Structure formation

Dark matter linear clustering

In the sub-Hubble regime (small scales), the evolution of DM perturbations is described by

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\,\dot{\delta}_k - 4\pi G\bar{\rho}\,\delta_k = 0$$

Inserting the dimensionless density Ω_m, and the evolution of the mean matter density (from the zeroth-order continuity equation)

$$\bar{\rho} = \rho_c\,\Omega_m\,a^{-3}$$

the evolution equation can also be written in the form:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k - \frac{3H_0^2\Omega_m}{2a^3}\delta_k = 0$$

Let us now consider the evolution in the various epochs of the Universe.

Matter-dominated epoch

To solve the evolution equation, we need first to insert the evolution of the scale factor.

From the Friedmann equation, in this regime,

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_m a^{-3} \quad \Leftrightarrow \frac{da}{dt} = H_0 \sqrt{\Omega_m} a^{-1/2}$$

The solution is,

$$a(t) = \left(\frac{3}{2} H_0 \sqrt{\Omega_m}\right)^{2/3} t^{2/3}$$

and

$$\dot{a}(t) = \left(\frac{3}{2} H_0 \sqrt{\Omega_m}\right)^{2/3} \frac{2}{3} t^{-1/3}$$

→ $\frac{\dot{a}}{a}(t) = \frac{2}{3t}$

Moreover, we also have, $\frac{3H_0^2\Omega_m}{2a^3} = \frac{2}{3t^2}$

Inserting in the evolution equation for δ we can write it as:

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0.$$

Interestingly, this equation has an equidimensional structure (a polynomial where the powers of the variable, t, follow the order of the derivatives of δ)

→ this implies we can look for a power law solution $\delta = At^n$

Inserting the solution, we get: $\dot{\delta} = A\,n\,t^{n-1}$

$$\ddot{\delta} = A\,n\,(n-1)\ t^{n-2}$$

and the evolution equation becomes an algebraic equation for the power-law index n:

$$An(n-1)t^{n-2} + \frac{4}{3}t^{-1}Ant^{n-1} - \frac{2}{3}t^{-2}At^{n} = 0$$

$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0$$

There are two possible values of n, i.e. 2 solutions for the evolution of δ:

n = 2/3 → a growing solution

and

n = -1 → a damping solution

usually written as,

$$\delta(t) = \delta(t_i)(D_+(t) + D_-(t))$$

Only the growing solution leads to structure formation.

$$D_+(t) \propto t^{2/3} \propto a(t)$$

This means that **during the matter-dominated epoch, sub-Hubble scales grow (cluster) at the same rate as the background expansion.**

At later times, still in the matter-dominated epoch, **dark energy** starts to play a role, and the background is no longer determined by Ω_m alone.
DE needs to be included in the Friedmann equation, resulting in a faster expansion a(t) and a slower decreasing rate of H(a) → this **increases the Hubble drag**.

The evolution of δ is also affected through the change in a(t) in the third term, brought by dark energy.

Only the mean matter density in the third term is not affected, because there is energy conservation per component, i.e., the decreasing rate of the mean matter density does not depend on the components present in the cosmological fluid, i.e., there is a zeroth-order continuity equation per component.

The solution for the growth of δ in the matter epoch (but in the presence of dark energy) deviates from being proportional to the scale factor.

In general, the growth becomes slower than δ ~ a → **DE works against structure formation**.

This also implies that **observations of structure formation - for example measurements of the matter power spectrum at various redshifts - are an indirect probe of dark energy.**
(two examples are **tomographic weak lensing** and **galaxy clustering** → this is in fact the main goal of the Euclid mission).

In order to emphasize the deviation from the standard **growth** rate δ ~ a, solutions δ(a) for the various cosmological models are usually written in terms of the growth function g

δ ~ a * g

Alternatively, it is also common to define the growth function f:

$$f = \frac{d \ln \delta}{d \ln a} = \frac{\dot{\delta}}{\delta H}$$

Note that f compares $\dot{\delta}/\delta$ with $\dot{a}/a$ instead of comparing δ with a (like g does)

Note also that for a power-law growth, the growth f is exactly the index of the power-law:

$\delta \sim a^f$

Note also that both f and g change with redshift.

Each dark energy model leads to a different growth rate of the dark matter clustering (different from $\delta \sim a$ **).**
For each case, the δ differential equation can be solved numerically (which implicitly also needs the solution of the Friedmann equation a(t) for that particular case of dark mater + dark energy).

The result for any dark energy model can be written with a general expression (a **fitting function for the growth rate):**

$$g(a) = \exp \int_{a_i}^{a} (\Omega_m^{\gamma}(a) - 1) \frac{da}{a}$$

The integral goes from a chosen normalization scale factor, a_i, to the a of interest, with g (a_i) = 1.

This fit introduces a new cosmological parameter: the **growth index** γ**.**

Each dark energy model has a given value of the growth index. This is not a new independent free parameter. Its value can be found from the equation-of-state parameter of the dark energy model. That relation is also given by a fitting function:

$$\gamma = 0.55 + 0.05[1 + w(z = 1)]$$

Naturally, the growth index is also related to the f growth function (besides being related with the g growth function). The relation is:

$$\gamma = \frac{\ln f}{\ln \Omega_m}$$

So, for a given dark energy model (characterized by w, or γ) and a given value of its density (Ω_Λ , or Ω_m from the Friedmann constraint) → we find the growth rate (g, or f) solution relying on these fitting functions.

In particular, we see that:

- higher Ω_m (lower Ω_Λ) → faster growth

- higher w_0 (less negative) → faster growth

For a generic model, γ encapsulates the solution for δ and parameterizes the deviation from the ΛCDM growth (for ΛCDM, γ = 0.55).

The result of the evolution of the density contrast on all scales is the **(theoretical) matter power spectrum**:

However, remember that the solution for δ is a solution for its growth rate, the equations do not compute the absolute amplitude of the growth. For this, we need to define the initial conditions of the primordial power spectrum:

Amplitude A_s
Shape (power-law index) n_s

The (theoretical) matter power spectrum for a given model is usually confronted with the **(observed) matter power spectrum**:

The observed power spectrum is measured within a certain range of scales (depending on the experiment) and at a given redshift (e.g. z=0). In general we can measure its

Shape: Γ
Amplitude within a range of scales (in particular at the scale 8 Mpc/h): $\boldsymbol{\sigma_8}$
Growth (measuring the amplitude at various redshifts): **f**

Note that these low-redshift cosmological parameters are not new independent parameters:

The shape of P(k, z=0) depends on n_s and the transfer function (which depends on the parameters of the homogeneous universe)

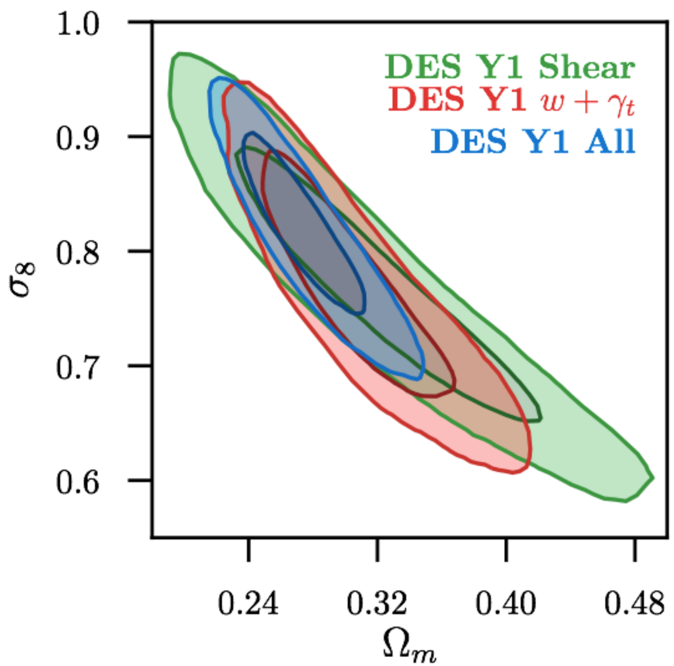
The amplitude σ_8 depends on A_s and on the evolution (which depends on the parameters of the homogeneous universe)

The growth f depends on on the evolution (which depends on the parameters of the homogeneous universe)

Note also that:

Cosmological probes that measure the power spectrum of matter at a single redshift (e.g: galaxy clustering) are **sensitive to the amplitude:** σ_8

Cosmological probes that measure the power spectrum of matter at various redshifts (e.g: weak lensing) are **sensitive to the combination of amplitude and growth**: $f\,\sigma_8$, or $S_8 = \sigma_8\,(\Omega_m/0.3)^{1/2}$



DES year 1 shear : weak lensing 2pt angular correlation

DES year 1 $w(\theta) + \gamma_t$: galaxy clustering 2pt angular correlation + galaxy-galaxy lensing (cross-correlation between wl and gc)

Now, remember that the parameter constraints (degeneracy directions) depend on the observed quantity.

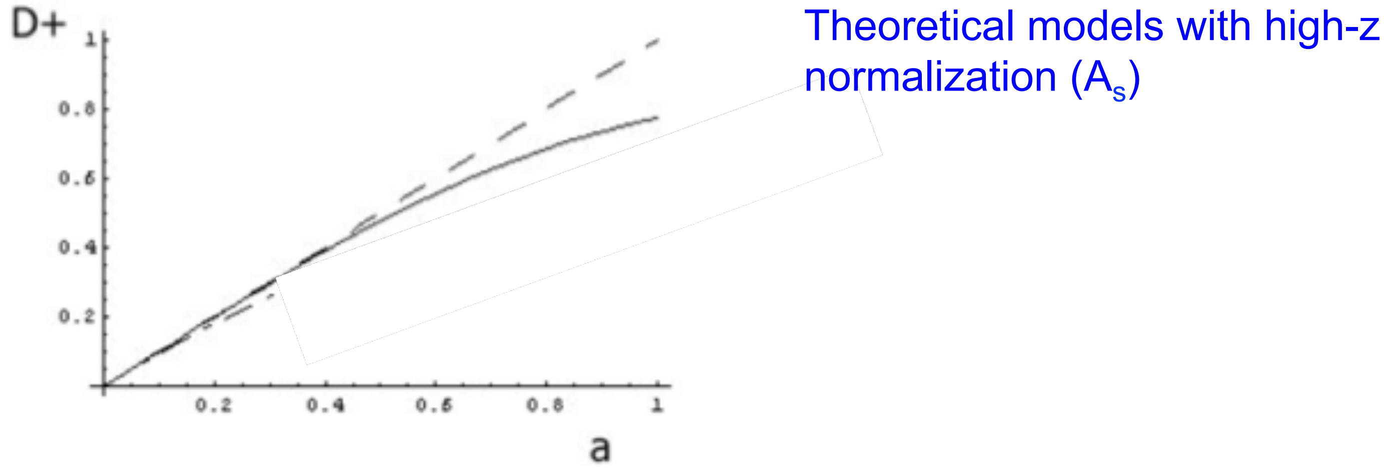
Example:

i) Let us consider that we measured the matter power spectrum at **high redshift** (observing the correlation function of the field of CMB temperature), and we want to fit the predictions of two cosmological models to those data. Consider that the models only differ in their values of Ω_m:

- *flat ΛCDM with* $\Omega_m = 0.3$ *(solid line)*
- *flat ΛCDM with* $\Omega_m = 0.5$ *(dashed line)*

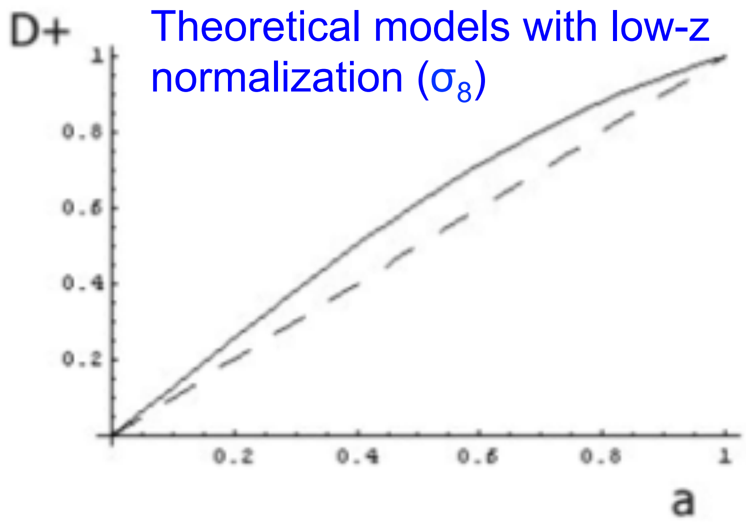
(the growth of structure is faster in the second model)

Since the data measures the matter power spectra at high z, we set the amplitude of the models by their A_s values → **since the two models start with the same amplitude A_s in the primordial universe, the one that grows faster is the one that produces more structure at z=1100.**



ii) Let us consider that we measured the matter power spectrum over a redshift range at **low redshift (**observing the correlation function of the field of galaxies positions, or the field of galaxies ellipticities) → we measure the amplitude σ_8 (z=0)

Since the two models reach the same amplitude σ_8 at z=0, the one that grows slower is the one that produces more structure at low redshift (i.e., in the redshift range measured in the neighborhood of z ~0)

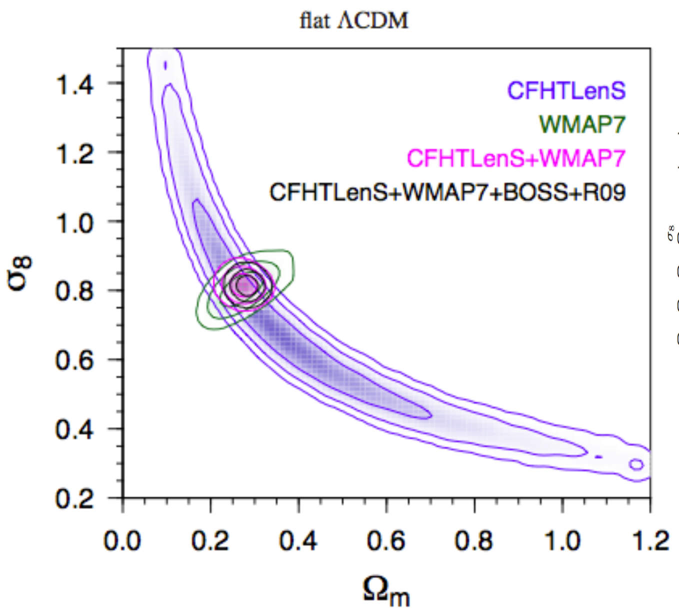


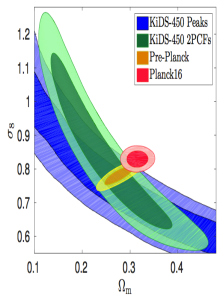
So, if we detect a power spectrum with high amplitude, the best-model fit may either be a model of fast structure growth (if we are measuring high-z data) or a model of slow structure growth (if we measure low-z data).

This explains the strong complementarity between CMB and Weak Lensing cosmological contours → the degeneracy direction found in high-redshift probes is the opposite of the one found in low-redshift probes.

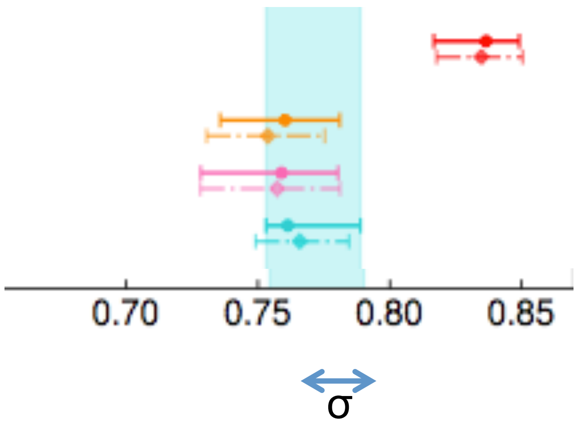
Weak lensing: CFHTLenS → KiDS
low-redshift probe

CMB: WMAP7 → Planck
high-redshift probe





However, current results are in "tension"



Dark Energy - dominated epoch

In the future of the universe, DE dominates → the Hubble drag term in the evolution equation of the dark matter density contrast starts dominating over the third term that goes with a^{-3}.

Neglecting the third term, and evaluating the second term using the Friedmann equation with only the cosmological constant, the equation becomes:

$$\ddot{\delta} + 2\,H_0\,\sqrt{\Omega_\Lambda}\,\dot{\delta} = 0$$

The solution is a sum of a constant and a decresing mode that decreases exponentially with time:

$$\delta = C_1 + C_2\,[\exp(H_0\,\Omega_\Lambda^{1/2}\,t)]^{-2} \quad \textit{(see homework)}$$

Given the exponential expansion at the DE-dominated epoch, the clustering rate of dark matter may also be written as,

$$\delta = C_1 + C_2\,a^{-2}$$

Remember that a linear perturbation may be thought of as a homogeneous part of the Universe that expands with its own rate.

If this rate is lower than the background expansion rate, then the density in the region becomes larger than the mean → clustering, δ increases

If this rate is higher than the background expansion rate, then the density in the region becomes smaller than the mean → δ decreases, and **linear structures already in the process of structure formation will eventually dilute.**

Note however that collapsed structures (non-linear structures) with high δ , have already decoupled from the evolution and will not get disrupted.

They are not part of the structure formation process anymore and they will just become increasingly separated from each other as the expanding universe accelerates.

Radiation-dominated epoch

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k - \frac{3H_0^2\Omega_m}{2a^3}\delta_k = 0$$

Now the third term deals both with dark matter density Ω_m (since we are computing the matter density contrast) and with radiation density (through the dependence on the scale factor).

The second term is determined by the expansion in the radiation epoch.

Let us consider the Friedmann equation, neglecting now the contribution of the matter density to the expansion:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\Omega_r a^{-4} \quad \rightarrow \quad a \propto t^{1/2}$$

Inserting these terms results in,

$$\ddot{\delta} + \frac{1}{2t}\dot{\delta} - \frac{3H_0^2\Omega_m}{2(2H_0)^{3/2}\Omega_r^{3/4}}\frac{1}{t^{3/2}}\delta = 0$$

which is no longer a simple equidimensional equation.

Since the approximation of neglecting Ω_m in the Friedmann equation did not result in a simple equation to solve, let us keep both matter and radiation mean densities in the Friedmann equation,

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3}\right) = \frac{H_o^2\,\Omega_m}{a_{\rm eq}^3}\frac{1}{y^3}\left(1 + \frac{1}{y}\right)$$

where we introduced the normalized scale factor y: $y = \dfrac{a}{a_{\rm eq}}$

and a_{eq} is the scale factor of radiation-matter equality: $a_{\rm eq} = \dfrac{\Omega_r}{\Omega_m}$

With these definitions, the equation for the density contrast evolution may be written using derivatives with respect to y as,

$$\delta'' + \left(\frac{H'}{H} + \frac{1}{y} + \frac{1}{Hy}\right)\delta' - \frac{3}{2}H^2\left(1 + \frac{1}{y}\right)^{-1}\delta = 0$$

where $\delta' = \frac{d\delta}{dy}$ and $H = \frac{\dot{a}}{a} = \frac{\dot{y}}{y}$ is given by the Friedmann equation.

Inserting H and H' from the Friedmann equation, and some manipulation, the equation can be written in the following form,

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0$$

This is known as the Meszaros equation and is approximately equidimensional. The second term is of type 1/y (even though it has an additional constant in the numerator) and the third term of type $1/y^2$

Note that, since no approximations were made, this equation is valid both for the radiation and the matter dominated epochs.

Indeed for the **(deep) matter epoch**, i.e., when $a >> a_{eq}$ → $1+y \sim y$ and $2+3y \sim 3y$

the equation becomes
$$\delta'' + \frac{3\delta'}{2y} - \frac{3\delta}{2y^2} = 0$$

Inserting a power law solution $\delta = A\, y^n$

the equation for the index will be $n(n-1) + \frac{3}{2}n - \frac{3}{2} = 0$

The solutions are thus: a growing solution n=1 and a damping solution n = -3/2.

So, the growing solution is (naturally) the same we found before,

$\delta \sim y$ → $\delta \sim a$

But we are interested in the (deep) radiation epoch, when $a << a_{eq}$ → we can neglect terms in y^2

The equation is then,

$$\delta'' + \frac{2+3y}{2y}\delta' - \frac{3}{2y}\delta = 0$$

It is no longer equidimensional (the third term has $1/y$ and not $1/y^2$) but it is easy to solve.

The solution is,

$$\delta = y + 2/3$$

This result is very important. It tells us that deep in the radiation epoch, δ remains constant → the (Newtonian, sub-Hubble) scales do not grow → **the amplitudes of the dark matter perturbations remain "frozen"** for a period, before starting growing with "a".

Super-Hubble scales

The Newtonian equation for the evolution of the dark matter density contrast is not valid for theses scales and the evolution of perturbations must be computed from the Einstein-Boltzmann equations.

However, it is possible to obtain an exact solution by using only zero-order background equations!

Let us consider that **a very large perturbation is a homogeneous region of the universe of radius R that expands slower than the background** → this region has a scale factor a_R slightly lower than the mean one 'a' → this region is clustering and its density is slightly larger than the mean density of the universe.

It is a mini-universe, inside the universe.

Let us first see how the mini-universe expands during the **radiation epoch**.

The density of the mini-universe is slightly larger than the mean density of the universe: $\Omega_{rR} = \Omega_r + \varepsilon$.

The density contrast of the mini-universe is then

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} = \frac{(\Omega_r + \epsilon)\, a_R^{-4} - \Omega_r a^{-4}}{\Omega_r a^{-4}} = \left(1 + \frac{\epsilon}{\Omega_r}\right)\left(\frac{a}{a_R}\right)^4 - 1$$

It is useful to write the Friedmann equations for the two "Universes" (in the radiation epoch, where Ω_r is the dominant component).

The background is flat, its Friedmann equation only involves $\Omega_{r.}$ However, the mini-universe is a confined region in the Universe → it must have a **spatial positive curvature** ($K = -\Omega_k H_0^2 < 0$). The Friedmann equation in the mini-universe is then:

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2(\Omega_{rR} a_R^{-4} - K a_R^{-2})$$

Let us consider an instant in time. In that instant, the mini-universe and the universe have different scale factors ‘a’ and a_R.

Expressions for time can be computed for both universes from the respective Friedmann equations.

For the mini-universe, time is given by,

$$t = \int_0^t dt = \int_0^a \left[\Omega_{rR} H_0^2 a_R^{-2} \left(1 - \frac{K a_R^2}{\Omega_{rR} H_0^2}\right)\right]^{-1/2} da$$

$$t\, H_0 = \Omega_{rR}^{-1/2} \left(\frac{1}{2} a_R^2 + \frac{1}{4} F_R a_R^4\right)$$

where

$$F_R = \frac{K}{2\Omega_{rR}\, H_0^2}$$

is the ratio between the curvature and radiation densities.

On the other hand, for the flat background universe time is given by

$$t\,H_0 = \Omega_r^{-1/2}\frac{1}{2}a^2$$

For any given instant t, the time computed in these two different ways, is of course, the same. This allows us to relate the quantities of the two systems:

$$(\Omega_r + \epsilon)^{-1/2}(a_R^2/2 + F_R a_R^4/4) = \Omega_r^{-1/2}\frac{1}{2}a^2$$

$$\left[\left(1+\frac{\epsilon}{\Omega_r}\right)\left(\frac{a}{a_R}\right)^4\right]^{1/2} = \left(1+\frac{1}{2}F_R a_R^2\right)$$

This equation is an expression for δ:

$$(1+\delta) = \left(1+\frac{1}{2}F_R a_R^2\right)^2$$

In the radiation epoch, “a” and also a_R are << 1, we can thus neglect the a^4 term and obtain

$$1+\delta \sim 1 + F_R\, a_R^2 + \tfrac{1}{4} F_R^2\, a_R^4 \rightarrow \delta \sim a_R^2$$

This means that the density contrast of the large relativistic scales (the super-Hubble scales) grows in the radiation era with a rate:

$$\delta \propto a^2 \quad \text{(or slightly slower than this)}$$

We found that while dark-matter perturbations on sub-Hubble scales are "frozen" (do not cluster in the radiation era), the large super-Hubble scale dark matter perturbations continue to cluster

→ this creates a scale-dependent effect in the general growth

i.e., this process produces a relative suppression of amplitude of the small scales with respect to the large scales.

This is the reason why the power spectrum today (z=0) has a peak, while the initial condition for the amplitudes of the density contrast (the primordial power spectrum) had larger amplitudes on small scales and smaller amplitudes on large scales.

Let us now see how the mini-universe expands during the **matter epoch**.

We can follow the exact same procedure, but now the Friedmann equation is dominated by Ω_m.

Now, matching the times leads to a different result:

$$(\Omega_m + \epsilon)^{-1/2}(a_R^{3/2} 2/3 + F_R a_R^{5/2} 2/5) = (\Omega_m)^{-1/2} \ \frac{2}{3} a^{3/2}$$

and the result is

$$\delta \propto a$$

This shows that **super-Hubble scales grow in the matter era,** with the same rate as the sub-Hubble scales
(and slower than they evolved in the radiation epoch)

→ during the matter era the growth is scale-independent.

And so, the shape of the (linear) power spectrum at z=0 is determined in the radiation era.

Dark matter linear power spectrum

We just saw that during the radiation era

large (super-Hubble) scales grow fast,

while small (sub-Hubble scales) do not grow at all.

The Hubble radius is the threshold between large (general relativistic) and small (Newtonian) scales.

Now, remember the Hubble radius $r_H(a) = c/H(a)$ increases as the Universe expands (at a different rate than the expansion)

So this threshold is not fixed (for this reason, the Universe is more relativistic in early times and becomes more Newtonian with time) → **Scales (perturbations) are gradually caught by the growing Hubble sphere, passing from super-Hubble to sub-Hubble and "freeze"**.

$\lambda 1 < \lambda 2 < \lambda 3$

scale

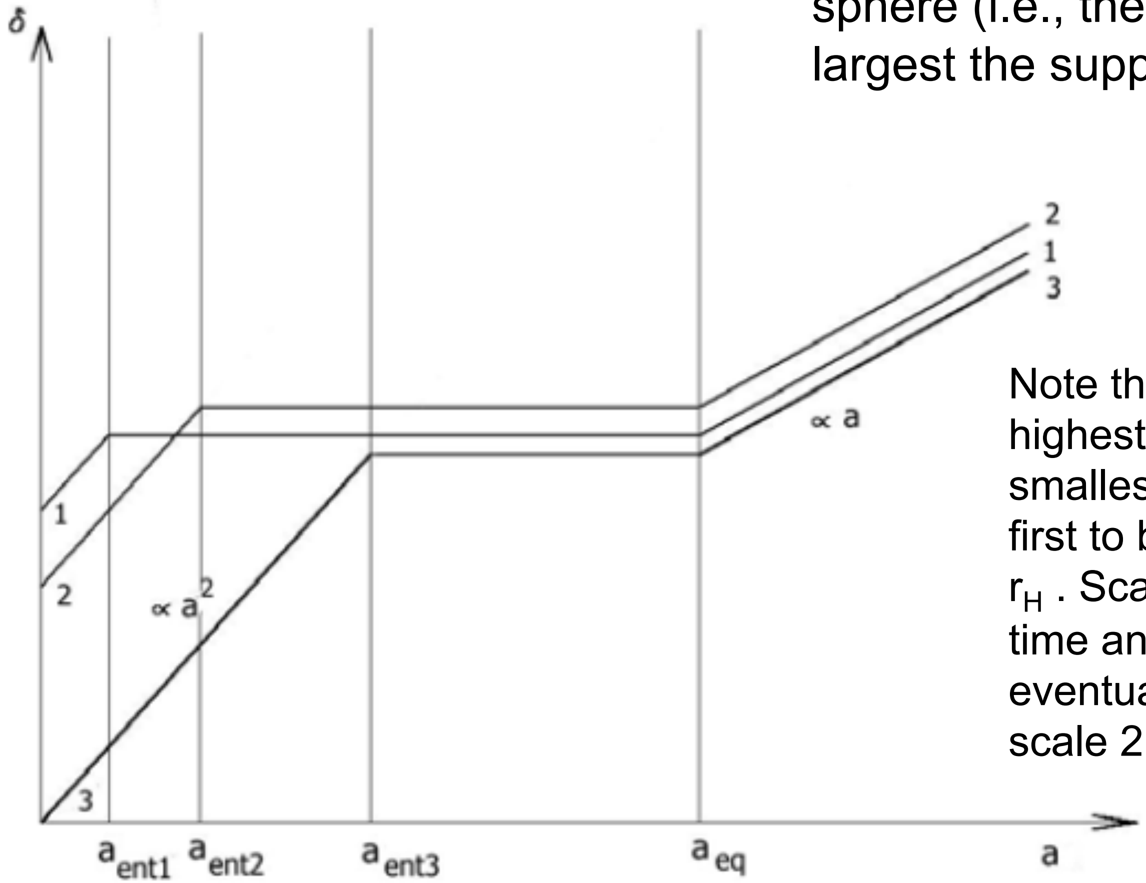
$\delta 1_i > \delta 2_i > \delta 3_i$

initial δ

The growth of a scale of size R is suppressed by a factor

$[a_{eq} / a\_enter\ (R)]^2$

→ the earlier it enters the Hubble sphere (i.e., the smallest it is), the largest the suppression factor.



Note that scale 1 starts with the highest amplitude, since it is the smallest scale, and thus it is the first to be caught by the growing r_H . Scale 1 remains frozen a long time and its amplitude is eventually overtaken by the larger scale 2 → a peak will be formed.

This process creates a peak in the matter power spectrum that corresponds to the largest scale overtaken by the Hubble radius during radiation era.

This scale is the largest scale to freeze → it has the size of the Hubble radius at matter/radiation equality.

$$r_H (a = a_{eq}) = c / H (a_{eq})$$

(its comoving size is 64 Mpc/h - concordance model).

The peak position is a distinctive feature of the matter power spectrum, useful to constrain cosmological parameters.

The relative suppression of amplitude of the small scales in relation to the large scales gives an effect of 'transfer of power' in the shape of the power spectrum. The change in the shape of the spectrum from the primordial one is known as the transfer function **T(k)**, which models the **scale-dependent evolution**.

There are fitting formulas to compute the transfer function as function of cosmological parameters, to avoid the need of solving the equations for each parameter value, allowing fast likelihood computations.

An example is the BBKS fit:

$$T(q) = \frac{\ln(1+2.34q)}{2.34q}\left[1+3.89q+(16.1q)^2+(5.46q)^3+(6.71q)^4\right]^{-1/4}$$

where the scale q contains the cosmological dependence: $q = \frac{k}{\Omega_m\, h^2\,\mathrm{Mpc}^{-1}} = \frac{k}{\Gamma}\,[h/\mathrm{Mpc}]$

The transfer process (and the peak position) depends mainly on the value of a_{eq}.

In terms of cosmological parameters, it directly depends on Ω_m h^2 (the physical matter density). The parameter combination $\Gamma = \Omega_m$ h is usually introduced to model the peak position and is called the shape parameter.

(Note that even though the dependence is on Ω_m h^2, Γ is used because one of the h factors is absorbed in the dimension of length -Mpc/h-)

The impact of a cosmological model on the peak position is the following: a model with higher Γ has a shorter radiation epoch → smaller r_H (a_{eq}) → the largest scale to freeze is smaller → peak moves to the right.

Then, after $a=a_{eq}$, all scales growth at the same rate, and the shape of the (linear) dark matter power spectrum remains fixed, with a **scale-independent evolution, D_+**.

The linear power spectrum can thus in general be written as,

$$P_\delta(k,a) = A\left(\frac{k}{k_0}\right)^{n_s} T(k) D_+^2(a)$$

the transfer function depends on the equality time a_{eq}, while
the scale-independent growth $D_+(a)$ depends on the dark energy model.

remember that the characteristic of dark energy (or modified gravity) directly relevant to structure formation is the **growth index γ**

$D_+ (a) = g(a)\, a$ or $D_+ (a) = a^{f(a)}$

where

$$g(a) = \exp \int_{a_i}^{a} \left(\Omega_m^\gamma(a) - 1\right) \frac{da}{a}$$

or

$f(a) = \Omega_m^\gamma (a)$

For $a < a_{eq}$

At first, the primordial power spectrum is linear in k.

Then the 'freezing' process changes the shape:

- Scales that remain always super-Hubble keep their relative power ~ k.

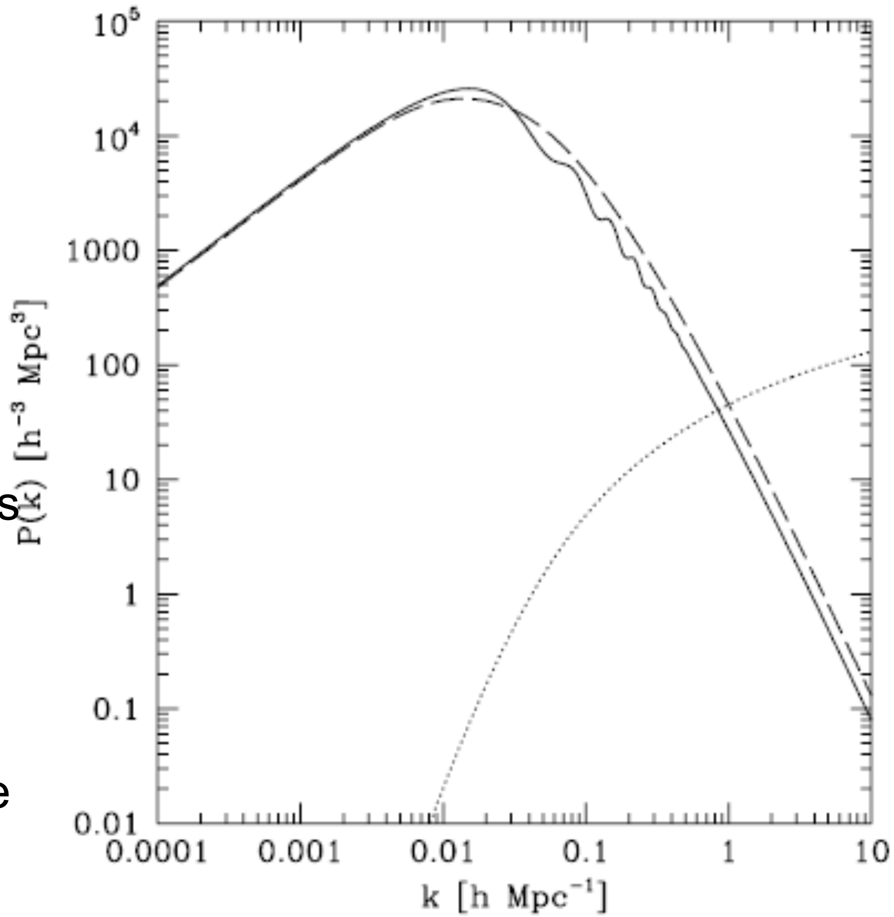
- Scales that are frozen during all radiation era end up with a relative power ~ k^{-3}.

- Scales in between, are frozen part of the time, get in-between power → a peak forms in the shape of the spectrum, showing the last scale to cross r_H (at a_{eq}).

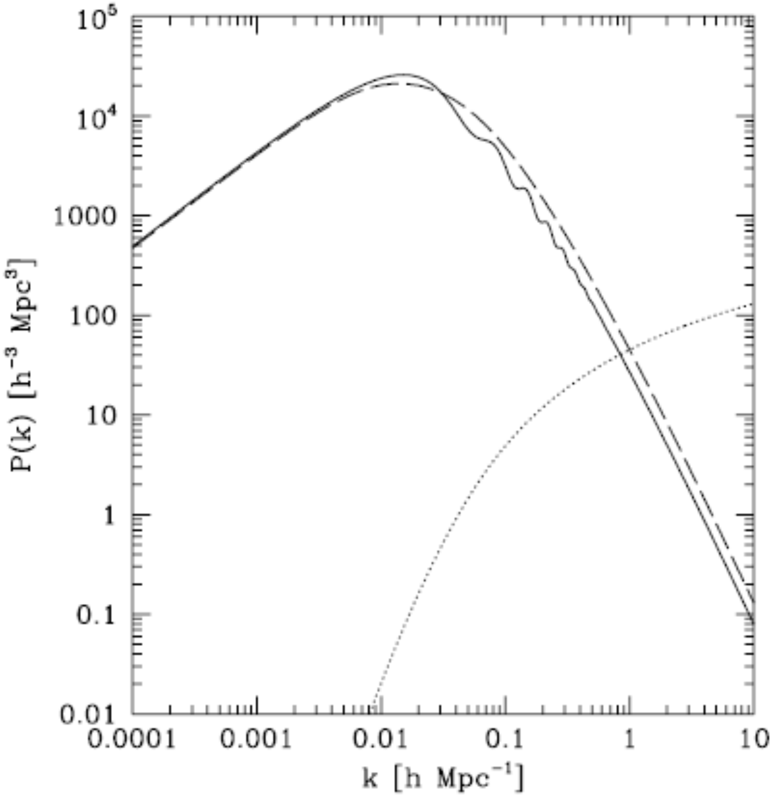
- For even smaller scales there is a cut-off, their amplitudes are completely erased due to free-streaming.

Matter linear power spectrum:

- Dark matter only (dashed line)
- DM + baryonic (solid line, contains wiggles) *(to be seen later)*
- Dimensionless (dotted line)



For $a > a_{eq}$



For $a > a_{eq}$, all scales grow with the same rate ~a or slower, depending on the dark energy model.

The power spectrum only changes in amplitude, not in shape.

The dimensionless power spectrum $\Delta^2(k)$ varies with $k^{(n_s+3)} \sim k^4$ on large scales and with $k^{(n_s-1)} \sim k^0$ on the smallest scales.

As such, it does not show a peak.

The dimensionless power spectrum shows that even with the freezing period small scales still reach the highest amplitude (except for the ones below the free-streaming scale) → they are the ones to reach first non-linearity $\delta > 1$ and form structure → **galaxies form first than clusters or large-scale structure.**

Free-streaming: which is the first scale to form structure?

In the early universe, dark matter particles are relativistic and their velocity is v=c

Later on, at a certain $t = t_{tr}$, they become non-relativistic and their dispersion velocities start decreasing with the temperature of the universe (i.e., proportional to the scale factor a).

The redshift at which this transition occurs depends on the mass of the particle → the lighter a particle is, the longer it remains relativistic.

A good approximation for the transition redshift is:

$$1 + z_{\rm tr} = 2 \times 10^3 \left(\frac{m_\nu}{1\,{\rm eV}}\right)$$

Now, while the DM particles are relativistic it is not possible to have growing localized densities of matter → **the particles move and any density contrast that starts to grow will quickly dilute.**

The maximum possible distance the dark matter particles can travel while they are relativistic is their particle horizon, i.e., the distance traveled with velocity v from the Big-Bang to the transition time.

This defines the largest scale where structure can be destroyed from free-streaming: the free-streaming scale (l_{FS})

(which is also the smallest scale where structure can form from the cosmological/gravitational process).

Its physical (proper) size is then:

$$l_{\mathrm{FS}}(t) = a(t) \int_0^t \frac{v(t')}{a(t')} dt'$$

Density values inside a region of this size will mix → **scales smaller than the free-streaming length are smoothed-out and structure does not form.**

The free-streaming scale is a characteristic of the **type of dark matter particle**.

The detection of the smallest cosmological structures is a probe of the type of dark matter in the cosmological model.

Assuming the transition occurs in the radiation epoch when a(t) ~ $t^{1/2}$ and that the particle's velocity is v(t) = c while relativistic, then the free-streaming scale is given by:

$$l_{\rm FS}(t_{\rm nr}) = a(t_{\rm nr})\, c \int_0^{t_{\rm nr}} \frac{dt'}{a_{\rm nr}} \left(\frac{t_{\rm nr}}{t'}\right)^{\frac{1}{2}} = 2\, c\, t_{\rm nr}$$

(notation: tr or nr, used for transition to non-relativistic)

Types of dark matter:

i) **WIMPS** (weakly interacting massive particles)

These particles have mass ~1 KeV → $z_{tr} \sim 2 \times 10^6$

In the concordance model this redshift corresponds to t_{tr} = 0.2 years (since the Big Bang).

They become non-relativistic in the very early universe and the density contrast can start to grow very early → for this reason they are called **cold dark matter** particles.

Their proper free-streaming size l_{FS} is

$$2 \times 3 \times 10^8 \times 0.2 \times 3.15 \times 10^7 \text{ m} = 4 \times 10^{15} \text{ m} = 0.2 \text{ pc}$$

The comoving size (equal to the proper size today given the expansion) is computed dividing this by the scale factor :

$$l_{FS}\ \textbf{(comoving)} = \textbf{0.2 Mpc}$$

Now, the 8Mpc scale (clusters) has a mass of 10^{15} M_Sun, and we just found out that the WIMP l_{FS} scale is a factor 8/0.2 = 40 smaller in size.

In a homogeneous universe, the mass in a region 40 times smaller than another one has a mass 40^3 (i.e. ~10^5) smaller → M ~ 10^{10} M_Sun → corresponding to galaxies.

With this rough approximation, we conclude that the DM density contrast on scales smaller than galactic scales are washed out → **no cosmological structure formation below galactic scales in a CDM model.**

This means **there is a cut-off in the power spectrum at galaxies scales** → astrophysical **objects smaller than galaxies (stars, planets, etc.) are not formed from the cosmological process** → they do not form inside an individual DM halo → they are not cosmological structures → **they are formed from astrophysical processes within the galaxies.**

ii) Hot Neutrinos

In principle neutrinos could also be candidates for dark matter particle.

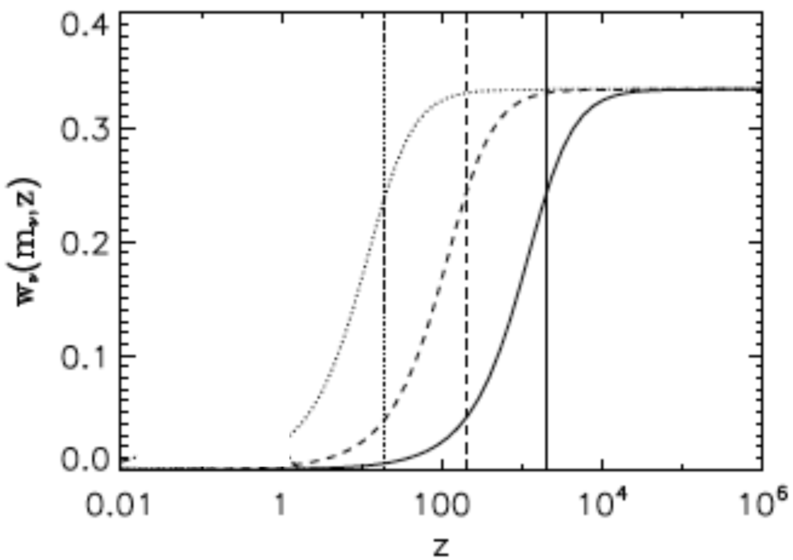
Neutrinos have much smaller masses than WIMPS. In some cases they can even still be relativistic particles today.

The figure shows the evolution of equation-of-state parameter for three cases of neutrinos masses: m=1 eV ; m=0.1 eV ; m=0.01 eV

In all cases w changes from 1/3 (relativistic) to 0 (matter) - the lightest case is the one that takes longer to reach the transition.

For example for m = 0.1 eV:
→ $z_{tr} \sim 200$
→ $t_{tr} = 5.7 \times 10^6$ years

They become non-relativistic late → they are **hot dark matter** particles.



The proper free-streaming size l_{FS} is

$$2 \times 3 \times 10^8 \times 5.7 \times 10^6 \times 3.15 \times 10^7 = 10^{23} \text{ m} = 3 \text{ Mpc}$$

and the comoving size is l_{FS} = 600 Mpc.

This would be the size of the smallest DM halo formed in this model, corresponding to a mass of 10^{20} M_Sun → **no structure formation possible in HDM models**

Structure could only form in the future.

Consider a **future evolution** of the HDM scenario, assuming there is no dark energy and the density contrast continues to evolve at a rate $\delta \sim a$:

Given that the initial condition for the amplitude is $P(k) \sim k$, and that there is a factor of 75 between the HDM l_{FS} and the cluster scale in CDM → the HDM l_{FS} scale would be the first structure to collapse but only in a = 75 (z = -0.987).

At that time in the future, a 600 Mpc structure would form and it would be the smallest cosmological DM halo → smaller objects (clusters, galaxies, stars, planets) could perhaps form in sub-regions of these large halos but only from astrophysical processes.

iii) **Warm Neutrinos**

HDM is an extreme scenario, with low mass neutrinos.

A more realistic case needs to consider observational constraints on the mass density.

The mass density of neutrinos is given by: $\Omega_m = 0.02 \times m(\text{in eV})$ → to get $\Omega_m = 0.3$ only from neutrinos we would need neutrinos of m=15 eV

In this case the free-streaming scale is $l_{FS} = 15$ Mpc,
corresponding to a $M_{FS} \sim 10^{15}$ M_Sun → this is roughly the scale of clusters.

If **warm dark matter particles** were the dominating type of dark matter particles, clusters would be the smallest structures formed from the cosmological process → galaxies would not form as a direct result of the evolution of perturbations but by later fragmentation of clusters.

This is called a top-down scenario of structure formation (first clusters then galaxies), while the CDM case is called a bottom-up scenario.

The two scenarios were in discussion in the 1970s. With astrophysical data showing that clusters are only observed at lower redshifts, while galaxies exist at higher redshifts (and also at low redshift) → the top-down scenario was ruled out.

Hierarchical structure formation

The result of the power spectrum evolution in the standard ΛCDM model is thus a **hierarchical structure formation**, i.e., scales move from the clustering phase to the collapsing phase in different times and sequentially.

In particular, since smaller scales collapse first than larger scales, this hierarchical procedure is also called a bottom-up scenario of structure formation.

The behaviour of the various scales is the following:

Very small scales (stars, star clusters, satellite galaxies)

These are not cosmological scales.

They are below the free-streaming limit and do not form from the structure formation process of cosmological evolution → they are sub-clumps of galactic-size dark matter halos.

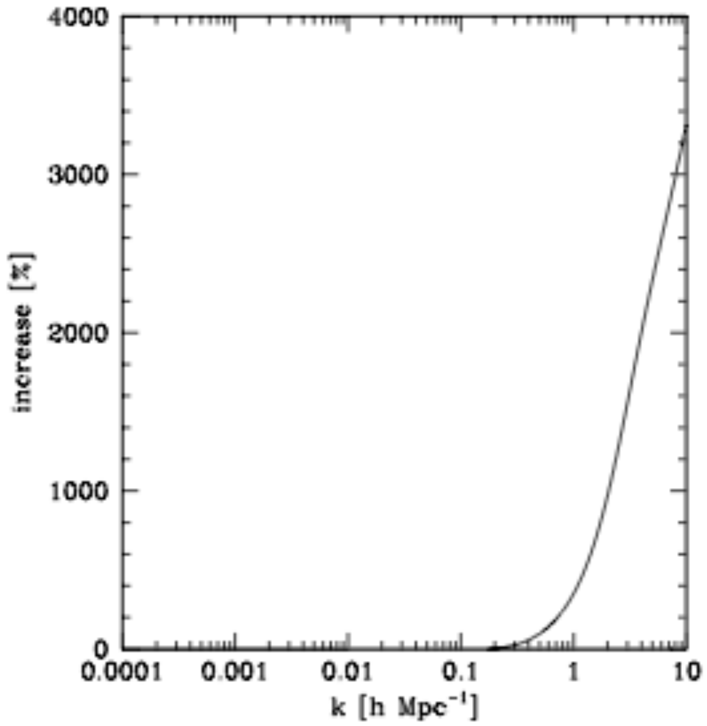
Small scales (galactic scale)

These are the first to reach non-linearity and to decouple from the expansion.

They are the first to form structures (those structures are the galaxy DM halos).

On these scales $\Delta^2 > 1$ today → non-linear scales are the smallest cosmological scales. For the ΛCDM concordance model the non-linear scales are the ones with **k > 0.2 h/Mpc.**

Note that the amplitude of the non-linear matter power spectrum is much larger than the value obtained with the linear equations.



The figure shows the increase (in %) of power (P_{NL}/P_L) as function of scale

In addition, when the evolution is non-linear, the statistical distribution of the density contrast (Gaussian) is not preserved and non-Gaussianities arise.

Intermediate scales (clusters scale)

These are the scales that are collapsing today at z ~0 (meaning that only today did Δ^2 at this scale reach the value ~1.

In the concordance model, the theoretical calculation finds that this scale is

$$k = 0.78 \text{ h/Mpc} \rightarrow R \sim 8 \text{ Mpc/h.}$$

Consider a homogeneous sphere of this size with density twice the mean density of the universe (i.e. $\delta = 1$). Inserting the standard values for the critical density and matter density, the mass of this region is

$$M = 4/3\pi\, 8^3\, \rho_c\, \Omega_m\, (1+\delta) \sim 10^{15}\ \text{M\_Sun / h}$$

which is a typical mass of a cluster, thus confirming that galaxy clusters are the largest collapsed structures.

Large scales (R > 8 Mpc/h)

like for example filaments.

These are still linear today ($\Delta^2 < 1$) and did not form bound collapsed structures yet → which implies they are not dense enough for star formation → they are non-luminous (because baryonic luminous matter did not yet associate with these structures).

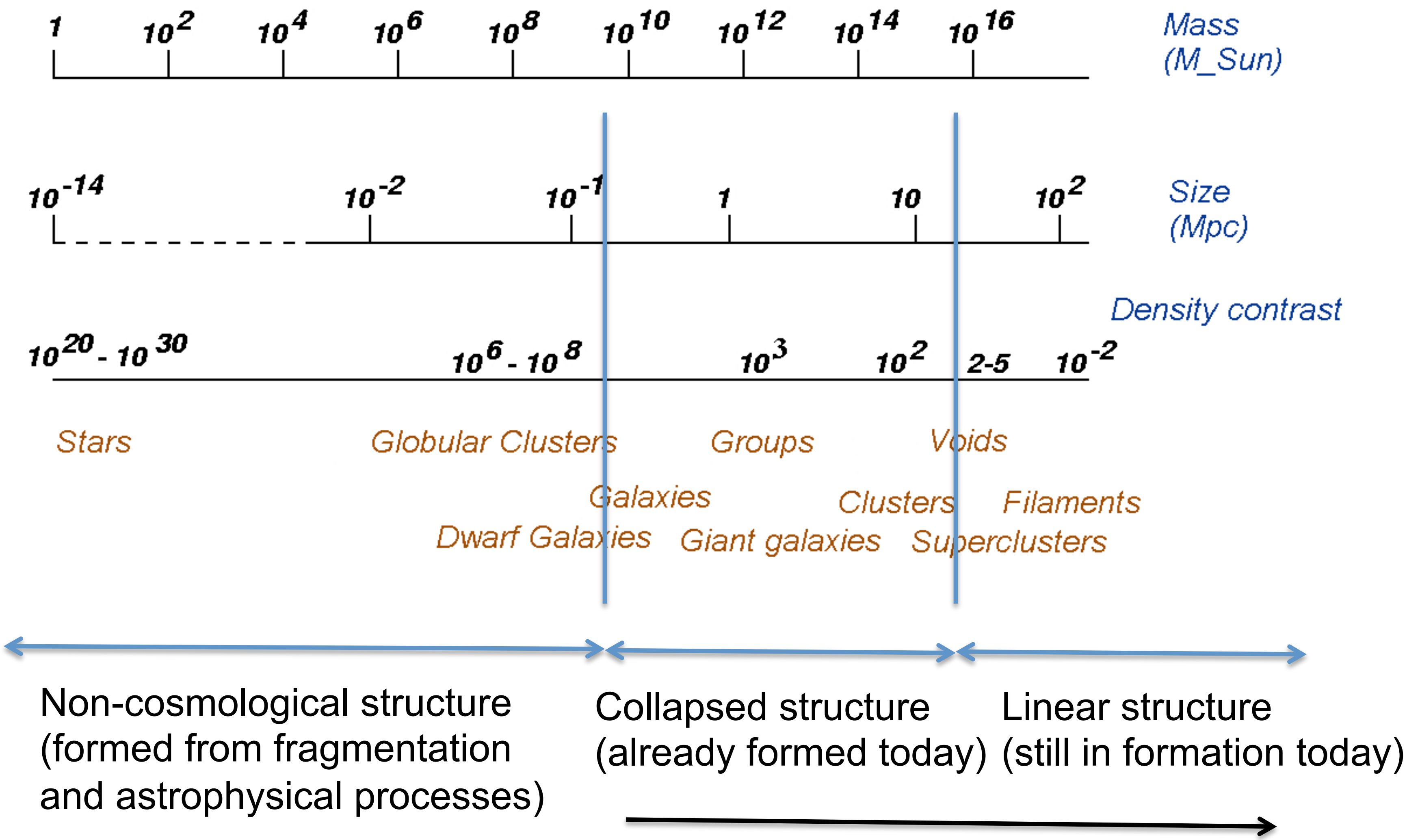
But they continue to evolve and will form large bound structures in the future.

Very large scales (R > 500 Mpc)

In these scales δ is so small that perturbations are still negligible today.

If $\Delta^2 \rightarrow 0$ on large scales, the Universe has a homogeneity scale.

The bottom-up diagram



Characteristic scales of the universe

are scales associated to physical processes that occur at certain transition redshifts and imprint features in the power spectrum.

Five important characteristic scales are:

- The particle horizon of dark matter at the relativistic - non-relativistic transition → the cut-off scale in the matter power spectrum (free-streaming)

- The characteristic scale at z=0 → the end of collapse reaching virialization *(to be seen later)*

- The turn-around scale at z=0 → the start of the non-linear regime

- The Hubble radius at radiation-matter equality → the peak in the matter power spectrum

- The sound horizon at z_dec → the first peak in the CMB power spectrum and also the baryon acoustic oscillations (BAO) peak in the matter power spectrum *(to be seen later)*

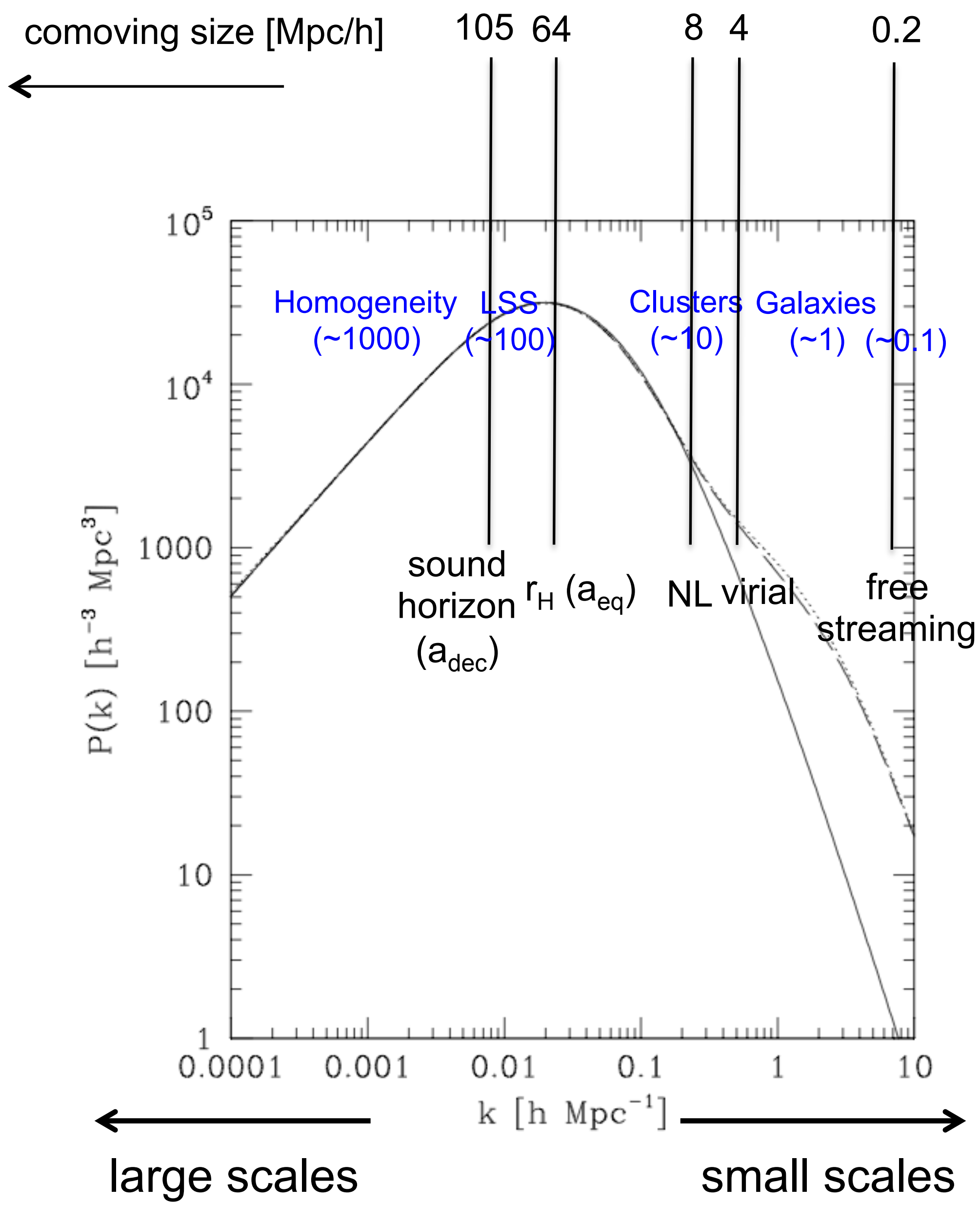
The dark matter power spectrum today (z=0)

(concordance model)

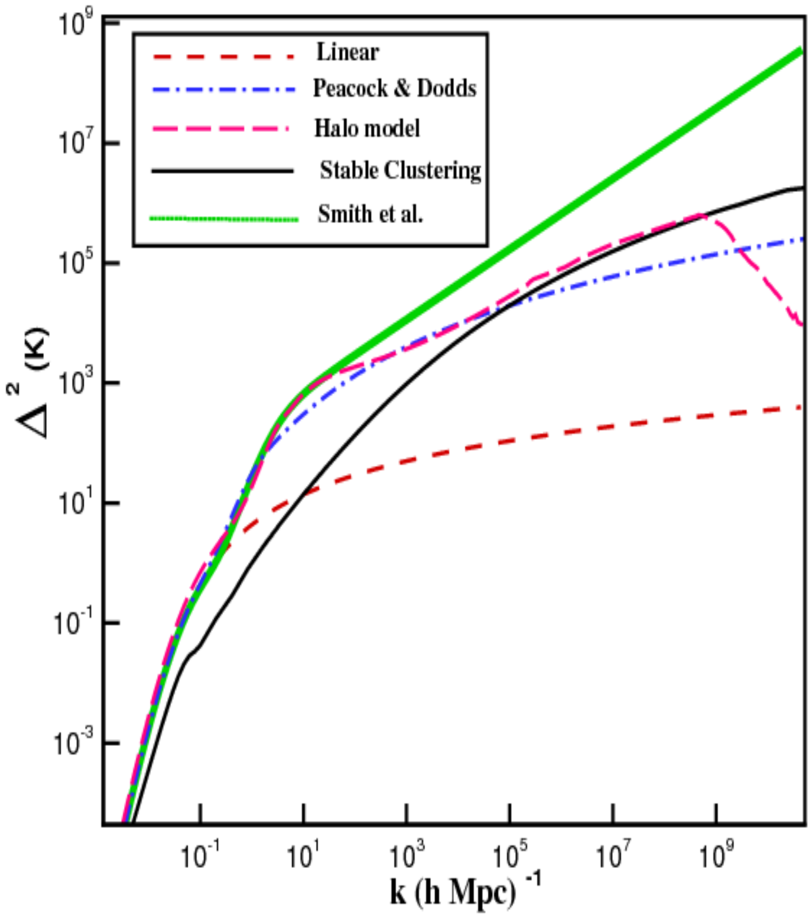
The structure formation process transforms the primordial matter power spectrum of the early Universe into the current matter power spectrum of z=0.

The shape of the power spectrum changed from the original power-law k^1 to a sequence of power-laws with indexes 1,0,-1,-2,-3

The dashed line shows the non-linear power spectrum. It deviates from the linear one for scales smaller than the NL scale.



The figure shows the **theoretical dimensionless dark matter power spectrum** at z=0, up to an extremely small scale.



Its shape is a a sequence of power-laws with indexes 4,3,2,1,0

The dimensionless power spectrum increases for small scales, while the power spectrum has a peak.

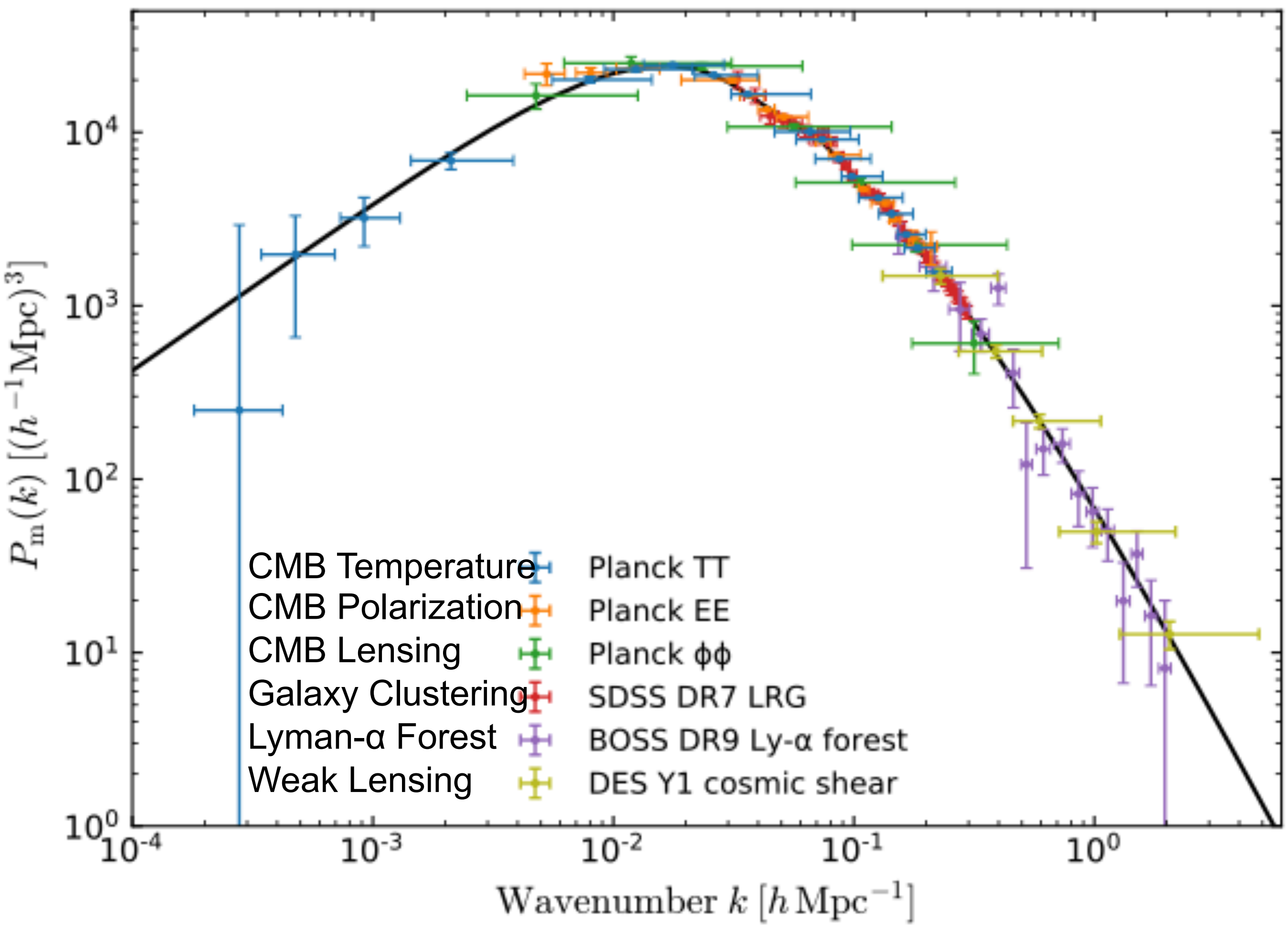
The amplitude of clustering is directly seen in the dimensionless power spectrum and not in the power spectrum. The reason is that the power spectrum has dimensions of volume (inverse of k-volume), so the amplitude it shows on small scales is greatly decreased by a factor of k^3.

The linear Δ^2 is the dashed red line. All the other lines show the non-linear Δ^2 computed in different ways. Notice that:

-The ratio between non-linear and linear is very large on these small scales.
- Different methods to compute the non-linear solution give quite different results.

The figure shows the **measured dark matter power spectrum** by various **cosmological probes of the inhomogeneous universe,** measured in 4 different **surveys**

Each survey may observe at a different redshift (but all results are transposed for z=0)



The various probes observe different scales:

- Probes of high redshift (CMB) measured in surveys of high resolution (Planck) may measure correlations on small angular scales, but even a small angular scale corresponds to a large size (given the high z) , and so to a large scale (small k) when transposed to z=0.
- Probes of lower redshift - DES measured correlations of WL on smaller scales than the GC measurements made by SDSS that covered a larger area of the sky.