Cosmological Observations

Cosmological Parameter Estimation

We saw how Bayesian inference methods allow us to find the posterior distribution in the parameters' space, from which we can find parameters' means, variances and best-fits, i.e., **estimates of the cosmological parameters.**

We saw two types of methods:

- Grid and Monte Carlo methods - where we sample the parameters' space in order to find the posterior PDF (**probability density function**)

- Fisher matrix - where we assume the PDF is a Gaussian centred on a fiducial value and just compute its covariance matrix.

We can now conclude *step 6* of a SN survey cosmological analysis and

Estimate the model parameters using SN observations

The cosmological information of the measured distance modulus is contained in the luminosity distance:

$$\mu(z) = 5 \log_{10} (D_L (z; H0, \Omega, w)) + 25$$

with $D_L = (1+z) D_C$ (for a flat Universe)

Let us investigate what is the **cosmological information** that the distance-modulus contains:

First, the comoving distance from the observer at t_0 to the source at t is computed from the metric as

$$D_c = \int_t^{t_0} \frac{1}{a} dt = \int_t^{t_0} (1+z) dt$$

a(t) (or H(t)) are usually computed using Einstein equations, and this expression will involve the density parameters (that define the cosmological model).

Alternatively, in order to access the cosmological information in a more fundamental **model-independent** way, let us consider first the **cosmographic approach,** where a(t) is expanded as:

$$a(t) = a_0 \left[1 + H_0 (t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \frac{1}{3!} j_0 H_0^3 (t-t_0)^3 + \frac{1}{4!} s_0 H_0^4 (t-t_0)^4 + \mathcal{O}(t-t_0)^5 \right]$$

From here, we can also write an expansion for z(t), since a-1 = - z/(z+1):

$$a - 1 = -\frac{z}{z+1}, \quad z = \frac{1}{a} - 1 \Rightarrow z = (z+1) \left[H_0 (t_0 - t) \overset{(-1)}{} + \frac{1}{2} q_0 H_0^2 (t_0 - t)^2 + \dots \right]$$

$$\Rightarrow z = \underbrace{H_0 (t_0 - t)}_{\text{the lowest-order term is order } \mathcal{O}(t)} + \frac{1}{2} q_0 H_0^2 (t_0 - t)^2 + \underbrace{z H_0 (t_0 - t)}_{\text{order } \mathcal{O}(t^2) + \dots} + \frac{1}{2} z \underbrace{q_0 H_0^2 (t_0 - t)^2}_{\text{order } \mathcal{O}(t^3) + \dots} + \dots$$

Keeping only order $\mathcal{O}(t^2)$ we may use $z \sim H_0 (t_0 - t)$ to insert here

$$\Rightarrow \boxed{z = H_0 (t_0 - t) + \left(\frac{1}{2} q_0 H_0^2 + H_0^2 \right) (t_0 - t)^2 + \mathcal{O}(t^3)}$$

Inserting the z(t) expansion in the comoving distance, we find, to second order:

$$D_c = \int_t^{t_0} \left[1 + H_0 (t_0 - t) \right] dt$$

(to second order we just need to consider order t in the integrand, because the integral will be order t^2)

$$\boxed{D_c = (t_0 - t) + \frac{H_0}{2} (t_0 - t)^2}$$

At this point it would be useful to invert the expansion z(t), to be able to find an expression for $D_c(z)$ instead of $D_c(t)$

Writing now for t :

$$\Rightarrow \quad t_0 - t = \frac{z}{H_0} - \left(\frac{q_0}{2} H_0^2 + H_0^2 \right) \frac{1}{H_0} (t_0 - t)^2 + \mathcal{O}(t^3)$$

$\downarrow$

again the same trick, using $(t_0 - t) \sim \frac{z}{H_0}$

$$\Rightarrow \quad t_0 - t = \frac{z}{H_0} - \left(\frac{q_0 H_0^2}{2} + H_0^2\right)\frac{z^2}{H_0^3} + \ldots$$

$$\text{or} \quad \boxed{t_0 - t = \frac{1}{H_0} z - \frac{1}{H_0}\left(1 + \frac{q_0}{2}\right) z^2 + \mathcal{O}(z^3)}$$

Now, Since

$$D_c = (t_0 - t) + \frac{H_0}{2}(t_0 - t)^2 + \mathcal{O}(t^3),$$

we get

$$D_c^{(z)} = \frac{1}{H_0} z - \left(1 + \frac{q_0}{2}\right)\frac{z^2}{H_0} + \frac{H_0}{2}\frac{z^2}{H_0^2} + \mathcal{O}(z^3)$$

$$\text{or} \quad D_c = \frac{1}{H_0} z - \left(1 + \frac{q_0}{2} - \frac{1}{2}\right)\frac{z^2}{H_0} + \mathcal{O}(z^3)$$

$$\Rightarrow \quad \boxed{D_c^{(z)} = \frac{1}{H_0}\left[z - \frac{1}{2}(1 + q_0) z^2\right]}$$

We arrive then at the expression for the luminosity distance that we were looking for:

$$D_L = \left[\frac{z}{H_0} - \frac{z^2}{H_0} \left(\frac{1}{2} + \frac{1}{2} q_0 \right) \right] (1+z) + \sigma(z^3)$$

$$= \frac{z}{H_0} - \frac{z^2}{H_0} \left(\frac{1}{2} + \frac{1}{2} q_0 \right) + \frac{z^2}{H_0} + \sigma(z^3)$$

or

$$\left[D_L = \frac{1}{H_0} \left(z + \frac{1}{2} (1 - q_0) z^2 \right) \right] \quad (\text{flat})$$

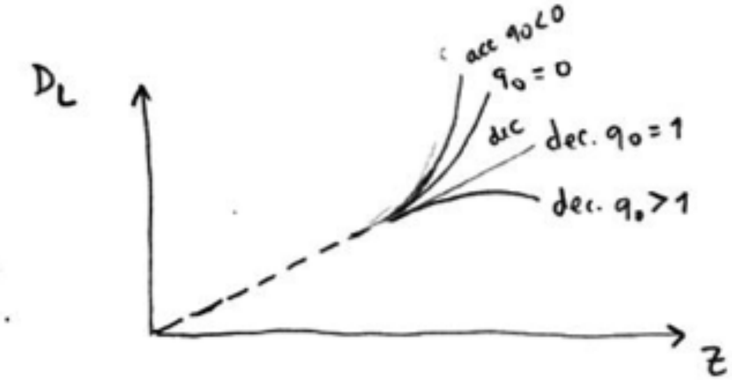
We see that, up to second-order, the luminosity distance:

- at low-z measures the (constant) **velocity of the Universe** (with (c)z/H_0 being the Doppler velocity)

- at high-z measures the (constant) **acceleration of the Universe** (q_0)

So, in order to detect an acceleration of the Universe we need to observe SNe at **high redshifts**.

SNe at **low redshifts** measure H_0 , i.e., the slope of the D_L (z) straight line



To determine **the value of the acceleration**, both high and low redshift measurements are needed, to break the **degeneracy** between H_0 (the Hubble law slope, important at lower z) and the acceleration (important at higher z)

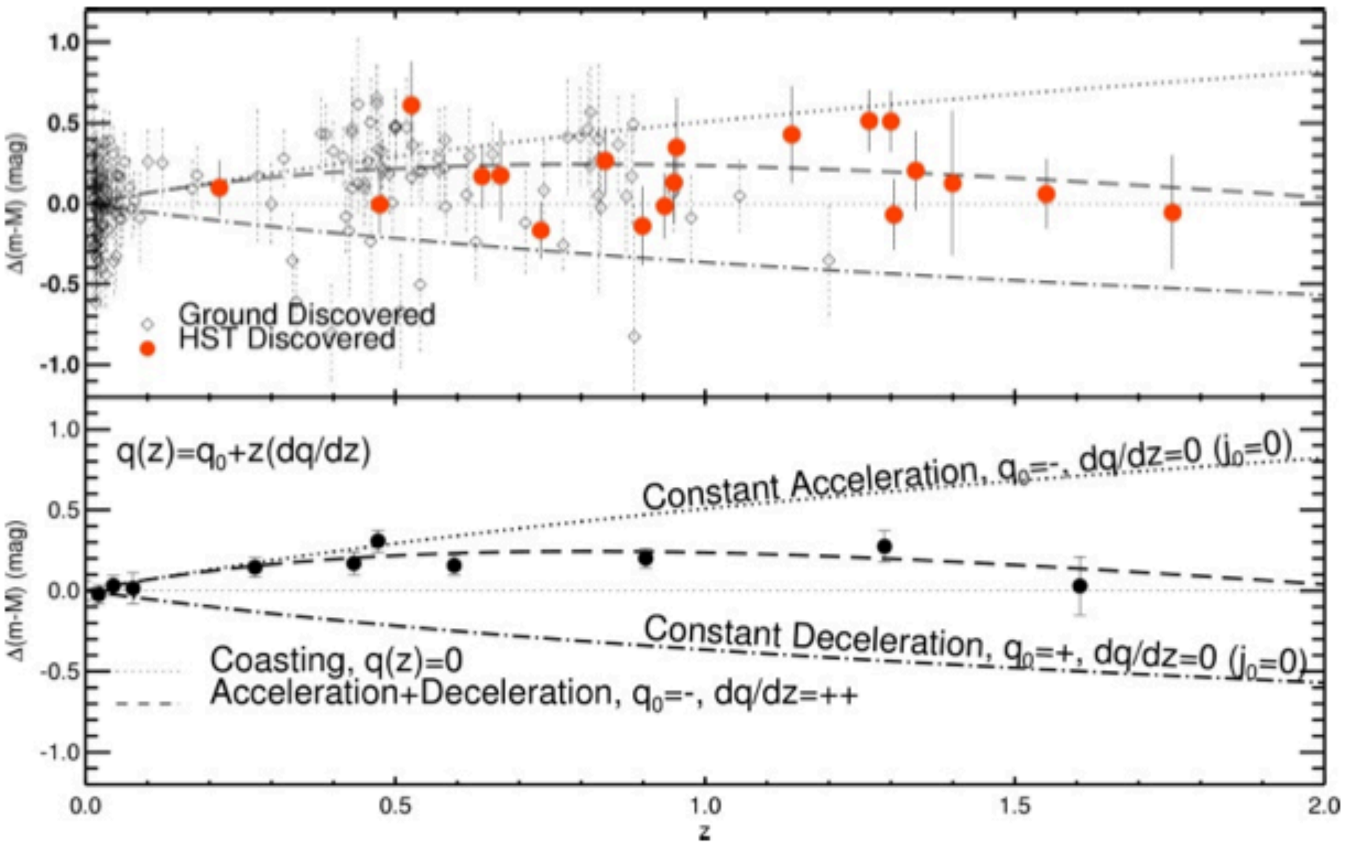
However, the evidence for acceleration is based only on the shape of the function.

So**,** even if the absolute values determined for the distances were not precise (i.e., if H_0 is estimated with a large uncertain) we can still find evidence for acceleration using only high-z SNe.

This result is only an approximation. **If we consider higher-orders**, we find

$$D_c(z) = \frac{z}{H_0}\left[1 - \left(1+\frac{q_0}{2}\right)z + \left(1+q_0+\frac{q_0^2}{2}-\frac{j_0}{6}\right)z^2 - \left(1+\frac{3}{2}q_0(1+q_0)+\frac{5}{8}q_0^3-\frac{1}{2}j_0-\frac{5}{12}q_0 j_0-\frac{s_0}{24}\right)z^3+O(z^4)\right]$$

The introduction of higher orders shows that the acceleration is not necessarily constant, i.e., the quadratic term depends also on j_0, i.e., there is a non-zero dq/dz.



The data seem to prefer a model with varying q_0 , such that **$q_0 > 0$ at high-z and $q_0 < 0$ at low-z.**

This plot shows the dynamic behaviour of the Universe (independently of the values of the density parameters) → clear **evidence for a model with acceleration for z < 1 and deceleration for z > 1** → proof of **late-time acceleration of the universe**

2011

Now, the cosmographic analysis was very useful to get an insight of the dynamic behaviour of the Universe, but in order to estimate cosmological parameters this analysis is not needed.

What we need is just to compute the observable cosmological function (i.e. the luminosity distance) from vectors of cosmological parameter values and compare the various theoretical D_L obtained with the observed one through the computation of likelihoods in the parameter space.

The dependence of the luminosity distance on the cosmological parameters can be most easily seen by writing the distance as an integral over redshift:

In general, $D_L(z) = (1+z)\, D_M$,

considering the case of flat Universe, we have

$$D_L(z) = (1+z)\, D_C(z) = (1+z) \int_{t}^{t_0} \frac{1}{a}\, dt = (1+z) \int_0^z \frac{c\,dz}{H(z)}$$

and the Hubble function is found in terms of the parameters of the cosmological fluid (densities of the various sources) through Friedmann's equation:

$$H^2(a) = H_0^2 \left(\Omega_r (1+z)^4 + \Omega_{\rm m} (1+z)^3 + \Omega_{\rm K} (1+z)^2 + \Omega_\Lambda \right)$$ (here including a cosmological constant)

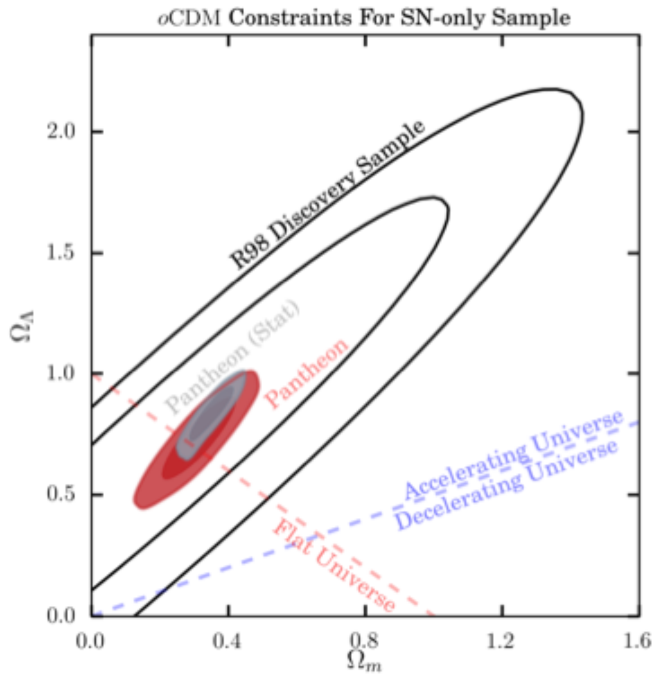
From this, the luminosity distance can be computed for any values of the vector of cosmological parameter, and for the redshifts of the various SN. Then all terms of the distance-modulus estimator are added (introducing a large number of nuisance parameters) and finally the theoretical distance-modulus μ(z) is found. Its likelihood can then be computed by comparing with the distance-modulus data. In a sampling method, the procedure is then repeated for millions of points in the parameter space. The cosmological parameters estimates are finally found by marginalizing over the nuisance parameters.

It is important to realize that the results depend on the cosmology assumed (it is a working hypothesis).

Let us consider the ΛCDM scenario.

flat ΛCDM → 2 independent background cosmological parameters: H_0 , $\Omega_{m,}$ since Ω_r and Ω_K are fixed and $\Omega_\Lambda = 1 - \Omega_{m.}$ If curvature is not fixed *a priori* then there are 3 free parameters: H_0 , Ω_m, $\Omega_{\Lambda,}$ and this is historically called **oCDM** (open CDM, even though the fit is free to have any curvature - flat, open or closed)

Constraints in the (Ω_m, Ω_Λ) plane: after marginalizing over the nuisance parameters and H_0, the constraints on the 2 density parameters are given as confidence contours in the 2D parameter space.

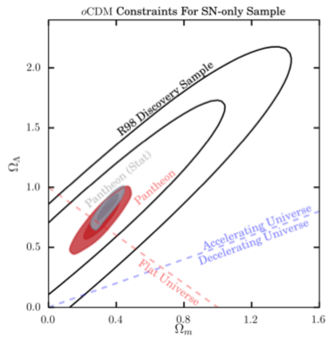


Some notes:

- The large contours are from the first SN results of 1998. They show quite large 1σ and 2σ probability contours, since the data has large error bars.

- The smaller contours (also showing 1σ and 2σ contours) are for the recent Pantheon results.

- Notice also the impact of considering or not the contribution of the systematic effects for the data error bars:



Grey contours - analysis done using Pantheon data with error bars including only the errors (statistical uncertainties)

Red contours - analysis done using Pantheon data with error bars including statistical + systematic uncertainties)

So, with larger error bars, a larger region of the parameter space has a "good likelihood" and is included inside the confidence contours (red larger than grey).

- If H_0 was known (fixed in the analysis instead of marginalized), the (Ω_m , Ω_Λ) contours would be smaller (tighter constraints)

- Models such that $\Omega_\Lambda = 1 - \Omega_m$ (i.e., $\Omega_K = 0$), lie on the straight line marked "flat"

- There is also a line dividing accelerating and decelerating models.

The contours are all aligned on a preferred direction. Why is this?

To answer this question, let us remember that to first approximation, we are measuring the acceleration of the Universe, i.e., as we saw, D_L depends directly on q_0

$$q_0 = - \frac{\ddot{a}}{a}\Big|_{t_0} \frac{1}{H_0^2}$$

From Raychadhuri equation, we can write the acceleration parameter in terms of the source parameters:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$= -\frac{8\pi G}{3H_0^2}\frac{H_0^2}{2}\rho(1+3W(a)) \Leftrightarrow \frac{\ddot{a}}{a} = -\frac{H_0^2}{2}\left[\Omega_m a^{-3} + \Omega_{DE}(1+3w(a))\right]$$

(for a general dark energy fluid)

For the case of a cosmological constant:

$$\Rightarrow \frac{\ddot{a}}{a}\Big|_{t_0} = -\frac{H_0^2}{2}\left[\Omega_m + \Omega_{DE}(1+3w)\right]$$

or, for $\Omega_{DE} = \Omega_\Lambda \Rightarrow \frac{\ddot{a}}{a}\Big|_{t_0} = -\frac{H_0^2}{2}(\Omega_m - 2\Omega_\Lambda)$

and so $$q_0 = \frac{1}{2}\Omega_m - \Omega_\Lambda$$

(note that q_0 is independent of H_0, which is consistent, since they are directly different orders of the Taylor expansion → acceleration is independent of velocity)

So, models with the same acceleration (same q_0) all lie in a line

$y = ax + b$

where $y = \Omega_\Lambda$, $x = \Omega_m$, $a = 1/2$, $b = -q_0$ (which is >0 for an accelerated model)

This line defines the direction of the contour (with some width due to the uncertainty on the measured acceleration)

This shows that Ω_m and Ω_Λ are correlated in the acceleration they produce.
The two parameters define a straight line along which all models have exactly the same acceleration and will have exactly the same likelihood values → a degeneracy direction

Consider a model 1. Now, if a model 2 has a higher Ω_Λ with respect to model 1, then by also increasing its Ω_m value the acceleration produced by model 2 will be the same as for model 1 → they are **correlated (positively correlated)** → contours from bottom-left to top-right.

If to keep the acceleration constant when one of the parameters increase, the other would need to decrease, then they would be → **anti-correlated (negatively correlated)** → contours from top-left to bottom-right

Two different ways of increasing luminosity distance:

1) Increase Ω_Λ
2) Decrease Ω_m

This causes the degeneracy between Ω_Λ and Ω_m

It is then impossible to distinguish those 2 models (or any model along the degeneracy direction) with SN measurements (or any other DL based method).

In general, **cosmological probes are very good in constraining degeneracy directions** (i.e. combinations of cosmological parameters) but not so good in constraining individual parameters.

In our case, SN measurements are good in constraining the orthogonal direction to the degeneracy direction i.e., the deviation from the acceleration line (or the width of the contours).

Note that a parameter defined along the width of the contours would be highly constrained - this parameter corresponds to the last principal components in a PCA analysis of the parameter space covariance matrix.

Notice that ($0.5\,\Omega_m - \Omega_\Lambda$ = constant) is a perfect degeneracy. Why then do the contours close and do not extend infinitely along the degeneracy direction?

This is because the linear dependence of D_L on q_0 is only a good approximation at second-order of the a (t) expansion. **In reality, there are other terms and degeneracy is not perfect → the contours close and show a preference for Ω_m < 1 (and Ω_Λ > 0)**

Could we get better estimates for individual parameters?

The Pantheon results are:

Analysis	Model	Ω_m	Ω_Λ
SN-stat	ΛCDM	0.284 ± 0.012	0.716 ± 0.01
SN-stat	oCDM	0.348 ± 0.040	0.827 ± 0.06
SN	ΛCDM	0.298 ± 0.022	0.702 ± 0.02
SN	oCDM	0.319 ± 0.070	0.733 ± 0.11

Notice that in the (flat) ΛCDM case, the result for Ω_Λ is just $1-\Omega_m$

Notice the constraints are looser (worse) if:

- systematics are included in the data error budget

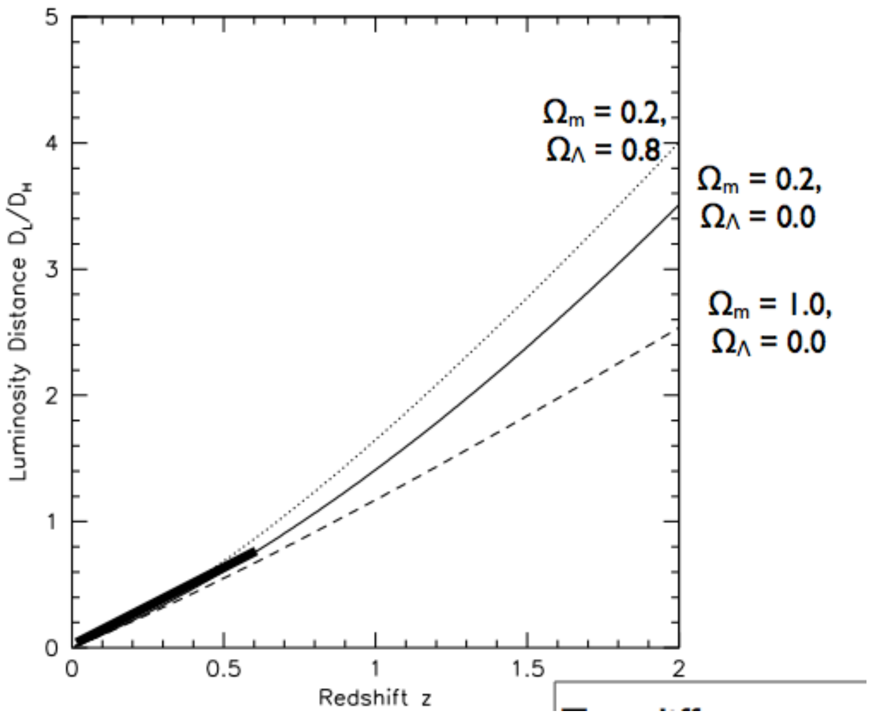
- curvature is left free (one more free parameter to add to the general degeneracies)

We can improve the constraints by combining various cosmological probes such as to break the degeneracy.

For example consider an observable that would depend directly on the curvature of the Universe. In the (Ω_m, Ω_Λ) plane we see that lines of constant curvature are more or less orthogonal (i.e. complementary) to lines of constant acceleration.

The joint likelihood analysis of those two datasets would produce contours in the intersection of the two directions → i.e. potentially small round contours → constraining simultaneously the two parameters Ω_m and Ω_Λ.

Do the cosmological observations prove the existence of dark energy?



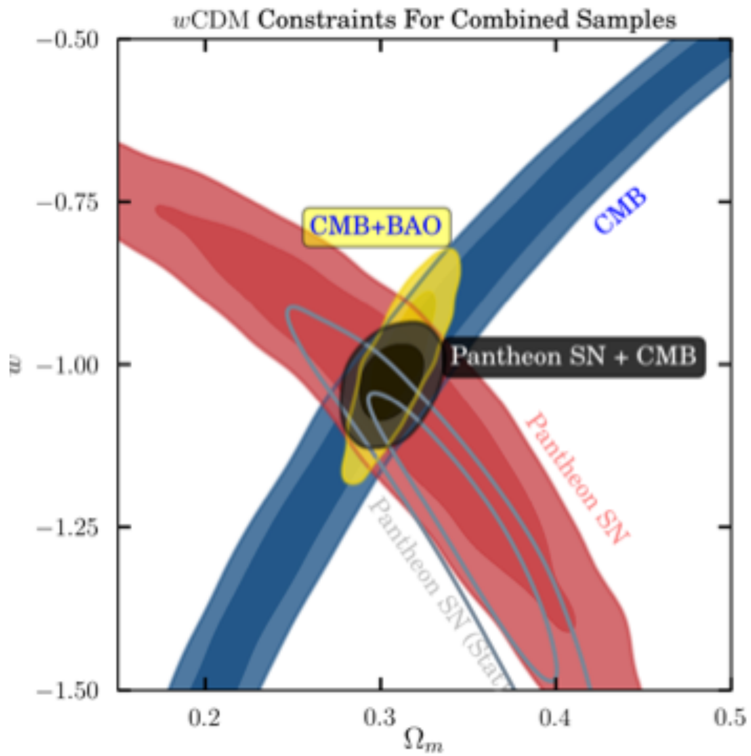
The evidence from the data is for acceleration (based on the shape of the D_L (z) function).

The "evidence for dark energy" is a model-dependent conclusion (i.e. based on the assumption of an underlying cosmology) and therefore less robust than the evidence for acceleration.

Let us consider now the wCDM scenario, where dark energy has a constant equation of state, but not necessarily equal to -1 (which would be ΛCDM).

wCDM → there are 4 independent background cosmological parameters: H_0 , Ω_m, $\Omega_{DE,}$ w (or alternatively H_0 , Ω_m, $\Omega_{K,}$ w), or only 3: H_0 , Ω_m, w, if flatness is also assumed ($\Omega_K = 0$ and $\Omega_{DE} = 1 - \Omega_m$)

Constraints on the (Ω_m, w) plane
(after marginalizing over the other parameters)



The SN-Pantheon contours (red) are in a very **different direction** than the contours in the $(\Omega_m, \Omega_\Lambda)$ plane that we saw previously.

This is because that (as before) they are determined by the acceleration parameter q_0 , which now (from Raychadhuri's eq.) is,

$$q_0 = \frac{1}{2}\Omega_m + \Omega_{DE}\frac{(1+3w)}{2}$$

i.e., Ω_m and w add, instead of subtracting (contrary to the relation between Ω_m and Ω_Λ), and so they are anti-correlated. (Note that this is just an effect of w being negative)

Moreover, **the contour is no longer an ellipse** (it is curved). This is because the line of constant luminosity distance (which in our $O(z^2)$ approximation is the line of constant acceleration) is no longer a straight line in the (Ω_m, w) plane.

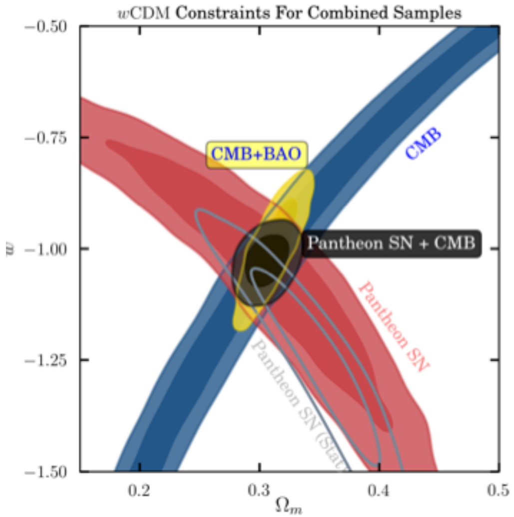
When we move along a straight line in this plane, a change on Ω_m induces a change on Ω_{DE} → the dependence of q_0 on the parameters is no longer linear.
Indeed, if we replace $\Omega_\Lambda = 1 - \Omega_m$ in the expression for q_0, we get

$$q_0 = \frac{1}{2} + \frac{3}{2}w\left(1 + \Omega_\mathrm{m}\right)$$

i.e., $y = a\,(1-x)^{-1}$

where $y = w$, $x = \Omega_m$, $a = 2/3\,(q_0 - 1/2)$

corresponding to the red curved contour.



We recover the result that only in the case the cosmological function (in this case, distance modulus, luminosity distance, acceleration) **depends linearly on the parameters, is the posterior distribution in the parameters space a Gaussian (leading to elliptical contours).**

The figure also shows:

- Contours from CMB-Planck measurements

orthogonal to the SN ones (they do not measure the luminosity distance or acceleration but different observables, like the angular size of the sound horizon at recombination) → they are complementary probes, and the joint contours are much reduced.

- Baryonic Acoustic Oscillations (BAO) measurements

similar to the SN ones (they measure the angular diameter distance) and also complementary to CMB.

Sample	w
CMB+BAO	-0.991 ± 0.074
CMB+H0	-1.188 ± 0.062
CMB+BAO+H0	-1.119 ± 0.068
SN+CMB	-1.026 ± 0.041
SN+CMB+BAO	-1.014 ± 0.040
SN+CMB+H0	-1.056 ± 0.038
SN+CMB+BAO+H0	-1.047 ± 0.038

SNe Ia distances combined with CMB and/or BAO remain the best probe to constraint the DE equation of state :

- a 5% measure of a constant DE EoS, w, is achievable
- currently little sensitivity to w(z)

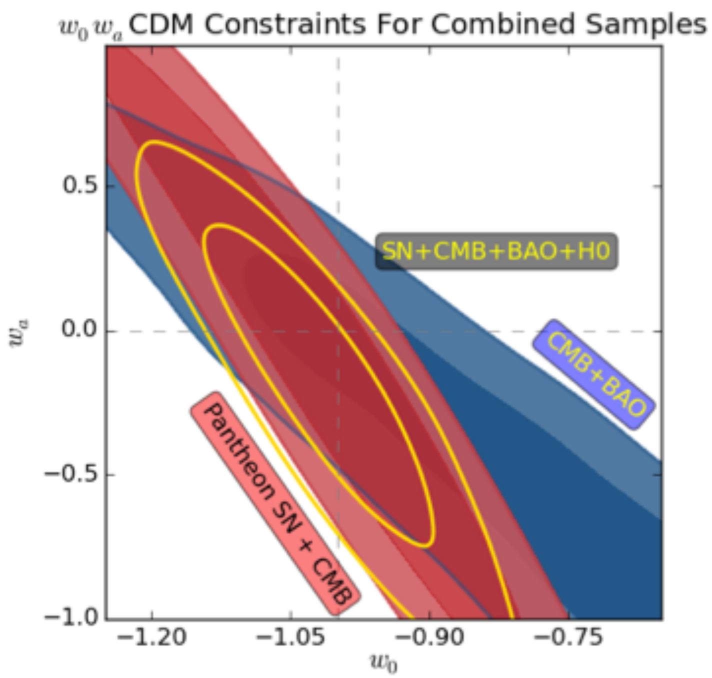
Including systematics and combined with BAO and CMB : w (cte) = -1.018 ± 0.057 (~6%) compatible with a cosmological constant

Let us consider now the w(z)CDM scenario, where dark energy has an evolving equation of state

w(z)CDM → there are now 5 independent cosmological parameters: H_0 , Ω_m, Ω_{DE}, w_0, w_a (or only 4 if flatness is assumed)

The evolution of the dark energy equation of state is parameterized as $w(z) = w_a$ which is a first-order Taylor expansion in the scale factor: $w_0 + w_a (1 - a)$

Constraints in the (w_0, w_a) plane
(after marginalizaing over the other parameters)



Some notes:

- The effect of w_a on the geometric observables is very weak → probes of structure are more useful, since the evolution of dark energy affects structure formation

- Due to the weak constraints the figure only shows combined contours: SN+CMB, BAO +CMB, SN+CMB+BAO+H0_prior

- ΛCDM is a point in this plane (w_0 = -1, w_a =0). and is inside all contours

Sample	w_0	w_a	Ω_m	H_0	FoM
CMB+BAO	-0.616 ± 0.262	-1.108 ± 0.771	0.343 ± 0.025	64.614 ± 2.447	14.5
CMB+H0	-1.024 ± 0.347	-0.789 ± 1.338	0.265 ± 0.015	73.397 ± 1.961	9.1
CMB+BAO+H0	-0.619 ± 0.270	-1.098 ± 0.781	0.343 ± 0.026	64.666 ± 2.526	14.5
SN+CMB	-1.009 ± 0.159	-0.129 ± 0.755	0.308 ± 0.018	68.188 ± 1.768	31.4
SN+CMB+BAO	-0.993 ± 0.087	-0.126 ± 0.384	0.308 ± 0.008	68.076 ± 0.858	65.0
SN+CMB+H0	-0.905 ± 0.101	-0.742 ± 0.465	0.287 ± 0.011	70.393 ± 1.079	54.2
SN+CMB+BAO+H0	-1.007 ± 0.089	-0.222 ± 0.407	0.300 ± 0.008	69.057 ± 0.796	63.2

The **dark energy figure-of-merit** (FOM) is defined as the inverse of the area of the 1-sigma contour - or more precisely, it is the area of an ellipse that fits the countour (since it is defined from the Fisher matrix approach the contour is necessarily an ellipse).

The larger the FoM → the smaller the contour → the stronger the constraint.

The most powerful combination in the table is SN+CMB+BAO.

step 7 - **Model selection** (**Goodness-of-fit**)

The estimation of the cosmological parameter credible intervals (mean values and uncertainties) is not the last step of the cosmological data analysis process.

Using the SN - Pantheon example, let us look at its results:

Nuisance parameters

(notice the uncertainties are much larger if only low-z SN are used)

Survey	α	β	γ
Pantheon	0.154 ± 0.006	3.02 ± 0.06	0.053 ± 0.009
Low-z	0.154 ± 0.011	2.99 ± 0.15	0.076 ± 0.030

Cosmological parameters

- constraints are worse if the full (stat+sys) errors are used (more realistic)

Analysis	Model	w	Ω_m	Ω_Λ
SN-stat	ΛCDM		0.284 ± 0.012	0.716 ± 0.01
SN-stat	oCDM		0.348 ± 0.040	0.827 ± 0.06
SN-stat	wCDM	-1.251 ± 0.144	0.350 ± 0.035	
SN	ΛCDM		0.298 ± 0.022	0.702 ± 0.02
SN	oCDM		0.319 ± 0.070	0.733 ± 0.11
SN	wCDM	-1.090 ± 0.220	0.316 ± 0.072	

From the table, it is clear that the results depend on the scenario assumed:

- ΛCDM (Ω_m) - with few free parameters, the constraints are tighter
- oCDM (Ω_m Ω_Λ) - not only parameter uncertainties are larger but the central values can change a lot (central values for ΛCDM are not even contained in the oCDM 1σ confidence intervals)
- wCDM (Ω_m w Ω_K) - constraints closer to the oCDM ones

So, what is the final result? What is our finding, is it Ω_m 0.30 or 0.32?

This is a question of **goodness-of-fit**. Among the various best-fits which one is the best?

We turn again to Bayesian inference to answer this question by performing **model comparison** tests.

There are different ways to evaluate the goodness-of-fit. The classic way is to look at the **chi-square**, while the most rigorous way is to use the **evidence**.

Chi-square

Criteria based on the chi-square values are standard in determining the best model in all branches of physics.

The most usual quantity is the reduced chi-square of the best-fit, i.e., the chi-square normalised by the number of degrees-of-freedom,

$N_{dof} = N_d - N_p$ (where N_d is the number of datapoints - for example the number of redshift bins in the SN data - and N_p is the number of parameters in the model)

In this criterium, **the best model** (i.e., the favoured one) **is the one where the best-fit has the lowest reduced chi-square,**

$$\chi^2_{red} = \chi^2 / N_{dof}$$

Evidence

It is the integral of the likelihood on the parameters space of a given cosmologcial model → it indicates the ‘average likelihood of a model“.

It may happen that a certain set of parameter values are a very good fit to the data (high likelihood values in that region of the parameter space), but overall this model can have a worse evidence than another one (for example because of having a larger number of parameters, or a large region of small likelihood values).

The evidence is thus a global way to characterize the goodness-of-fit of a model, beyond the simple assessment of finding which model has the "best best-fit".

The evidence is a good number to show the balance between **best-fit vs. model complexity.**

In this approach, the best model is the one with the highest Bayes factor, computed from the evidences of the 2 models under comparison:

B = (Evidence_1 * Prior_1) / (Evidence_2 * Prior_2)

The **Jeffrey's scale** classifies the degree of preference for a model over another, based on the values of lnB:

<1 → unconclusive
1 - 2.5 → substantial evidence for one of the models
2.5 - 5 → strong evidence
> 5 → decisive evidence

The evidence is difficult to compute in practice with high precision, since it is **a multi-dimensional integral** of a possibly complex posterior distribution function.

Moreover, by sampling the posterior with a grid or an MCMC method, we only know a rough sample of it, which may be good enough to find the parameter constraints, but not precise enough to compute the total integral.

By design, MCMC algorithms only sample with high resolution the region near the maximum of likelihood. The tails of the distribution are usually badly sampled because they are not needed for parameter inference.

So the sample obtained with MCMC is not complete enough to compute the evidence. We need other Monte Carlo sampling methods to solve the multi-dimensional integral.

A popular algorithm for this is the **Nested Sampling:**

1. Sample N points randomly from within the prior, and evaluate their likelihoods. Initially we will have the full prior range available, i.e., $(0, X_0 = 1)$.

2. Select the point with the lowest likelihood (L_j). The prior volume corresponding to this point, X_j, can be estimated probabilistically. The average volume decrease is given as $X_j/X_{j-1} = t$, where t is the expectation value of the largest of N random numbers from uniform (0, 1), which is $N/(N + 1)$.

3. Increment the evidence by $E_j = L_j(X_{j-1} - X_{j+1})/2$.

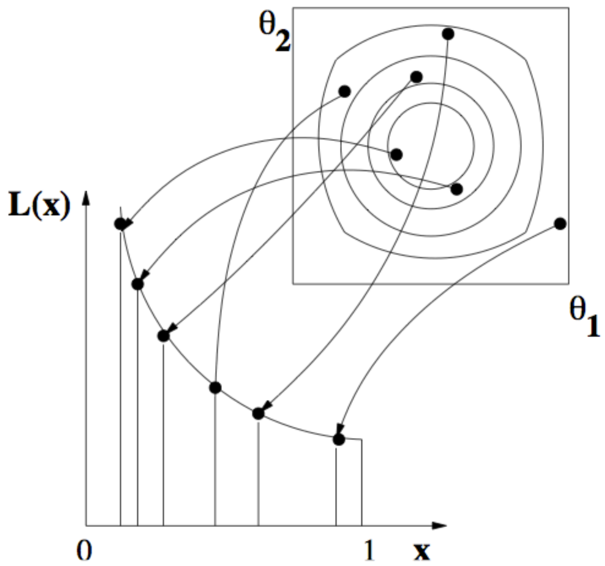
4. Discard the lowest likelihood point and replace it with a new point, which is uniformly distributed within the remaining prior volume $(0, X_j)$. The new point must satisfy the hard constraint on likelihood of $L > L_j$.

5. Repeat steps 2–4, until the evidence has been estimated to some desired accuracy.

This means: find iso-regions of likelihood. If they are 'nested' the integrand is monotonic → the integral reduces to 1-dimension.

For each layer → $$E_j = \frac{L_j}{2}(X_{j-1} - X_{j+1}).$$

The total evidence is → $$E = \sum_{j=1}^{m} E_j.$$



Information criteria

Besides the evidence, there are alternative approximate methods, much simpler to compute, that can also be used for model selection and quantify the balance of best-fit vs. model complexity. Some popular of these **information criteria** are:

Akaike information criterion: AIC = -2 ln L_bestfit + 2 n_p = χ^2_bestfit + 2n_p
(this formula is the result of a minimisation of entropy criterium)

Bayesian information criterion: BIC = -2 ln L_bestfit + n_p ln(n_d)
(based on an approximation of the evidence)
BIC penalizes more the complexity than AIC does.

Deviance information criterion: DIC = 2 χ^2_mean - χ^2_bestfit
(it is like an effective χ^2 , sensitive to the difference between the best-fit and the full distribution).

For all information criteria, **the best model is the one with the lowest value**.

Results of model selection

In this example, SN data was used to test two very different scenarios: ΛCDM and UDM (model where DM and DE are one single fluid.

This model has one density parameter less, but 2 new additional parameters - so one parameter more than ΛCDM in total).

Two different UDM models were tested and (like ΛCDM) both are able to produce D_L (z) functions that allow for good fits to the SN data for certain values of their parameters.

Various model selection criteria were computed:

The question is, is there enough evidence to select UDM over ΛCDM?

	UDM	ΛCDM	UDM_{ph}
χ^2_{min}	552.59	552.77	552.75
χ^2_{red}	0.9478	0.9449	0.9481
$\ln \text{B}_{\text{U}\Lambda}$	−0.0196	0	0.6850
BIC	584.485	571.902	584.644
DIC	553.250	552.770	552.814

- The first UDM model is the one with the smallest best-fit χ^2 , i.e., it contains a vector of parameter values that produced the closest fit to the data.

However, since this model has more cosmological parameters than ΛCDM it is penalized and the lowest reduced chi-square turns out to be the one of ΛCDM. The complexity of the model (having more free parameters) is always penalized in these criteria. This is because increasing the number of parameters naturally helps in finding a closer fit (in a potentially artificial way).

- UDM_ph is the model with largest evidence. Indeed, the Bayes factor of the second UDM model with respect to ΛCDM is positive, although smaller than one → the analysis shows a very slight unconclusive preference for this model UDM_ph

- BIC shows a reasonable preference for ΛCDM.

- DIC shows a slight preference for ΛCDM.

The analysis does not show a conclusive preference for any of the models

(but given the close results, it shows that it is useful to compute all the criteria).

The status of the standard cosmological model in 2023

Cosmological tensions: joint constraints from different probes

We saw that by combining cosmological probes, it is possible to break degeneracies and find tight constraints for individual parameters, as it is seen in the table below.

Sample	Ω_m	Ω_Λ	Ω_K	H_0
CMB+BAO	0.310 ± 0.008	0.689 ± 0.008	0.001 ± 0.003	67.900 ± 0.747
CMB+H0	0.266 ± 0.014	0.723 ± 0.012	0.010 ± 0.003	73.205 ± 1.788
CMB+BAO+H0	0.303 ± 0.007	0.694 ± 0.007	0.003 ± 0.002	68.723 ± 0.675
SN+CMB	0.299 ± 0.024	0.698 ± 0.019	0.003 ± 0.006	69.192 ± 2.815
SN+CMB+BAO	0.309 ± 0.007	0.690 ± 0.007	0.001 ± 0.002	67.985 ± 0.699
SN+CMB+H0	0.274 ± 0.012	0.717 ± 0.011	0.009 ± 0.003	72.236 ± 1.572
SN+CMB+BAO+H0	0.303 ± 0.007	0.695 ± 0.007	0.003 ± 0.002	68.745 ± 0.684

analysis done assuming oCDM

Note that when combining the SN data with CMB and BAO, **the constraints improve by an order of magnitude**. For example the precision on Ω_m improves from 0.07 (~ 23% of the mean value 0.3) to 0.007 (~2%).

Note also that the combinations SN+CMB, CMB+BAO, SN+CMB+BAO, all point to a flat ΛCDM Universe: Ω_m ~0.31, Ω_Λ ~0.69, Ω_K ~0 → this became known as the concordance model.

However, not all datasets are consistent.
When these are combined with direct measurements of H_0, the joint constraints can be quite different. For example, the case SN+CMB+H_0 finds a lower $\Omega_m \sim 0.27$. The discrepancy comes from the fact that while combing the usual cosmological probes finds a lower value of $H_0 \sim 69 \pm 0.6$ Km/s/Mpc, direct measurements of H_0 systematically find a higher value of $H_0 \sim 73 \pm 1.5$ Km/s/Mpc. Given the small size of the uncertainties, the two results are inconsistent. This is known as the **Hubble tension**.

The smaller value of H_0 found in CMB (and in combination with SN or BAO), pushes Ω_m to the higher values found because Ω_m and H_0 are anti-correlated in CMB data (the dominant dataset in the combination).

Note that **H_0 direct measurements** are mostly measurements at low redshift, which as we saw do not depend on the density parameters, but only on H_0. They need to be carefully calibrated through a **distance ladder** procedure. Currently there is much effort in these calibration methods, by doing precise distance measurements to nearby calibrators such as : parallax measurements of Cepheids, mega-maser systems, distances to the Large Magellanic Cloud, or the tip-of-the-red-giant-branch method.

(In addition H_0 can also be directly measured by detecting **time-delays** in multiple lensed images, like it is done in the H_0LiCoW survey).

What is the reason for the Hubble tension?

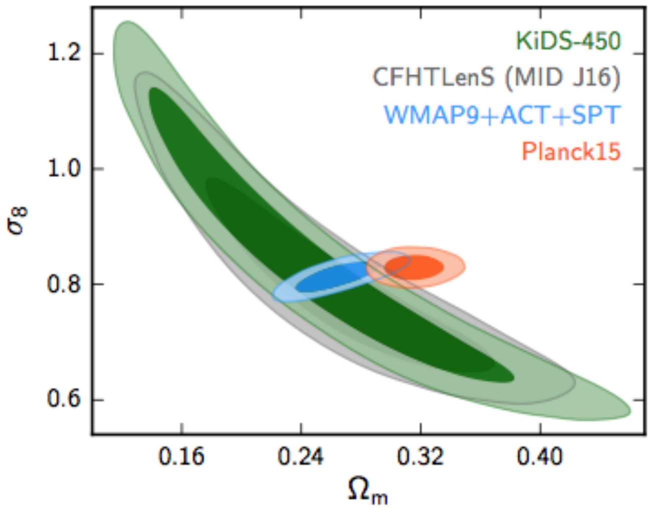
- perhaps one or various datasets have **unidentified systematic** effects → biased results

- perhaps we live in a "**local void**", with slightly lower mean density than the average over the Universe → it would be normal that local measurements give different results than high-redshift ones

- perhaps it is an indication of **new physics**, i.e., it could just be that ΛCDM is not the right solution after all. We should test different alternatives → **Beyond ΛCDM**

There is today a great debate on systematics vs. new physics, especially that in addition to the Hubble tension, there is also another tension: the **sigma-8 tension**. The σ_8 parameter parameterizes the amplitude of the matter perturbations in the inhomogeneous Universe.

Structure formation probes of the high-z Universe (CMB) find a higher value of this parameter ($\sigma_8 \sim 0.9$), while lower-z probes (weak lensing, galaxy clustering) find lower values ($\sigma_8 \sim 0.8$):



Tension between CMB (red) and weak lensing (green)

Again, is this an indication of not well controlled systematics, or new physics?

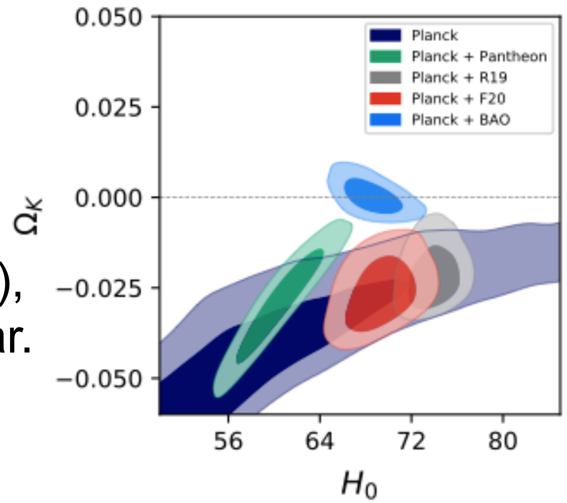
The simplest way to test new physics is to consider **extensions of ΛCDM**, i.e., to allow some fundamental properties of ΛCDM to vary:

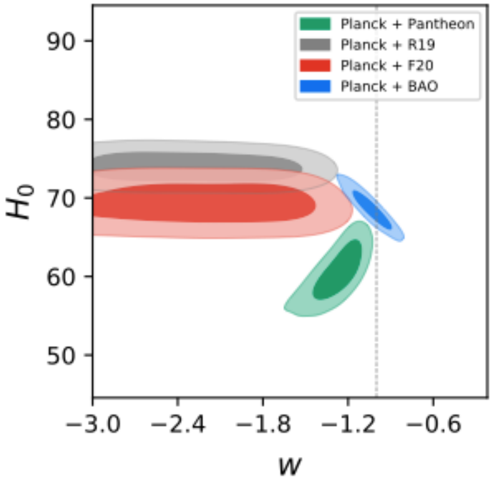
- consider more free parameters by allowing curvature, dark energy equation of state w, non-zero neutrino masses, a scale-varying index of the inflation power spectrum.

Assuming all this simultaneously, the inconsistencies between datasets become more clear:

- when combining CMB (Planck) with different datasets (SN-Pantheon, BAO or R19 and F20 H_0 direct measures), the inconsistency between various datasets become clear.

- in most cases a closed universe is preferred.





- again, there is a large variety of H values. Even though some cases have tight constraints, the contours do not overlap.

- some cases have a preference for w < -1, away from ΛCDM.

It seems that with nowadays more precise data and better calibrations, we are moving from the cosmic concordance of the last decade (2010's) to a **cosmic discordance** *(crisis in cosmology).*

This makes research in **"beyond ΛCDM"** models, which has been the main activity of theoretical cosmology since the beginning of the century, even more relevant today.

There is a large number of models proposed. Some popular ones are: unified dark_matter/dark_energy (where there is only one "dark component"), fuzzy DM, warm DM, quintessence, k-essence, coupled dark energy models, modifications of General Relativity such as massive gravity or f(R) gravity where the Einstein tensor is not constructed directly from the Ricci scalar but from a function of it f(R).

See the review papers:

"In the realm of the Hubble tension - a review of solutions", Di Valentino et al 2021, https://arxiv.org/abs/2103.01183).

"Beyond ΛCDM: Problems, solutions, and the road ahead", Bull et al 2016, https://arxiv.org/abs/1512.05356 - ***including a curious opinion poll at the end***

1. What are the biggest deficits and challenges of the ΛCDM paradigm?	**No.**	**%**
The cosmological constant problem	40	66
ΛCDM will remain the best-fit model to the data while not being understood theoretically	29	48
There are no compelling alternatives	19	31
It can't explain small-scale structure (e.g. dwarf galaxies)	14	23
Cold dark matter	13	21
The model is fine-tuned	13	21
Inflation is a general idea with no clear implementation in particle physics	13	21
Baryonic effects are too difficult to model	11	18
Confirmation bias	9	15
The coincidence problem	8	13
Effects of inhomogeneities and anisotropies	8	13
Inflation isn't predictive enough	7	12
The Big Bang singularity and high energy description of gravity	6	10
GR cannot be relied upon on large length scales	4	7
(Nothing is wrong with ΛCDM)	3	5
There is no dark matter	2	3
Cosmic variance on ultra-large scales	1	2
Other	1	2