Cosmologia Física

Homework 4

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Exercise 1: Newtonian perturbed fluid equations

(In the following, "boldface"denotes a vector).

1.1) Consider the continuity equation for the dark matter fluid in an expanding background in comoving coordinates:

$$\frac{\partial \rho}{\partial t} - \frac{\dot{a}}{a}\mathbf{x}.\nabla\rho + \frac{1}{a}\nabla.(\rho\mathbf{u}) = 0\,.$$

a) Inserting the density and velocity perturbations through, $\rho = \bar{\rho}(1+\delta)$ and $\mathbf{u} = \dot{a}\mathbf{x} + \mathbf{v}$, derive the perturbed linearized comoving continuity equation (i.e., the first equation given in 1.3 below).

1.2) Consider the Euler equation for the dark matter fluid in an expanding background in comoving coordinates:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{\dot{a}}{a}(\mathbf{x}.\nabla)\mathbf{u} + (\mathbf{u}.\frac{1}{a}\nabla)\mathbf{u} = -\frac{1}{a}\nabla\Phi\,.$$

a) Inserting the density and velocity perturbations through, $\rho = \bar{\rho}(1+\delta)$ and $\mathbf{u} = \dot{a}\mathbf{x} + \mathbf{v}$, derive the perturbed linearized comoving Euler equation (i.e., the second equation given in 2.3 below).

1.3) Consider the system of perturbed linearized comoving Newtonian equations of fluid mechanics (continuity, Euler and Poisson):

$$\dot{\delta} + \frac{1}{a}\nabla.\mathbf{v} = 0\,, \qquad \frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} = -\frac{1}{a}\nabla\Phi\,, \qquad \nabla^2\Phi = 4\pi G a^2\bar{\rho}\delta\,.$$

a) Combine the linearized equations to derive the equation of motion of δ,

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi\, G\,\bar{\rho}\,\delta = 0\,.$$

Exercise 2: Dark matter linear perturbations

2.1) Consider a flat Universe with some amount of non-relativistic pressureless massive neutrinos and no baryonic matter. In this Universe, the mean matter density Ω_m has two contributions: cold dark matter $\Omega_{\rm cdm}$ and neutrinos Ω_ν, i.e., $\Omega_m = \Omega_{\rm cdm} + \Omega_\nu$. We assume that only the cold dark matter component clusters and forms structure.

a) Compute the evolution of the scale factor $a(t)$ in the matter-dominated epoch for this universe.

b) Compute the evolution of the cold dark matter density contrast for this universe in the matter-dominated epoch.

c) Assume the neutrino fraction is 10% of the total matter density. In this case, using the result found in b), what is the growth rate (n) of the dark matter overdensities: $\delta_{\rm cdm} \propto a^n$?

2.2) Consider a flat Universe with dark matter and dark energy (where Ω_ν is now 0) where the dark matter growth rate is exactly the same found in 2.1 (i.e. n=0.94).

a) If dark energy is a cosmological constant, what is the value of Ω_Λ needed to produce that growth rate?

b) In the case that dark energy has a varying w=w(z) and the model is indistinguishable from the concordance ΛCDM at z=0, show that w(z) changed by roughly an order of magnitude since $z = 1$.

2.3) Consider a flat Universe with dark matter and a cosmological constant, such that the transition from matter to dark energy epochs occurs only today, at $a = 1$. In the matter epoch, the matter density contrast grows as $\delta \propto a$. Assume that $\delta(t)$ keeps that rate all the way until the transition (i.e., dark energy does not make the rate to decrease during the matter epoch). This means that we can write $\delta(a) = a\,\delta_0$, where δ_0 is the clustering amplitude today. After the transition, the universe is dominated by the cosmological constant. Assume that the mean dark matter density can be neglected immediately after the transition: $\Omega_m(a > 1) = 0$.

a) Solve the equation of motion for the dark matter density constrast δ in the dark energy dominated epoch $a > 1$.

b) In a) you must have found that the solution for $\delta(t)$ is the sum of a decaying solution plus a constant term (an integration constant). Show that the constant term in the solution can be written in the form $n\delta_0$. Find out the value of n.

c) Assume that no collapsed (non-linear) structures have yet formed in this universe at $a = 1$, i.e., $\delta_0 < 1$. What is the minimum value that δ_0 must have for non-linear structure to be able to form in this universe in the future?

Exercise 3: Dark matter non-linear perturbations

3.1) The linearized Newtonian fluid equations are not valid to describe the non-linear clustering (when $\delta > 1$). However N-body simulations use Newton's law and Poisson equation to evolve the density field until $z = 0$ obtaining very large values of overdensities and a valid non-linear power spectrum. Do you think this is a valid approach or is there an inconsistency here? Justify your answer.

3.2) Consider an overdensity that can be treated as a curved mini-universe with mean matter density Ω_m and positive curvature $\Omega_K < 0$ (and also no dark energy $\Omega_\Lambda = 0$ and negligible radiation) that evolves in a background Einstein-de Sitter Universe.

a) Show that the scale factor of this overdensity evolves according to the parametric solution

$$a(\theta) = \frac{\Omega_m}{2(\Omega_m - 1)}(1 - \cos\theta)$$

$$t(\theta) = \frac{\Omega_m}{2H_0(\Omega_m - 1)^{3/2}}(\theta - \sin\theta)$$

Hint: Check that the parametric solution satisfies the Friedmann equation.

b) Show that the density contrast of the overdensity when it reaches virial equilibrium (assuming the virial state is reached at $\theta = 3\pi/2$) is $\delta_c = 132$.

3.3) Consider the halo mass function

$$\frac{dn}{dM} = -\left(\frac{2}{\pi}\right)^{1/2} \frac{\Omega_m \rho_c}{M} \frac{\delta_c}{\sigma_M^2} \frac{d\sigma_M}{dM} \exp\left(-\frac{\delta_c^2}{2\sigma_M^2}\right).$$

The halo mass function is valid for a wide range of spherical halos of sizes R and corresponding masses $M = (4/3)\pi R^3 \bar{\rho}$. The correlation function of those halos is measured to be $\xi \sim r^{-(3+n)}$, where n is the power spectrum index (remember exercise 2 of Homework 3). The clustering amplitude σ_R (also called σ_M) of a scale of size R and mass M has the same dependence in R as the correlation function, i.e.,

$$\sigma_R^2 = \sigma_0^2 \left(\frac{R}{R_0}\right)^{-(3+n)} = \sigma_8^2 \left(\frac{R}{R_8}\right)^{-(3+n)},$$

where R_0 or R_8 are just examples of reference scales (any scale can be used for reference).

a) What is the dimension (i.e. the units) of the mass function?

b) Show that the amplitude in function of mass is

$$\sigma_M = \sigma_0 \left(\frac{M}{M_0}\right)^{-(3+n)/6}.$$

c) Show that the mass function given above can be written as

$$\frac{dn}{dM} = \frac{1}{\sqrt{\pi}} \Omega_m \rho_c \frac{3+n}{3} \frac{1}{M^2} \left(\frac{M}{M_0}\right)^{\frac{3+n}{6}} \exp\left[-\left(\frac{M}{M_0}\right)^{\frac{3+n}{3}}\right].$$

d) As you realized in your derivation, the result in c) is only valid for one particular reference scale M_0. What is the name and physical meaning of that scale? Compute its numerical value in terms of the reference mass M_8, using the standard values for the linear critical density for virialization ($\delta_c = 1.68$), for the clustering amplitude ($\sigma_8 = 0.9$) and power spectrum index ($n = -1.3$).

e) Compute the total mass density in the Universe (in the assumption that all mass is contained in halos).

Hint: Use the mass function derived in c).