Aula 30-mar

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1) $|\psi\rangle = N e^{-\alpha r}$

$\langle\psi|\psi\rangle = 1 \Leftrightarrow \int \psi^* \psi \, d\tau = 1 \Leftrightarrow \int N e^{-\alpha r} \cdot N e^{-\alpha r} dr = 1 \Leftrightarrow$

$\Leftrightarrow \underbrace{\int_0^{2\pi} d\varphi}_{[\varphi]_0^{2\pi} = 2\pi} \underbrace{\int_0^{\pi} \operatorname{sen}\theta \, d\theta}_{[-\cos\theta]_0^{\pi} = -\cos\pi + \cos 0 = 1 + 1 = 2} \int_0^{\infty} \underbrace{N e^{-\alpha r} \cdot N}_{N^2} e^{-\alpha r} r^2 dr = 1 \Leftrightarrow$

$\Leftrightarrow 4\pi N^2 \int_0^{\infty} \underbrace{e^{-2\alpha r} r^2}_{m = 2 \;\; \alpha = 2\alpha} dr = 1$

$\int_0^{\infty} x^m e^{-\alpha x} dx = \frac{m!}{\alpha^{m+1}}$

$\Leftrightarrow 4\pi N^2 \frac{2!}{(2\alpha)^{2+1}} = 1 \Leftrightarrow \frac{\cancel{4}\pi N^2 \times \cancel{2}}{\cancel{2}^3 \alpha^3} = 1 \Leftrightarrow \pi N^2 = \alpha^3 \Leftrightarrow$

$\Leftrightarrow N = \sqrt{\frac{\alpha^3}{\pi}}$

Logo $|\psi\rangle = \sqrt{\frac{\alpha^3}{\pi}} e^{-\alpha r}$

b) $T|\psi\rangle = -\frac{1}{2} \nabla^2 |\psi\rangle = -\frac{1}{2} r^{-2} \frac{d}{dr}\left[r^2 \frac{d}{dr} N e^{-\alpha r}\right] =$

$\frac{d}{dx} e^{m} = (m)' e^{m}$

$= -\frac{1}{2} r^{-2} \frac{d}{dr}\left(r^2 \cdot (-\alpha) \cdot N e^{-\alpha r}\right) =$

$(x_1 x_2)' = x_1 \times (x_2)' + (x_1)' \times (x_2)$

$= \frac{N}{2} r^{-2} \left[\alpha r^2 (-\alpha) e^{-\alpha r} + 2r\alpha e^{-\alpha r}\right] =$

$= \frac{N}{2}\left[\frac{\alpha \cancel{r^2} (-\alpha) e^{-\alpha r}}{\cancel{r^2}} + \frac{2\cancel{r}\alpha e^{-\alpha r}}{r^{\cancel{2}}}\right] = \frac{N}{2}\left[\frac{2\alpha e^{-\alpha r}}{r} - \alpha^2 e^{-\alpha r}\right]$

$\langle T\rangle = \langle\psi| -\frac{1}{2}\nabla^2 |\psi\rangle = \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_0^{\pi} \operatorname{sen}\theta \, d\theta}_{2} \int_0^{\infty} \overbrace{N e^{-\alpha r} \cdot \frac{N}{2}}^{N^2} \left[\frac{2\alpha e^{-\alpha r}}{2} - \alpha^2 e^{-\alpha r}\right] r^2 dr$

$$\underbrace{\quad}_{2\pi} \quad \underbrace{\quad}_{2}$$

$$= \frac{4\pi N^2}{2} \int_0^\infty \left[\frac{2\alpha e^{-2\alpha r}}{r} - \alpha^2 e^{-\alpha r 2} \right] r^2 dr =$$

$$= 2\pi N^2 \int_0^\infty 2\alpha r e^{-2\alpha r} - \alpha^2 r^2 e^{-2\alpha r} dr =$$

$$= 2\pi N^2 \left[2\alpha \underbrace{\int_0^\infty r e^{-2\alpha r} dr}_{n=1 \quad \alpha = 2\alpha} - \alpha^2 \underbrace{\int_0^\infty r^2 e^{-2\alpha r} dr}_{n=2 \quad \alpha = 2\alpha} \right] =$$

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

$$= 2\pi N^2 2\alpha \cdot \frac{1}{(2\alpha)^2} - 2\pi N^2 \alpha^2 \frac{2!}{(2\alpha)^3} =$$

$$= \frac{\cancel{4}\pi N^2 \cancel{2\alpha}}{\cancel{2}^2 \alpha^{\cancel{2}}} - \frac{\cancel{8}\pi N^2 \cancel{\alpha^2} \cancel{2}}{2^3 \alpha^{\cancel{3}}} = \frac{\pi N^2}{\alpha} - \frac{\pi N^2}{2\alpha} = \frac{\pi N^2}{2\alpha}$$

como $N^2 = \frac{\alpha^3}{\pi}$

$$\frac{\cancel{\pi} \alpha^{\cancel{3}\,2}}{2 \cancel{\alpha} \cancel{\pi}} = \frac{\alpha^2}{2} = \langle T \rangle$$

c) $$\langle V \rangle = \langle \psi | -\tfrac{1}{r} | \psi \rangle = \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_0^{\pi} \operatorname{sen}\theta \, d\theta}_{2} \int_0^\infty \underbrace{N e^{-\alpha r} \cdot \left(-\tfrac{1}{\cancel{r}}\right) \cdot N e^{-\alpha r}}_{N^2} r^{\cancel{2}} dr =$$

$$= -4\pi N^2 \int_0^\infty \underbrace{r e^{-2\alpha r} dr}_{n=1 \quad \alpha = 2\alpha} = -4\pi N^2 \frac{1!}{(2\alpha)^2} = -\frac{\cancel{4}\pi N^2}{\cancel{2^2}\alpha^2} = \frac{-\pi N^2}{\alpha^2} =$$

$N^2 = \frac{\alpha^3}{\pi}$

$$= -\frac{\cancel{\pi} \alpha^{\cancel{3}}}{\cancel{\alpha^2} \cancel{\pi}} = -\alpha = \langle V \rangle$$

D) $$\langle E \rangle = \langle T \rangle + \langle V \rangle = \frac{\alpha^2}{2} - \alpha$$

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E) $\frac{\partial E}{\partial \alpha} = 0 \Rightarrow \frac{\partial}{\partial \alpha}\left(\frac{\alpha^2}{2} - \alpha\right) = 0 \Leftrightarrow$

$\left(\frac{\alpha^2}{2}\right)' = \left(\frac{1}{2}\right)' \alpha^2 + \frac{1}{2}\left(\alpha^2\right)' = \frac{\cancel{2}\alpha}{\cancel{2}} = \alpha$

$\Leftrightarrow \alpha - 1 = 0 \Leftrightarrow \alpha = 1$

$\langle E \rangle = \frac{\alpha^2}{2} - \alpha = \frac{1^2}{2} - 1 = -\frac{1}{2} \equiv -0{,}5\, Ha$