Structure Formation

The Einstein-Boltzmann equations

Einstein equations

scalar perturbations

We will consider metric scalar perturbations and derive the **Einstein equations** in the **Newtonian gauge** to linear order

In this case, the perturbed Robertson-Walker metric is:

$$ds^2 = a(\eta)^2 \left[-(1+2\Phi)d\eta^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\right]$$

Note that there are different sign conventions (+,-) (-,+) (+,+) and different naming conventions found in the literature:

For example

$$ds^2 = a(\eta)^2 \left[-(1+2\Psi)d\eta^2 + (1+2\Phi)\delta_{ij}dx^i dx^j\right]$$ is used in Dodelson

$$ds^2 = a(\eta)^2 \left[-(1+2\Psi)d\eta^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right]$$ is used in Liddle & Lyth and Baumann

In order to write the Einstein equations, we need first to compute the following quantities:

Einstein tensor: $G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu$

Ricci scalar: $R \equiv R^0_0 + R^i_i$

Ricci tensor: $R_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\alpha\mu,\nu} + \Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}$

Connection: $\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2}g^{\mu\lambda}(g_{\lambda\beta,\mu} + g_{\alpha\lambda,\beta} - g_{\alpha\beta,\lambda})$

given our metric: $$g_{\mu\nu} = a^2 \begin{pmatrix} -1-2\Phi & 0 \\ 0 & (1-2\Psi)\delta_{ij} \end{pmatrix}$$

and its inverse: $$g^{\mu\nu} = a^{-2} \begin{pmatrix} -1+2\Phi & 0 \\ 0 & (1+2\Psi)\delta_{ij} \end{pmatrix}$$

We need first to compute the **connections**.

For example, for the term 000 we have,

$$\Gamma^0_{00} = \frac{1}{2} g^{00} g_{00,0} = \frac{1}{2a^2}(-1+2\Phi)[-a^2(1+2\Phi)]' =$$

$$= (1-2\Phi)[\Phi' + \frac{a'}{a}(1+2\Phi)] =$$

$$= \Phi' + \mathcal{H} + 2\mathcal{H}\Phi - 2\Phi\Phi' - 2\mathcal{H}\Phi - 4\mathcal{H}\Phi^2 \simeq \Phi' + \mathcal{H}$$

The results for all terms are:

$$\Gamma^0_{00} = \mathcal{H} + \Phi', \qquad \Gamma^0_{0k} = \Phi_{,k}, \qquad \Gamma^0_{ij} = (\mathcal{H} - 2\mathcal{H}(\Phi + \Psi) + \Psi')\delta_{ij},$$

$$\Gamma^i_{00} = \Phi_{,i}, \qquad \Gamma^i_{0j} = (\mathcal{H} - \Psi')\delta_{ij}, \qquad \Gamma^i_{kl} = -(\Psi_{,l}\delta^i_k + \Psi_{,k}\delta^i_l) + \Psi_{,i}\delta_{kl}.$$

Note the results show **a natural separation between the background RW and perturbation**:

$$\Gamma^{\alpha}_{\beta\gamma} = \bar{\Gamma}^{\alpha}_{\beta\gamma} + \delta\Gamma^{\alpha}_{\beta\gamma}$$

where

$$\bar{\Gamma}^{0}_{00} = \mathcal{H} \qquad \bar{\Gamma}^{0}_{0k} = 0 \qquad \bar{\Gamma}^{0}_{ij} = \mathcal{H}\delta_{ij}$$

$$\bar{\Gamma}^{i}_{00} = 0 \qquad \bar{\Gamma}^{i}_{0j} = \mathcal{H}\delta^{i}_{j} \qquad \bar{\Gamma}^{i}_{kl} = 0$$

and

$$\delta\Gamma^{0}_{00} = \Phi' \qquad \delta\Gamma^{0}_{0k} = \Phi_{,k} \qquad \delta\Gamma^{0}_{ij} = -\left[2\mathcal{H}(\Phi+\Psi) + \Psi'\right]\delta_{ij}$$

$$\delta\Gamma^{i}_{00} = \Phi_{,i} \qquad \delta\Gamma^{i}_{0j} = -\Psi'\delta^{i}_{j} \qquad \delta\Gamma^{i}_{kl} = -\left(\Psi_{,l}\delta^{i}_{k} + \Psi_{,k}\delta^{i}_{l}\right) + \Psi_{,i}\delta_{kl}$$

From this we can compute the **Ricci tensor**

$$R_{\mu\nu} \equiv \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\alpha\mu,\nu} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}$$

(it includes sums over all t,x derivatives and sums of products of two connections)

Note: some useful sums are:

$$\begin{aligned} \Gamma^{\alpha}_{0\alpha} &= 4\frac{a'}{a} + \Phi' - 3\Psi' \\ \Gamma^{\alpha}_{i\alpha} &= \Phi_{,i} - 3\Psi_{,i} \end{aligned}$$

Now, computing for example for 00:

$$R_{00} = \partial_\rho \Gamma^{\rho}_{00} - \partial_0 \Gamma^{\rho}_{0\rho} + \Gamma^{\alpha}_{00}\Gamma^{\rho}_{\alpha\rho} - \Gamma^{\alpha}_{0\rho}\Gamma^{\rho}_{0\alpha}$$

in this case, a term with ρ=0 always cancels out some other term, and so:

$$\begin{aligned} R_{00} &= \partial_i \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{0i} + \Gamma^{\alpha}_{00}\Gamma^{i}_{\alpha i} - \Gamma^{\alpha}_{0i}\Gamma^{i}_{0\alpha} \\ &= \partial_i \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{0i} + \Gamma^{0}_{00}\Gamma^{i}_{0i} + \underbrace{\Gamma^{j}_{00}\Gamma^{i}_{ji}}_{\mathcal{O}(2)} - \underbrace{\Gamma^{0}_{0i}\Gamma^{i}_{00}}_{\mathcal{O}(2)} - \Gamma^{j}_{0i}\Gamma^{i}_{0j} \end{aligned}$$

The results for all terms are:

$$\begin{aligned}
R_{00} &= -3\mathcal{H}' + 3\Psi'' + \nabla^2\Phi + 3\mathcal{H}(\Phi' + \Psi') \\
R_{0i} &= 2(\Psi' + \mathcal{H}\Phi)_{,i} \\
R_{ij} &= (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij} \\
&\quad + \left[-\Psi'' + \nabla^2\Psi - \mathcal{H}(\Phi' + 5\Psi') - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)\right]\delta_{ij} \\
&\quad + (\Psi - \Phi)_{,ij}
\end{aligned}$$

To compute the Einstein tensor, we also need the Ricci scalar:

$$R \equiv R^0_0 + R^i_i$$

This requires to **raise an index**. Note that this needs to be done using the full metric, we cannot just raise the index of the background and perturbative parts separately:

$$R^\mu_\nu = g^{\mu\alpha}R_{\alpha\nu} = (\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha})(\bar{R}_{\alpha\nu} + \delta R_{\alpha\nu}) = \bar{R}^\mu_\nu + \delta g^{\mu\alpha}\bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta R_{\alpha\nu}$$

(i.e., there are cross-terms)

The results for all terms are:

$$
\begin{aligned}
R^0_0 &= 3a^{-2}\mathcal{H}' + a^{-2}[-3\psi'' - \Delta\Phi + -3\mathcal{H}(\Phi' + \Psi') - 6\mathcal{H}'\Phi], \\
R^0_i &= -2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
R^i_0 &= 2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
R^i_j &= a^{-2}(\mathcal{H}' + 2\mathcal{H}^2)\delta^i_j \\
&\quad + a^{-2}[-\Psi'' + \Delta\Psi - \mathcal{H}(\Phi' + 5\Psi) - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)]\delta_{ij} \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij}.
\end{aligned}
$$

(Note: here the results are given for $R^\mu{}_\nu$ and not for $R_{\mu\nu}$, hence the a^{-2} factors)

and the **Ricci scalar** is thus:

$$
\begin{aligned}
R &= R^0_0 + R^i_i \\
&= 6a^{-2}(\mathcal{H}' + \mathcal{H}^2) \\
&\quad + a^{-2}\left[-6\Psi'' + 2\nabla^2(2\Psi - \Phi) - 6\mathcal{H}(\Phi' + 3\Psi') - 12(\mathcal{H}' + \mathcal{H}^2)\Phi\right]
\end{aligned}
$$

Finally, the **Einstein tensor** is:

$$
\begin{aligned}
G^0_0 &= -3a^{-2}\mathcal{H}^2 + a^{-2}[2\Delta\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\Phi], \\
G^0_i &= R^0_i = -2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
G^i_0 &= R^i_0 = 2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i}, \\
G^i_j &= R^i_j - \frac{1}{2}R\delta^i_j \\
&= a^{-2}(-2\mathcal{H}' - \mathcal{H}^2)\delta^i_j \\
&\quad + a^{-2}[2\Psi'' + \Delta(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi]\delta^i_j \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij}.
\end{aligned}
$$

This is the linearized Einstein tensor for the scalar-perturbed Robertson-Walker metric in the conformal Newtonian gauge.

It depends on :

a(t) and its time derivative,

the two metric potentials and their time and spatial derivatives.

Note that the off-diagonal components only have perturbations, while the diagonal components have both perturbations and background terms.

We can now write the Einstein equations $G^{\mu}_{\nu} = 8\pi G T^{\mu}_{\nu}$

considering the energy-momentum tensor background + perturbations

$$T^{\mu}{}_{\nu} = \bar{T}^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu}$$

$$\bar{T}^{\mu}{}_{\nu} = (\bar{\rho} + \bar{P})\bar{U}^{\mu}\bar{U}_{\nu} - \bar{P}\delta^{\mu}_{\nu}$$

$$\delta T^{\mu}{}_{\nu} = (\delta\rho + \delta P)\bar{U}^{\mu}\bar{U}_{\nu} + (\bar{\rho} + \bar{P})(\delta U^{\mu}\bar{U}_{\nu} + \bar{U}^{\mu}\delta U_{\nu}) - \delta P\,\delta^{\mu}_{\nu} - \Pi^{\mu}{}_{\nu}$$

Remember: the perturbations are density contrast δ, pressure δ_p , peculiar velocity, anisotropy tensor

The **velocity** 4-vector is $u^{\mu} = \bar{u}^{\mu} + \delta u^{\mu}$ and its norm is -1

in the background: $\bar{u}_{\mu}\bar{u}^{\mu} = \bar{g}_{\mu\nu}\bar{u}^{\mu}\bar{u}^{\nu} = -a^2(\bar{u}^0)^2 = -1$

the perturbation defines the **peculiar velocity**: $u^{i} = \bar{u}^{i} + \delta u^{i} = \delta u^{i} \equiv \frac{1}{a}v_i$

Hence, the 4-velocity vector is: $$u^\mu = \frac{1}{a}\begin{pmatrix} 1-\Phi \\ v_{N,i} \end{pmatrix}$$

Note the 0 component does not introduce a new perturbation because of the norm constraint.
The perturbation is the spatial part v_i

-- v_i, δ, δ_p are 5 components = 3S+2V

(for scalar perturbations, we just consider the scalar perturbation v associated with the vector $v_i \rightarrow v_i$ = grad(v))

-- the **traceless anisotropic stress** Π_{ij} accounts for the remaining 5 components = 1S+2V+2T

$$\delta T^i_j = \delta p \delta^i_j + \Sigma_{ij} \equiv \bar{p}\left(\frac{\delta p}{\bar{p}} + \Pi_{ij}\right)$$ (the 3x3 spatial tensor)

In conclusion, **the perturbed part of the energy-momentum tensor** is:

$$\delta T^\mu_\nu = \begin{bmatrix} -\delta\rho^N & -(\bar{\rho}+\bar{p})v^N_{,i} \\ (\bar{\rho}+\bar{p})v^N_{,i} & \delta p^N \delta^i_j + \bar{p}(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Pi) \end{bmatrix}$$

Note that the velocity perturbation does not contribute to the diagonal at linear order because it would contribute with a quadratic term vδ.

We can now write the Einstein equations

showing only linearized perturbations, i.e.,

- no background zero-order terms present

- no higher-order terms present → not valid for non-linear evolution

$$
\begin{aligned}
\delta G^0_0 &= a^{-2}\left[-2\nabla^2\Psi + 6\mathcal{H}(\Psi' + \mathcal{H}\Phi)\right] = -8\pi G\delta\rho^N \\
\delta G^0_i &= -2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = -8\pi G(\bar{\rho} + \bar{p})v^N_{,i} \\
\delta G^i_0 &= 2a^{-2}\left(\Psi' + \mathcal{H}\Phi\right)_{,i} = 8\pi G(\bar{\rho} + \bar{p})v^N_{,i} \\
\delta G^i_j &= a^{-2}\left[2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi\right]\delta^i_j \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij} = 8\pi G\left[\delta p^N\delta^i_j + \bar{p}(\Pi_{,ij} - \tfrac{1}{3}\delta_{ij}\nabla^2\Pi)\right] .
\end{aligned}
$$

The ij equations can be separated in diagonal and off-diagonal parts, and **the full set of equations** is,

$$
\begin{aligned}
3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \nabla^2\Psi &= -4\pi G a^2 \delta\rho^N \\
(\Psi' + \mathcal{H}\Phi)_{,i} &= 4\pi G a^2 (\bar{\rho} + \bar{p}) v^N_{,i} \\
\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \tfrac{1}{3}\nabla^2(\Phi - \Psi) &= 4\pi G a^2 \delta p^N \\
(\partial_i\partial_j - \tfrac{1}{3}\delta^i_j\nabla^2)(\Psi - \Phi) &= 8\pi G a^2 \bar{p}(\partial_i\partial_j - \tfrac{1}{3}\delta^i_j\nabla^2)\Pi
\end{aligned}
$$

The equations can also be written in **Fourier space**:

$$
\begin{aligned}
\left(\frac{k}{\mathcal{H}}\right)^2 \Psi &= -\tfrac{3}{2}\left[\delta^N + 3(1+w)\frac{\mathcal{H}}{k} v^N\right] \\
\left(\frac{k}{\mathcal{H}}\right)^2 (\Psi - \Phi) &= 3w\Pi \\
\mathcal{H}^{-1}\Psi' + \Phi &= \tfrac{3}{2}(1+w)\frac{\mathcal{H}}{k} v^N \\
\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}\left(\Phi' + 2\Psi'\right) + \left(1 + \frac{2\mathcal{H}'}{\mathcal{H}^2}\right)\Phi - \tfrac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2 (\Phi - \Psi) &= \tfrac{3}{2}\frac{\delta p^N}{\bar{\rho}},
\end{aligned}
$$

In the case of a perfect fluid ($\Pi_{ij} = 0$) and only scalar fluid perturbations, there are only 4 independent Einstein equations (00, 0i, ii, ij) since all spatial i are identical.

In this case, the 4 **first-order linearized Einstein equations** in the Newtonian gauge reduce to:

$$\nabla^2\Psi - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi G a^2 \bar{\rho}\delta$$ "Friedmann / Poisson"

$$\Psi' + \mathcal{H}\Phi = -4\pi G a^2(\bar{\rho} + \bar{p})v$$ new "velocity"

$$\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p$$ "Raychaudhuri / eq. movement"

$$\Psi - \Phi = 0$$ new "anisotropy"

We see that there are 4 fundamental Einstein equations at first-order perturbative level, in contrast with only 2 at background level.

For dark matter (no pressure or pressure perturbations) they can be used to solve for the 4 unknowns: Φ, Ψ, δ, v

We can also write separate **zeroth-order Einstein equations,** i.e., for the homogeneous background.

Since $T_{\mu\nu}$ is the sum of background + matter perturbations and only two of the Einstein tensor components (G_{00} and G_{ll}) are a sum of background + metric perturbations, there are only 2 background Einstein equations.

These are:

$$\mathcal{H}^2 = \frac{8\pi G}{3}\bar{\rho}a^2;$$

$$\mathcal{H}' = -\frac{4\pi G}{3}(\bar{\rho} + 3p)\,a^2$$

notice that

$$(\mathcal{H}' + \mathcal{H}^2)\,a^{-2} = \frac{\ddot{a}}{a} + H^2$$

$$2\mathcal{H}' = -\mathcal{H}^2(1+3w)$$

$$2\mathcal{H}' + \mathcal{H}^2 = -3w\mathcal{H}^2$$

i.e., we recover **Friedmann and Raychaudhuri equations.**

Let us go through the equations one by one.

00 - the Hamiltonian constraint

This equation relates the Laplacian of the potential with the matter density → it is a relativistic Poisson equation.

The two new terms, Ψ' and Φ, function of the potentials, are relativistic corrections to the Newtonian Poisson equation.

The corresponding background equation is the Friedmann equation

→ so Friedmann equation is a kind of Poisson equation, relating the density with gravity (metric) properties.

In the homogeneous case the metric property is the scale factor and not the potential. The potential is a perturbation and does not appear in the homogeneous FRW universe.

The scale factor is related to the "potential of the homogeneous universe", being responsible for the redshift (like the potential is responsible for a gravitational redshift). The potential has dimensions of velocity square → the Hubble flow.

0i - the momentum constraint

This is the peculiar velocity equation.

It has no background counterpart.

Combining equations 00 and 0i, we can cancel out the relativistic corrections and obtain a Poisson equation for the gauge-invariant Δ

$$\nabla^2\Psi = 4\pi G a^2 \bar{\rho}(\delta + 3\mathcal{H}v)$$

that thus defines the GR meaningful “**effective density contrast**”.

ii - the pressure constraint (potential evolution equation)

This equation involves second-order time derivative of the potential → it is an equation of movement of the potential, describing the evolution of the metric perturbation.

The corresponding background equation is the Raychaudhuri equation → it is the equation of movement for the scale factor.

ij - the anisotropy constraint

This equation tells us that the two Bardeen potentials are equal → it is called the anisotropy equation.

If there is anisotropic stress, the two potentials are no longer equal → in GR, a perfect fluid always induces a metric with equal potentials.

It has no background counterpart.

Let us see a few results of these equations.

Equation 4 (ij): anisotropy equation Ψ

A detection of a difference between the potentials (in the case of a perfect fluid) is a possible **signature of modified gravity**.

This signature is usually parameterized introducing the gravitational slip parameter η

$$\Psi = (1 + \eta)\Phi$$

Since there are 2 independent scalar metric perturbations → 2 scalar dof → 2 gravitational potentials in a relativistic theory of gravitation → there is room for a second independent modified gravity signature.
This is usually parameterized by the mass screening parameter Q, or equivalently by an effective gravitational constant G_eff.

This means that G would be different in that theory → it would be equivalent to consider that the same value of the potential is created by a different value of density, through a modified Poisson equation:

$$\nabla^2\Psi = -4\pi G Q a^2 \bar{\rho}\Delta$$

Equation 3 (ii): evolution of the potential Φ

Let us start by introducing the definition of **sound speed** in the equation

$$c_s = \left(\frac{\delta_p \, \bar{p}}{\delta \, \bar{\rho}} \right)^{1/2}$$

We see that the right-hand sides of equations 00 and ii only differ by a factor c_s^2, i.e. → 00 = ii c_s^2

Inserting eq. 00 in eq. ii, and using eq. ij ($\Psi = \Phi$) , we obtain an equation of motion for Φ:

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' + \left[2\mathcal{H}' + \mathcal{H}^2(1 + 3c_s^2) - c_s^2\nabla^2\right]\Phi = 0$$

On **small scales**

$$\frac{1}{k} << \mathcal{H}$$

the evolution of the metric perturbation Φ can be approximated by (in the harmonic space)

$$\Phi'' - c_s^2 k^2 \Phi = 0$$

i.e., all terms with H are neglected.

This is a wave equation → Φ oscillates in time, propagating with a velocity given by c_s.

This equation confirms that **the ratio of the pressure to the density perturbations is the velocity of propagation in the fluid.**

On **large scales**

the terms with k are neglected

In the case of a barotropic fluid: p = wρ

→ $c_s^2 = w$

In the case of a adiabatic fluid: $c_s^2 = \partial p / \partial \rho$

In this case, the evolution of the potential is given by:

$$\Phi'' + 3\mathcal{H}(1 + c_s^2)\Phi' = 0$$

(since $2\mathcal{H}' = -\mathcal{H}^2(1 + 3w)$)

This second-order differential equation has 2 solutions:

-- a constant → ***the potential remains constant in time***

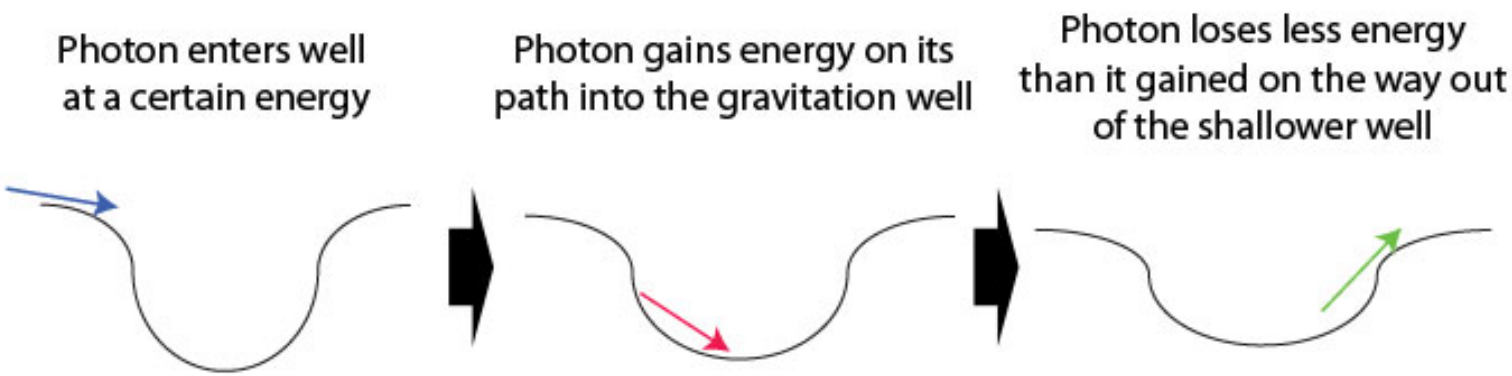
-- a **decaying solution**

The actual solution Φ (t) depends on the background evolution H(t).

In the late universe when dark energy becomes important, the dominating behaviour is the decaying solution → the potential decreases with time.

That evolution can be used to **test dark energy models** → when CMB photons cross an evolving LSS potential they are blue-shifted (gain energy when entering) and then redshifted (lose energy when leaving).

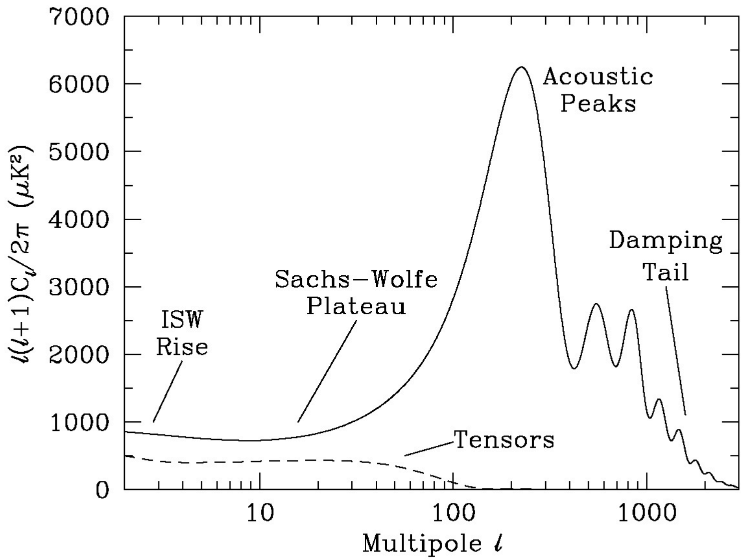
The energy balance is not zero, they gain energy if the potentials decay → their temperature increases with respect to their original temperature.



Gravitational well of galaxy supercluster - the depth shrinks as the universe (and cluster) expands

The effect is larger on large scales (because photons take longer to cross the larger potentials) → **it is measurable as a change in the amplitude of the CMB power spectrum at large scales.**

It is a test of dark energy (or also a signature of modified gravity), called the Integrated Sachs-Wolfe effect.



Equation 2 (0i): evolution of the peculiar velocity v

Taking now the 0i equation, and inserting the constant potential solution, and the Friedmann equation, the equation for the velocity becomes,

$$v_N = -\frac{2\Phi}{3\mathcal{H}}.$$

In the matter-dominated epoch, the conformal Hubble function decreases as $a^{-1/2}$ → ***the peculiar velocity grows with $a^{1/2}$*** *as dark matter clusters* in the matter-dominated epoch.

Equation 1 (00): evolution of the density contrast δ

Inserting the constant potential solution in the 00 equation (Poisson), and using Friedmann's equation, the equation for the density becomes,

$$\delta_N = -2\Phi - \frac{2k^2}{3\mathcal{H}^2}\Phi$$

Small-scales → the k^2 term dominates.

In the matter-dominated epoch, the conformal Hubble function decreases as $a^{-1/2}$
→ ***the density contrast grows with a***

Large-scales → the constant term dominates.

δ does not grow.
However, on large scales we need to consider the comoving gauge-invariant density contrast Δ. This is the one that enters the relativistic Poisson equation and is the quantity that has physical meaning in a general relativistic covariant framework.

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\,\Delta$$

From this Poisson equation, we see that:

- radiation epoch → $\mathbf{\Delta \sim a^{-2}\, a^{4} \sim a^{2}}$

- matter epoch → $\mathbf{\Delta \sim a^{-2}\, a^{3} \sim a^{1}}$

This result can also be found with the mini-universe approach.

Now: the Einstein equations do not contain differential equations for the source perturbations, but only for the metric perturbations.

However, observations measure parameters of the source (not of the metric potential) → **it would be more convenient to study the evolution of δ from a differential equation for δ, defining initial conditions (cosmological parameters) for δ.**

Energy conservation equations

Like it is done for the background, we can obtain more equations by considering the energy conservation of the energy-momentum tensor:

$$\nabla_\mu T^\mu_\nu = 0.$$

i.e.,

$$T^\mu_{\nu;\mu} = T^\mu_{\nu,\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha = 0$$

At first-order we obtain 2 conservation equations (instead of a single one as was the case for the background)

ν = 0

$$\partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 + \underbrace{\Gamma^\mu_{\mu i} T^i{}_0}_{\mathcal{O}(2)} - \Gamma^0_{00} T^0{}_0 - \underbrace{\Gamma^0_{i0} T^i{}_0}_{\mathcal{O}(2)} - \underbrace{\Gamma^i_{00} T^0{}_i}_{\mathcal{O}(2)} - \Gamma^i_{j0} T^j{}_i = 0$$

This case has a time derivative of T_{00} and a spatial derivative of T_{0i} ,
plus dependence on the potential through the metric (covariant derivative).

Inserting the energy-momentum components and the connection coefficients, the result is an **energy conservation equation**.

Collecting the pure background terms, the result is the zero-order continuity equation, that accounts for the energy conservation in the expanding background:

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{P})$$

The remaining terms are the first-order relativistic continuity equation:

$$\delta' + 3\mathcal{H}(c_s^2 - w)\,\delta + (1+w)(\nabla . v + 3\Phi') = 0$$

note that the divergence of the peculiar velocity is usually denoted $\theta = \nabla . v$

We can compare it with the Newtonian first-order (linearized) comoving continuity equation (for dark matter):

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla . \mathbf{v} = 0$$

For dark matter ($w=0$, $c_s^2 = 0$), the only difference (i.e. the relativistic correction) is the term with the derivative of the potential, that is negligible for slow-varying or constant potentials.

ν = 1

$$\partial_0 T^0{}_i + \partial_j T^j{}_i + \Gamma^\mu_{\mu 0} T^0{}_i + \Gamma^\mu_{\mu j} T^j{}_i - \Gamma^0_{0i} T^0{}_0 - \Gamma^0_{ji} T^j{}_0 - \Gamma^j_{0i} T^0{}_j - \Gamma^j_{ki} T^k{}_j = 0$$

This case has a time derivative of T_{0i} and spatial derivatives, plus dependence on the potential through the metric (covariant derivative).

At background level there is no T_{0i} term and thus there is just one conservation equation.

At perturbative level we get a **momentum conservation equation:**

$$\theta' + \left[\mathcal{H}(1-3w) + \frac{w'}{1+w}\right]\theta = -\nabla^2\left(\frac{c_s^2}{1+w}\delta + \Phi\right)$$

This is also a fundamental equation in fluid dynamics - the **Euler equation** - it is the (acceleration) equation of movement of a Newtonian fluid.

We can compare it with the Newtonian first-order (linearized) comoving Euler equation (for dark matter):

$$\frac{\partial v}{\partial t} + \frac{\dot{a}}{a} v = -\frac{1}{a} \nabla_x \delta\Phi$$

For dark matter (w=0, c_s^2 = 0, w'=0), the Newtonian and relativistic equations are identical.

It tells us that the rate of change of velocity depends on the background expansion, and of the gradients of pressure and gravitational potential ("forces").

Like we saw, it has no counterpart in homogeneous cosmology.

These two fluid evolution equations are not independent of the Einstein equations, but they can be used instead of the two Einstein evolution equations, or in combination with them.

They have the interest of introducing explicitly differential equations for the density contrast and peculiar velocity.

Up to now, the results we found in the relativistic approach are not very different from the ones in the Newtonian approach.

The main differences were:

- the Friedmann equation appears as a Poisson equation (no need to introduce it by hand)

- the Raychadhuri equation appears as an equation for the evolution of the potential (was not part of the set of Newtonian equations)

- the relativistic terms of those equations contain new information that allows us to compute the evolution on large scales, and define a gauge-invariant density contrast

- the continuity and Euler equation appear naturally as before

Perturbed Boltzmann equation

However, the energy-momentum fluid description is not always valid.

Beyond background level, radiation is not well described by a cosmological fluid approach.

The perturbations in the plasma density cannot be described by a coherent fluid with a well defined velocity → various particle fluxes intersect in the global fluid (multi-streams).

Even for dark matter, in the radiation epoch, the evolution is not accurately computed by using an energy-momentum fluid in the Einstein equations.

The energy-momentum conservation must be studied at the level of particles and not at fluid level, using a kinetic approach (statistical physics) → a transport equation that describes the evolution of a distribution function f(x,p,t) of the cosmological species in the phase space.

The evolution of a distribution function f(x,p,t) is described by the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(\dot{x} f) + \frac{\partial}{\partial p}(\dot{p} f) = \left(\frac{df}{dt}\right)_c$$

or the Vlasov equation if the total derivative of f is conserved (the **collisionless** case):

$$\frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \dot{p}\frac{\partial f}{\partial p} = 0$$

The perturbations - density contrast = n(1) / n(0); and velocity v - **are moments of the energy-momentum distribution.**

Remember that the α- order moment of a distribution of a variable, is the integral of the variable over its space weighted by its distribution function.

$$n \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} f \qquad\qquad nv^i \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3} f \frac{p\hat{p}^i}{E}$$

(the normalization of f) (the weighted mean of the velocity)

Since the Boltzmann equation describes the evolution of the distribution f in the phase space → the moments of this equation will be equations that describe the evolution of the moments of particles that follows that distribution → i.e. equations for the evolution of energy density and momentum → i.e. conservation equations.

This description implies a hierarchy of equations, corresponding to the moments of the Boltzmann equation.

In particular, for **cold dark matter**, the energy and momentum of particles of mass m in the perturbed scalar RW metric, are written as

$$g_{\mu\nu}P^{\mu}P^{\nu} = -m^2 \qquad P^i = \frac{p}{a}(1-\Phi)\hat{p}^i \qquad P^0 = E(1-\Psi)$$

(Notation: here the naming of the potentials is inverted)

The collisionless Boltzmann equation is then:

$$\frac{\partial f}{\partial t} + \frac{p}{aE}\hat{p}^i\frac{\partial f}{\partial x^i} - \left(H\frac{p^2}{E} + \frac{p^2}{E}\dot{\Phi} + \frac{p\hat{p}^i}{a}\partial_i\Psi\right)\frac{\partial f}{\partial E} = 0$$

The **zeroth-order moment** of the collisionless CDM Boltzmann equation for dark matter is found by computing the integral of each term :

$$\dot{n} + \frac{1}{a}\partial_i(nv^i) - (H + \dot{\Phi}) \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\partial f}{\partial E} \frac{p^2}{E} - \frac{1}{a}\partial_i \Psi \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\partial f}{\partial E} p\hat{p}^i = 0$$

Integrating all terms, the result is:

$$\dot{n} + \frac{1}{a}\partial_i(nv^i) + 3(H + \dot{\Phi})n = 0$$

this is the continuity equation

The **first-order moment** of the collisionless CDM Boltzmann equation is its integration in momentum space with its terms multiplied by $p\hat{p}^i/E$

The result is:

$$\frac{\partial v^i}{\partial t} + Hv^i + \frac{1}{a}\partial_i \Psi = 0$$

this is the Euler equation

For cold dark matter, this approach just provided an alternative method that led to the same conservation equations (alternative to using the conservation of the T_{ab} tensor).

However, for perturbations in the radiation component this approach is really needed, since they cannot be described by a fluid.

It is the correct procedure to compute the density perturbations in the radiation-baryonic plasma (needed to compute the CMB power spectrum) or the velocity radiation perturbations (needed to compute dark matter perturbations in multi-fluid coupled equations)

For example for **photons**, we need to consider the Bose-Einstein distribution function:

$$f(t, p, \boldsymbol{x}, \hat{p}) = \left\{ \exp\left[\frac{p}{T(t)[1 + \Theta(t, \boldsymbol{x}, \hat{p}^i)]} \right] - 1 \right\}^{-1} .$$

where the temperature fluctuations are $\Theta(t, \boldsymbol{x}, \hat{p}^i) \equiv \delta T / T$

The Boltzmann equation leads to the **differential equation for the evolution of the temperature fluctuations**:

$$\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} = n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{\boldsymbol{p}} \cdot \boldsymbol{v}_b]$$

For **baryons**, this approach is also needed, but since they are massive particles, the distribution function is different, as well as the relation between energy and momentum.

This allows us to derive the full set of **photon-baryon coupled equations**:
(written in Fourier space)

$$\Theta' = -\Phi' - ik\mu(\Theta + \Psi) - \tau'_{\rm op}(\Theta_0 - \Theta + \mu v_b),$$

$$\delta'_b = -ikv_b - 3\Phi',$$

$$v'_b = -\mathcal{H}v_b - ik\Psi + \frac{\tau'_{\rm op}}{R_s}(3i\Theta_1 + v_b),$$

The public codes **(CAMB, CLASS)** that compute the linear evolution of cosmological structures for all cosmological species and for a large range of scales and redshift, implement this approach → they solve the system of Einstein-Boltzmann differential equations.

Other types of perturbations

Up to know we focused on scalar perturbations.

However, remember that there are also vector and tensor perturbations, and a possible total of 10 Einstein equations.

Vector perturbations

There are 4 V perturbations in the metric and 2 V components of the transformation vector ξ (the 2 vector components $\xi_i^{\perp}$) → the choice of gauge fixes 2 V components of the metric.

This can be done, for example, by setting the vector part of h to 0, (in addition to fixing 2 scalar components, for example w = h = 0)

There remains 2 V components of the metric (the vector parts of w).

In the Einstein equations, there are 3 equations that involve the vector metric perturbations and the vector source perturbations.

Those 3 equations are:

$$\begin{aligned}
(v_i^{\perp} + w_i^{\perp})' + \mathcal{H}(v_i^{\perp} + w_i^{\perp}) &= 0, \\
\frac{1}{2}\Delta w_i^{\perp} &= 8\pi G a^2 \bar{\rho}(v_i^{\perp} + w_i^{\perp}), \\
\left(w_{(i,j)}^{\perp}\right)' + 2\mathcal{H} w_{(i,j)}^{\perp} &= 0.
\end{aligned}$$

The solution, from the first equation is:

$$v_i^{\perp} + w_i^{\perp} \sim a^{-1}$$

This shows that the vector perturbations decay with time.

The vector part of initial velocity perturbations eventually disappear and they are not relevant in the standard cosmological model.

Tensor perturbations

There are 2 T perturbations in the metric and no T components of the transformation vector ξ → **tensor perturbations are gauge-invariant by construction** → no gauge fixing needed.

Even if the energy-momentum tensor has no tensor part (no anisotropic stress) there exists still one equation in the Einstein - energy conservation system that involves only tensor metric perturbations

(in fact 2 equations, since there are 2 T components) → these 2 components may also be written as a polarization vector (a polar vector).

The equations are:

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \Delta h_{ij} = 0$$

This is a second-order differential equation in time and space: a wave equation, also containing a first-order derivative term (a friction term, known as the Hubble drag).

The solution is:

$$h_{ij} \sim e^{i(\omega\tau + \mathbf{k}\cdot\mathbf{x})}$$

This means that the tensor perturbations evolve in time and space in a coherent way, as a propagating wave.

Even with no sources, initial tensor metric perturbations do not vanish and propagate as a wave.
We can say it is an **intrinsic property of GR** → these are the gravitational waves.

It seems a more fundamental property than gravity being attractive, because attraction depends on the source → with no initial sources (δ), there would be no structure formation, but there would still exist gravitational waves.

The amplitude of the wave does not remain constant, it decreases in time due to the Hubble drag term.

Remember that **inflation** sets the **initial conditions** for scalar and tensor metric perturbations:

For scalar perturbations → sets the slope of the primordial power spectrum of the curvature potential (the scalar index n_s) → sets the slope of the primordial matter power spectrum (through Poisson equation).

For tensor perturbations → sets the slope of the primordial power spectrum of tensor perturbations (the tensor index n_t) → no equivalence in a source power spectrum.

These are the **primordial gravitational waves**.

Local interactions of strong gravity can produce **secondary gravitational waves** → produced by periodic movement of compact objects: black holes, neutron stars binaries, etc.
(These are the ones that have been detected, not related to cosmology).

Observationally, there are two main **signatures of (cosmological) primordial gravitational waves** that are being explored:

- The metric at a location changes as the wave passes → produces a periodical change in the size (or distance) of objects

- GW polarize the CMB photons → could be detected in the CMB polarization power spectra.

Being a fundamental property of gravitation, GW can also be used to test modified gravity. Some theories of gravity may have a different number of tensor modes → different types of polarization in their gravitational waves.