operator correspondences

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \mathrm{p} \to -i\hbar\nabla.$$

Derivation of the Schrödinger equation

Start from the Hamiltonian

$$H = T + V, \qquad H\psi = E\psi,$$

with kinetic energy

$$T = \frac{p^2}{2m}.$$

Replacing the classical quantities by operators,

$$\mathrm{p} \to -i\hbar\nabla, \qquad E \to i\hbar\frac{\partial}{\partial t},$$

we obtain

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}.$$

$$\boxed{i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi}$$

Time-dependent Schrödinger equation.

Free particle solution and probability density

For a free particle ($V = 0$) and setting $\hbar = 1$:

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2m}\nabla^2\psi = 0$$

A plane-wave solution is

$$\psi = Ne^{i(\mathrm{p}\cdot\mathrm{x} - Et)}.$$

The probability density is

$$\rho = |\psi|^2 = |N|^2 \geq 0.$$

Normalization condition:

$$\int d^3x\,\rho = 1.$$

Thus $|\psi|^2$ can be interpreted as a probability density.

Continuity equation

Starting from the Schrödinger equation and its complex conjugate, one obtains

$$\frac{\partial}{\partial t}(\psi^*\psi) - \frac{1}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = 0.$$

Using the identity

$$\psi^*\nabla^2\psi - \psi\nabla^2\psi^* = \nabla\cdot(\psi^*\nabla\psi - \psi\nabla\psi^*),$$

we obtain the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0,$$

with probability current

$$\vec{j} = \frac{1}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*).$$

Continuity Equation from Klein–Gordon

Start from the Klein–Gordon equation:

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

and its complex conjugate:

$$(\partial_\mu \partial^\mu + m^2)\phi^* = 0$$

Multiply:

$$i\phi^*(\partial_\mu \partial^\mu \phi + m^2\phi) = 0; \qquad -i\phi(\partial_\mu \partial^\mu \phi^* + m^2\phi^*) = 0$$

Subtract the two equations:

$$i\phi^* \partial_\mu \partial^\mu \phi - i\phi \partial_\mu \partial^\mu \phi^* = 0$$

Rewrite as a total derivative and get the conserved current

$$\partial_\mu \big[i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \big] = 0 \qquad j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*),$$

Plane waves and the problem of probability

For a plane–wave solution

$$\phi = N e^{i(\mathrm{p}\cdot\mathrm{x} - Et)}$$

we obtain

$$\frac{\partial \phi}{\partial t} = -iE\phi \qquad \frac{\partial \phi^*}{\partial t} = +iE\phi^*.$$

The density becomes

$$\rho = i\left(\phi^* \partial_t \phi - \phi \partial_t \phi^*\right) = 2E|\phi|^2 = 2E|N|^2.$$

From the relativistic dispersion relation

$$E^2 = p^2 + m^2 \qquad E = \pm\sqrt{p^2 + m^2}.$$

Thus ρ is not positive definite, so it cannot represent a probability density.

$$\rho = j^0$$

is instead interpreted as a charge density.

The Dirac Equation: Linearizing the Relativistic Relation

Dirac wanted an equation that was relativistically invariant, first order in time, had a positive probability density and would reduce to Schrödinger's at low energies.

$$i\partial_t\psi = (\boldsymbol{\alpha}\cdot \mathrm{p} + \beta m)\psi$$

We want

$$H^2 = \mathrm{p}^2 + m^2$$

to reproduce the relativistic energy relation.

Conditions on the Matrices

Expanding

$$H = \boldsymbol{\alpha} \cdot \mathrm{p} + \beta m$$

gives

$$H^2 = \alpha_i \alpha_j p_i p_j + \alpha_i \beta p_i m + \beta \alpha_i m p_i + \beta^2 m^2$$

To obtain

$$H^2 = \mathrm{p}^2 + m^2$$

we require (and the constants have to be matrices)

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}; \qquad \{\alpha_i, \beta\} = 0; \qquad \beta^2 = 1$$

Define

$$\gamma^0 = \beta; \qquad \gamma^i = \beta \alpha^i$$

Gamma Matrices

Then

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

This is the Clifford algebra of Minkowski spacetime.
The Dirac equation becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Dirac/Bjorken–Drell Representation - many possibilities exist for choosing the γ matrices.

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using these matrices, the Klein–Gordon operator factorizes:

$$(\gamma^\mu p_\mu - m)(\gamma^\nu p_\nu + m) = 0.$$

This acts on a 4-component wavefunction (Dirac spinor)

$$\psi^T = (\psi_1, \psi_2, \psi_3, \psi_4).$$

From $(i\gamma^\mu \partial_\mu - m)\psi = 0$ we obtain the Dirac equation.

Taking the Hermitian conjugate:

$$(i\gamma^\mu \partial_\mu \psi - m\psi)^\dagger = 0; \qquad \Rightarrow -i(\partial_\mu \psi^\dagger)(\gamma^\mu)^\dagger - m\psi^\dagger = 0.$$

By inspection:

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^0 \gamma^i \gamma^0.$$

Conserved Current of the Dirac Field

Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Adjoint equation (multiply by γ^0 and define $\bar{\psi} = \psi^\dagger \gamma^0$):

$$\bar{\psi}(i\overleftarrow{\partial_\mu}\gamma^\mu + m) = 0$$

Multiply the Dirac equation by $\bar{\psi}$ and the adjoint equation by ψ:

$$i\bar{\psi}\gamma^\mu(\partial_\mu \psi) - m\bar{\psi}\psi = 0; \qquad i(\partial_\mu \bar{\psi})\gamma^\mu \psi + m\bar{\psi}\psi = 0$$

Adding the two equations:

$$\partial_\mu(\bar{\psi}\gamma^\mu \psi) = 0$$

Conserved current

$$j^\mu = \bar{\psi}\gamma^\mu \psi$$

Consistency Check: Dirac ⇒ Klein–Gordon

Start from the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

Multiply on the left by $(i\gamma^\nu\partial_\nu + m)$:

$$(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi = 0$$

Expanding:

$$-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu\psi - m^2\psi = 0$$

Using the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

we obtain

$$(\partial_\mu\partial^\mu + m^2)\psi = 0$$

Every component of the Dirac spinor satisfies the Klein–Gordon equation.

The Dirac Spinor

The Dirac wavefunction has four components:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

It can be written as two two-component spinors:

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}; \qquad \phi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \chi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}.$$

Physical interpretation

- Two components $\rightarrow$ spin up / spin down
- Positive-energy solutions $\rightarrow$ particles
- Negative-energy solutions $\rightarrow$ antiparticles

Solutions of the Dirac Equation

Rest frame: choose $\vec{p} = 0$

$$(i\gamma^0\partial_t - m)\psi = 0 \quad \Rightarrow \quad i\frac{\partial}{\partial t}\psi = m\gamma^0\psi$$

Component form:

$$i\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = m\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

Solutions:

$$\psi_{1,2} \sim e^{-imt} \quad \text{(positive energy)}$$

$$\psi_{3,4} \sim e^{+imt} \quad \text{(negative energy)}.$$

General plane-wave solutions:

$$\psi_u(x) = e^{-ip\cdot x} u(p,\lambda), \quad (\not{p} - m) u(p,\lambda) = 0$$

$$\psi_v(x) = e^{+ip\cdot x} v(p,\lambda), \quad (\not{p} + m) v(p,\lambda) = 0$$

Dirac equation: plane-wave solutions

Start from the Dirac equation

$$(\gamma^\mu p_\mu - m)\, u(p) = 0.$$

In matrix form:

$$\begin{pmatrix} E - m & -\boldsymbol{\sigma} \cdot \mathrm{p} \\ \boldsymbol{\sigma} \cdot \mathrm{p} & -(E + m) \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0.$$

This gives the coupled equations:

$$(E - m) u_A = (\boldsymbol{\sigma} \cdot \mathrm{p})\, u_B,$$

$$(E + m) u_B = (\boldsymbol{\sigma} \cdot \mathrm{p})\, u_A.$$

Solving:

$$u_B = \frac{\boldsymbol{\sigma} \cdot \mathrm{p}}{E + m} u_A.$$

Choose basis spinors:

$$u_A^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_A^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Full solutions:

$$u^{(s)} = N \begin{pmatrix} u_A^{(s)} \\ \frac{\boldsymbol{\sigma} \cdot \mathrm{p}}{E + m} \nu_A^{(s)} \end{pmatrix}.$$

Dirac spinors: normalization and spin sums

Choose normalization

$$N(E) = \sqrt{\frac{E+m}{2m}}.$$

Positive-energy spinors:

$$u^{(s)}(p) = N(E) \begin{pmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma} \cdot \mathrm{p}}{E+m} \chi^{(s)} \end{pmatrix}, \qquad s = \pm\tfrac{1}{2}.$$

Negative-energy spinors:

$$v^{(s)}(p) = N(E) \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathrm{p}}{E+m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}.$$

Spin sums (completeness relations):

$$\sum_s u^{(s)}(p)\, \bar{u}^{(s)}(p) = \not{p} + m,$$

$$\sum_s v^{(s)}(p)\, \bar{v}^{(s)}(p) = \not{p} - m.$$

Dirac equation and negative energy solutions

From the relativistic dispersion relation:

$$E^2 = \mathrm{p}^2 + m^2 \quad \Rightarrow \quad E = \pm\sqrt{\mathrm{p}^2 + m^2}.$$

Problem: the Dirac equation has negative-energy solutions.

$$(\gamma^\mu p_\mu - m)\psi = 0 \quad \Rightarrow \quad E < 0 \text{ solutions exist.}$$

Naively, this would allow transitions to arbitrarily low energy.
Resolution (Dirac / Feynman–Stückelberg):
Negative-energy solutions are reinterpreted as

positive-energy antiparticles moving forward in time.

- Particle with $E > 0$ moving forward in time
- Antiparticle with $E > 0$ moving forward in time

Physical interpretation

Plane-wave solutions:

$$\psi \sim e^{-iEt + i\mathrm{p}\cdot\mathrm{x}}$$

For negative energy:

$$E < 0 \quad \Rightarrow \quad e^{+i|E|t}$$

Reinterpretation:

$$\text{negative energy} + \text{backward time} \iff \text{positive energy antiparticle.}$$

Dirac equation predicts antiparticles.

Spinors: Rest Frame vs Moving Frame

Rest frame ($\mathrm{p} = 0$):
Positive-energy Dirac spinors:

$$u^{(s)} = \begin{pmatrix} \chi^{(s)} \\ 0 \end{pmatrix}$$

with Pauli spinors

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Moving particle ($\mathrm{p} \neq 0$):

$$u(p) = \begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma}\cdot\mathrm{p}}{E+m}\chi \end{pmatrix}$$

Projectors and physical interpretation

The spin sums act as projectors:

$$\Lambda_+(p) = \frac{\not{p} + m}{2m}, \qquad \Lambda_-(p) = \frac{\not{p} - m}{2m}.$$

They project onto:

- positive-energy states (particles)
- negative-energy states (antiparticles)

Properties:

$$\Lambda_\pm^2 = \Lambda_\pm, \qquad \Lambda_+\Lambda_- = 0.$$

Dirac spinors form a complete basis of solutions.

Derivation of the spin sum

Start from the Dirac equation:

$$(\not{p} - m)u^{(s)}(p) = 0.$$

We want to compute:

$$\sum_s u^{(s)}(p)\,\bar{u}^{(s)}(p).$$

Step 1: Ansatz

The result must be Lorentz covariant, so it can only depend on:

$$\not{p}, \quad m.$$

Thus write

$$\sum_s u^{(s)}\,\bar{u}^{(s)} = A\,\not{p} + B\,m.$$

Step 2: Use the Dirac equation

Multiply on the right by $(\not{p} - m)$:

$$\sum_s u^{(s)} \bar{u}^{(s)} (\not{p} - m) = 0.$$

Insert the ansatz:

$$(A\not{p} + Bm)(\not{p} - m) = 0.$$

Using $\not{p}^2 = p^2 = m^2$, solve:

$$A = B.$$

Step 3: Fix normalization

Using $\bar{u}^{(s)} u^{(s)} = 2m$ gives:

$$A = 1.$$

Result:

$$\boxed{\sum_s u^{(s)}(p)\, \bar{u}^{(s)}(p) = \not{p} + m}$$

Spin sum for antiparticles

Similarly, for v spinors:

$$(\not{p} + m)v^{(s)}(p) = 0.$$

Repeating the same steps:

$$\boxed{\sum_s v^{(s)}(p)\,\bar{v}^{(s)}(p) = \not{p} - m}$$

Interpretation:

- $\not{p} + m$ projects onto particle states
- $\not{p} - m$ projects onto antiparticle states

Chirality and Helicity

Helicity operator:

$$h = \vec{\Sigma} \cdot \hat{p} \tag{1}$$

- Measures spin projection along direction of motion
- $\hat{p} = \vec{p}/|\vec{p}|$

Matrix form:

$$h = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & 0 \\ 0 & \vec{\sigma} \cdot \hat{p} \end{pmatrix} \tag{2}$$

- Helicity eigenvalues:

$$h = \pm 1$$

- Right-handed (RH): $h = +1$
- Left-handed (LH): $h = -1$

Chirality Operator γ^5

Definition:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3}$$

Key property:

$$(\gamma^5)^2 = 1 \tag{4}$$

Eigenvalue relations:

$$\gamma^5\psi_R = +\psi_R \tag{5}$$

$$\gamma^5\psi_L = -\psi_L \tag{6}$$

- Eigenvalues: ± 1
- Defines chirality (left vs right)

Properties of Projectors

Projection operators:

$$P_R = \frac{1}{2}(1+\gamma^5), \quad P_L = \frac{1}{2}(1-\gamma^5)$$

$$P_R^2 = P_R; \qquad P_L^2 = P_L; \qquad P_R P_L = 0$$

$$P_R + P_L = 1 \tag{7}$$

- They form a complete set of orthogonal projectors
- Any spinor can be decomposed into LH and RH parts

$$\psi = (P_R + P_L)\psi = \psi_R + \psi_L \tag{8}$$

$$\psi_R = P_R\psi, \quad \psi_L = P_L\psi \tag{9}$$

Chirality vs Helicity

- For massless particles:
  - Chirality = Helicity
- For massive particles:
  - Helicity is frame-dependent
  - Chirality is Lorentz invariant

Important:

- Weak interactions couple only to left-handed fermions

Parity Transformation

Definition:

$$\vec{x} \to -\vec{x}, \quad t \to t$$

Operator form:

$$P: \quad f(\vec{x}, t) \to f(-\vec{x}, t)$$

- Discrete spacetime symmetry
- Reverses spatial coordinates

Consider Lagrangian:

$$\mathcal{L} = (\partial_\mu \phi)^2 - m^2 \phi^2 \tag{10}$$

Under parity:

- $\partial_i \to -\partial_i$
- $\partial_0 \to \partial_0$

- Kinetic term invariant
- Mass term invariant

Electromagnetic Field Example

Vector potential:

$$\vec{A}(\vec{x}, t) \to -\vec{A}(-\vec{x}, t) \tag{11}$$

Fields:

$$\vec{E} = -\partial_t \vec{A} - \nabla \phi \tag{12}$$

$$\vec{B} = \nabla \times \vec{A} \tag{13}$$

Under parity:

$$\vec{E} \to -\vec{E} \tag{14}$$

$$\vec{B} \to \vec{B} \tag{15}$$

Parity Transformation and Dirac Bilinears

Parity transformation:

$$P: \quad t \to t, \quad \vec{x} \to -\vec{x} \qquad \psi(t,\vec{x}) \xrightarrow{P} \gamma^0 \psi(t,-\vec{x})$$

Dirac bilinears and their transformation properties:

- **Scalar:**

$$\bar{\psi}\psi \quad \to \quad \text{even (P = +1)}$$

- **Pseudoscalar:**

$$\bar{\psi}\gamma^5\psi \quad \to \quad \text{odd (P = -1)}$$

- **Vector:**

$$\bar{\psi}\gamma^\mu\psi \quad \to \quad \begin{cases} \text{time component: even} \\ \text{space components: odd} \end{cases}$$

- **Axial vector (pseudovector):**

$$\bar{\psi}\gamma^\mu\gamma^5\psi \quad \to \quad \begin{cases} \text{time component: odd} \\ \text{space components: even} \end{cases}$$

Tensor Bilinear: $\bar{\psi}\sigma^{\mu\nu}\psi$

Definition:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Case 1: σ^{0i}

$$\sigma^{0i} \to \frac{i}{2}[\gamma^0, -\gamma^i] = -\sigma^{0i}$$

$$\Rightarrow \bar{\psi}\sigma^{0i}\psi \quad \text{is odd}$$

Case 2: σ^{ij}

$$\sigma^{ij} \to \frac{i}{2}[-\gamma^i, -\gamma^j] = \sigma^{ij}$$

$$\Rightarrow \bar{\psi}\sigma^{ij}\psi \quad \text{is even}$$

Complete Classification of Dirac Bilinears

Bilinear	Name	Parity
$\bar{\psi}\psi$	Scalar	$+$
$\bar{\psi}\gamma^5\psi$	Pseudoscalar	$-$
$\bar{\psi}\gamma^\mu\psi$	Vector	$(+,-)$
$\bar{\psi}\gamma^\mu\gamma^5\psi$	Axial vector	$(-,+)$
$\bar{\psi}\sigma^{\mu\nu}\psi$	Tensor	$\begin{cases} - & (0i) \\ + & (ij) \end{cases}$

- Time indices behave differently from spatial ones
- γ^0 controls all parity properties
- Tensor splits into two distinct parity sectors

Dirac Bilinears: Lorentz, P and C (Cheat Sheet)

Bilinear	Lorentz Type	P	C
$\bar{\psi}\psi$	Scalar	$+$	$+$
$\bar{\psi}\gamma^5\psi$	Pseudoscalar	$-$	$+$
$\bar{\psi}\gamma^\mu\psi$	Vector	$\begin{cases} + & \mu = 0 \\ - & \mu = i \end{cases}$	$-$
$\bar{\psi}\gamma^\mu\gamma^5\psi$	Axial Vector	$\begin{cases} - & \mu = 0 \\ + & \mu = i \end{cases}$	$+$
$\bar{\psi}\sigma^{\mu\nu}\psi$	Tensor	$\begin{cases} - & (0i) \\ + & (ij) \end{cases}$	$-$

Key patterns:

- γ^0 controls parity (time vs space components)
- C flips sign of vector-like objects (charge flow)
- Tensor splits: $(0i)$ like E, (ij) like B

Role in the Standard Model

Weak interactions couple only to

$$\psi_L$$

Example:

$$\mathcal{L}_{weak} \sim \bar{\psi}_L \gamma^\mu \psi_L W_\mu$$

Thus the weak interaction is

- maximally parity violating
- purely left–chiral

This explains why neutrinos are observed to be left-handed.

Summary

Helicity:

$$h = \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|}$$

Chirality:

$$\psi_L = \frac{1 - \gamma^5}{2} \psi$$

$$\psi_R = \frac{1 + \gamma^5}{2} \psi$$

- helicity depends on reference frame
- chirality is a property of the spinor
- for massless particles they coincide

Discrete Symmetries of the Dirac Equation

Besides P, relativistic field theories may have other **discrete symmetries**:

- Charge conjugation (C) — particle $\leftrightarrow$ antiparticle
- Time reversal (T)

Now we have

- charge conjugation
- invariance of QED under P and C

Charge Conjugation

Charge conjugation transforms particles into antiparticles.
Define the charge conjugated field

$$\psi^C = C\bar{\psi}^T$$

where C is the charge conjugation matrix.
It satisfies

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T$$

A common representation is

$$C = i\gamma^2\gamma^0$$

Dirac Equation under Charge Conjugation

Start with

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

Take complex conjugate and use the properties of C.
One finds

$$(i\gamma^\mu \partial_\mu - m)\psi^C = 0$$

Thus the Dirac equation is invariant under charge conjugation.
Interpretation:

- ψ describes a particle
- ψ^C describes its antiparticle

Charge Conjugation in QED

Under charge conjugation

$$\psi \to \psi^C$$

The electromagnetic current transforms as

$$J^\mu = \bar{\psi}\gamma^\mu\psi$$

$$J^\mu \to -J^\mu$$

The photon field transforms as

$$A_\mu \to -A_\mu$$

Thus

$$J^\mu A_\mu$$

remains invariant.

Physical Interpretation

Charge conjugation reverses electric charge.

$$e \to -e$$

Thus

- electron $\leftrightarrow$ positron
- current changes sign
- photon field also changes sign

The interaction

$$-e\bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

remains unchanged.

Discrete Symmetries of the Dirac Theory

In relativistic quantum field theory the fundamental discrete symmetries are

$$P \quad \text{Parity}$$

$$C \quad \text{Charge Conjugation}$$

$$T \quad \text{Time Reversal}$$

Each acts on spacetime coordinates and on the spinor field.

Understanding these symmetries allows us to study

- particle–antiparticle relations
- spatial inversion properties
- time reversal behavior
- fundamental symmetry constraints of quantum field theory

Time Reversal

Time reversal transforms

$$T: \quad (t, \mathrm{x}) \to (-t, \mathrm{x})$$

The Dirac field transforms as

$$\psi'(t, \mathrm{x}) = \eta_T T \psi(-t, \mathrm{x})$$

where

$$T = i\gamma^1\gamma^3$$

and complex conjugation is required.

Time reversal is anti-unitary, meaning

$$TiT^{-1} = -i$$

This ensures the Dirac equation remains invariant.

Combined CPT Transformation

Applying the three transformations CPT together gives

$$x^\mu \to -x^\mu$$

and the Dirac field transforms as

$$\psi(x) \to i\gamma^5\psi(-x)$$

up to a phase.
Thus

$$\psi(x) \xrightarrow{CPT} \gamma^5 C\bar{\psi}^T(-x)$$

The combined transformation maps

- particle $\to$ antiparticle
- spin $\to$ reversed
- spacetime $\to$ inverted

The CPT Theorem

One of the most fundamental results in quantum field theory:

CPT Theorem

Any quantum field theory that satisfies

- Lorentz invariance
- locality
- Hermitian Hamiltonian

is invariant under the combined transformation

$$CPT$$

Physical Consequences of CPT

The CPT theorem implies several fundamental equalities:

$$m_{\text{particle}} = m_{\text{antiparticle}}$$

$$\tau_{\text{particle}} = \tau_{\text{antiparticle}}$$

$$|q_{\text{particle}}| = |q_{\text{antiparticle}}|$$

Experimental tests confirm CPT symmetry to extremely high precision.

Violation of CPT would imply

- violation of Lorentz invariance
- or breakdown of locality

making it a powerful probe of fundamental physics.