Probes of Structure Formation

Cosmic Microwave Background

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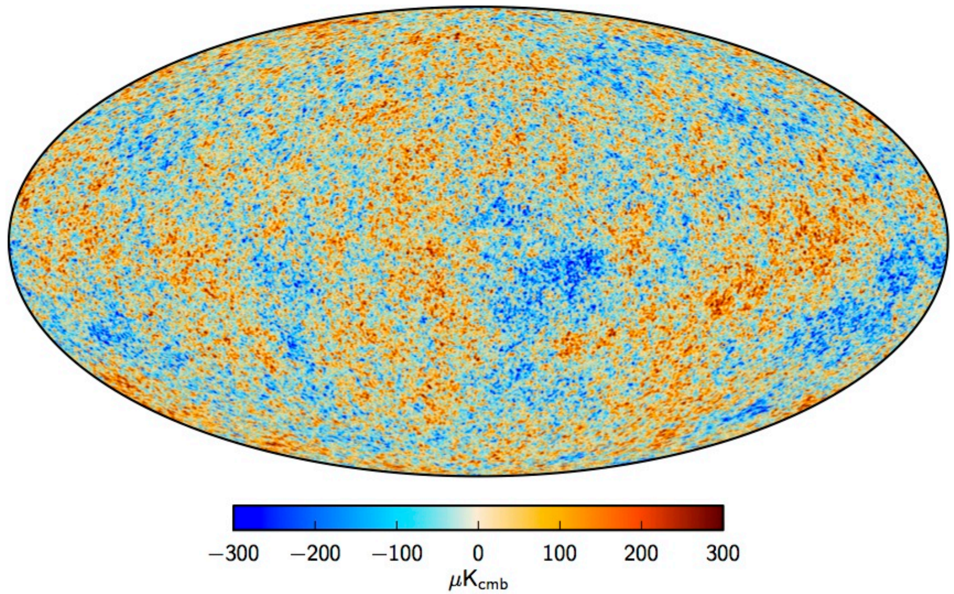
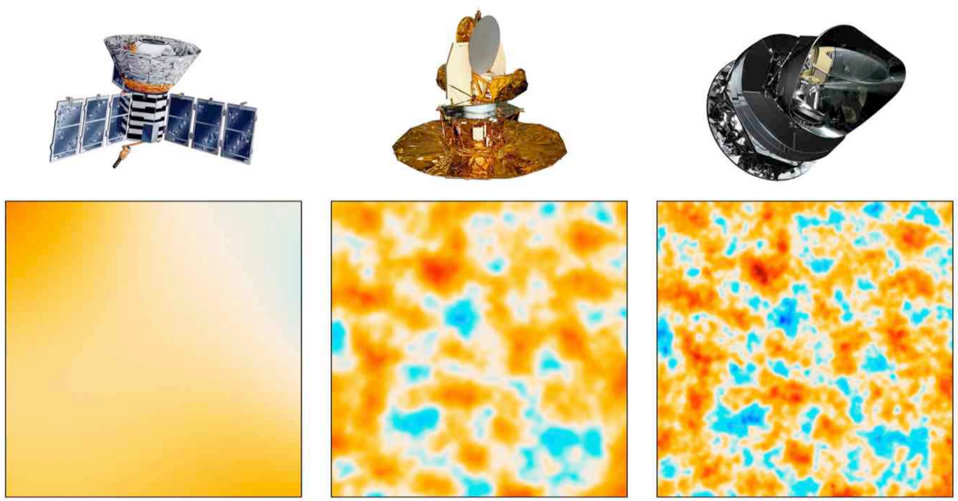
Observationally → the **CMB power spectrum** (cosmological function) is estimated from the observed correlation function of the temperature anisotropies δ_T (observable)

Theoretically → need to compute the temperature power spectrum from the plama perturbations





COBE WMAP Planck

δ_b/r → δ_T

The observed δ_T is an unbiased tracer of δ_b/r

CMB anisotropies: estimator

We want to estimate the CMB two-point angular function C_l from the observed map of CMB temperature contrast

Note that direct measurements of a power spectrum from a map, $<\delta_k \delta_k>$, cannot be directly compared to the theoretical predicted power spectrum.

This is because "**experimental complications**" such as finite resolution of the instruments (**PSF**, **beam**), or incomplete coverage of the sky (**masks**), or observing window functions, **bias the measured power spectrum**.

The way to proceed is to derive an expression (function of the measured power spectrum) that gives a power spectrum as close as possible to the true one → i.e., **unbiased** and with **high S/N**.

This expression is called an estimator of the true (the theoretical) power spectrum from the measured one.

We already saw some examples of estimators:

- The **distance modulus** estimator

In this case there are "**experimental complications**" , such as the observing filter function (that requires the **K-correction**) , or the flux-limited observations (that introduces the **Malmquist bias**) and "**astrophysical complications**", such as the non-universality of the SNe luminosity (that requires to shift the light-curves and the introduction of a magnitude response model with **nuisance parameters**).

- The **galaxy power spectrum** estimator

In this case, there are "**experimental complications**" like the fact that the measurements are made on discrete positions (that requires to subtract the shot noise to the measurement), and "**astrophysical complications**", such as the fact that galaxies form on dark matter halos, which are a biased representation of the dark matter density field (that requires the introduction of a bias model with **nuisance parameters).**

In the case of estimating the true CMB C_l from the data, we have to deal with effects introduced by the finite resolution of the instrument (the presence of a **beam**) and of **noise** in the data.

On each direction **n** on the sky there is a value of δT - the true (**theoretical**) "temperature overdensity" (usually denoted by Θ in the literature and not δT)

The **measured** value (denoted Δ) $\Delta \doteq (T - T_0)/T_0$

of the **clean map** in a direction (pixel) **n** is given by

$$\Delta(\hat{\boldsymbol{n}}) = \int d\Omega' \Theta(\hat{\boldsymbol{n}}') B(\hat{\boldsymbol{n}}, \hat{\boldsymbol{n}}') + \eta(\hat{\boldsymbol{n}})$$

This means that due to the finite angular resolution of the detector there is a **smearing** of the signal (like a **point-spread-function** in optical telescopes), described by the function B.

In a radio telescope this is called the **beam**.

The effect of the beam is a convolution in real space.

There is also **noise** $\eta(\hat{\boldsymbol{n}})$ affecting the measurement.

The full-sky transform of the measured δT in **spherical harmonics** is:

$$a_{lm}^{\mathrm{obs}} = \int d\Omega \, Y_{lm}^{*}(\hat{\boldsymbol{n}}) \Delta(\hat{\boldsymbol{n}}).$$

and so, it involves the transforms of the beam and of the noise:

$$a_{lm}^{\mathrm{obs}} = \sum_{l'm'} B_{lm,l'm'} a_{l'm'} + \eta_{lm},$$

the beam transform is a 2-pt quantity
because it connects two positions

true a_{lm}

For a constant and isotropic beam, e.g.: $B_l = \exp(-l^2 \theta_{\mathrm{beam}}^2 / 2)$

the measured a_{lm} simplifies to:

$$a_{lm}^{\mathrm{obs}} = a_{lm} B_l + \eta_{lm}$$

and hence depends on the **power spectrum of the bean** B_l and the **power spectrum of the noise**

$$\langle \eta_{lm} \eta_{l'm'}^{*} \rangle = N(l) \delta_{ll'} \delta_{mm'}.$$

In real space, the value of δT in a location is the convolution of the true δT with the beam, plus the noise at that location.

But in harmonic space we lose the one-to-one local relation. So, the **estimator of the true CMB power spectrum $|a_{lm}^2|$** is not simply found by inverting

$$a_{lm}^{\text{obs}} = a_{lm} B_l + \eta_{lm}$$

It is found with **a likelihood procedure**:

what is the probability of getting the data a_{lm}^{obs} given the true a_{lm}

Assuming a Gaussian distribution, this probability is:

$$P(a_{lm}^{\text{obs}} | a_{lm}) = \frac{1}{\sqrt{2\pi N(l)}} \exp\left[-\frac{1}{2N(l)} |a_{lm}^{\text{obs}} - B_l a_{lm}|^2 \right]$$

(assuming the noise has zero mean, and it contributes only to the variance)

After an analytical marginalization over the true a_{lm},

$$\mathcal{L} \equiv P\left(\{a_{lm}^{\rm obs}\}|C(l)\right) = \left(2\pi\left[C(l)B_l^2 + N(l)\right]\right)^{-(2l+1)/2} \exp\left\{-\frac{1}{2}\sum_{m=-l}^{l}\frac{\left|a_{lm}^{\rm obs}\right|^2}{C(l)B_l^2 + N(l)}\right\}$$

and maximizing the likelihood $\frac{d\ln\mathcal{L}}{dC(l)} = 0$

we can derive the **C_l estimator:**

$$\hat{C}(l) = B_l^{-2}\left(\frac{1}{2l+1}\sum_{m=-l}^{l}\left|a_{lm}^{\rm obs}\right|^2 - N(l)\right)$$

So, from the map we need to:

- measure δT,
- compute its harmonic transform a_{lm},
- compute the corresponding power spectrum,
- sum over m,
- subtract the Noise power spectrum,
- divide by the Beam power spectrum squared.

This is the maximum likelihood estimator of the true CMB power spectrum.

Its **variance** is given by:

$$\mathrm{Cov}_{ll'} = \frac{2}{2l+1}\left[C(l) + N(l)B_l^{-2}\right]^2 \delta_{ll'}$$

cosmic variance   noise

It is a diagonal matrix (independent modes)

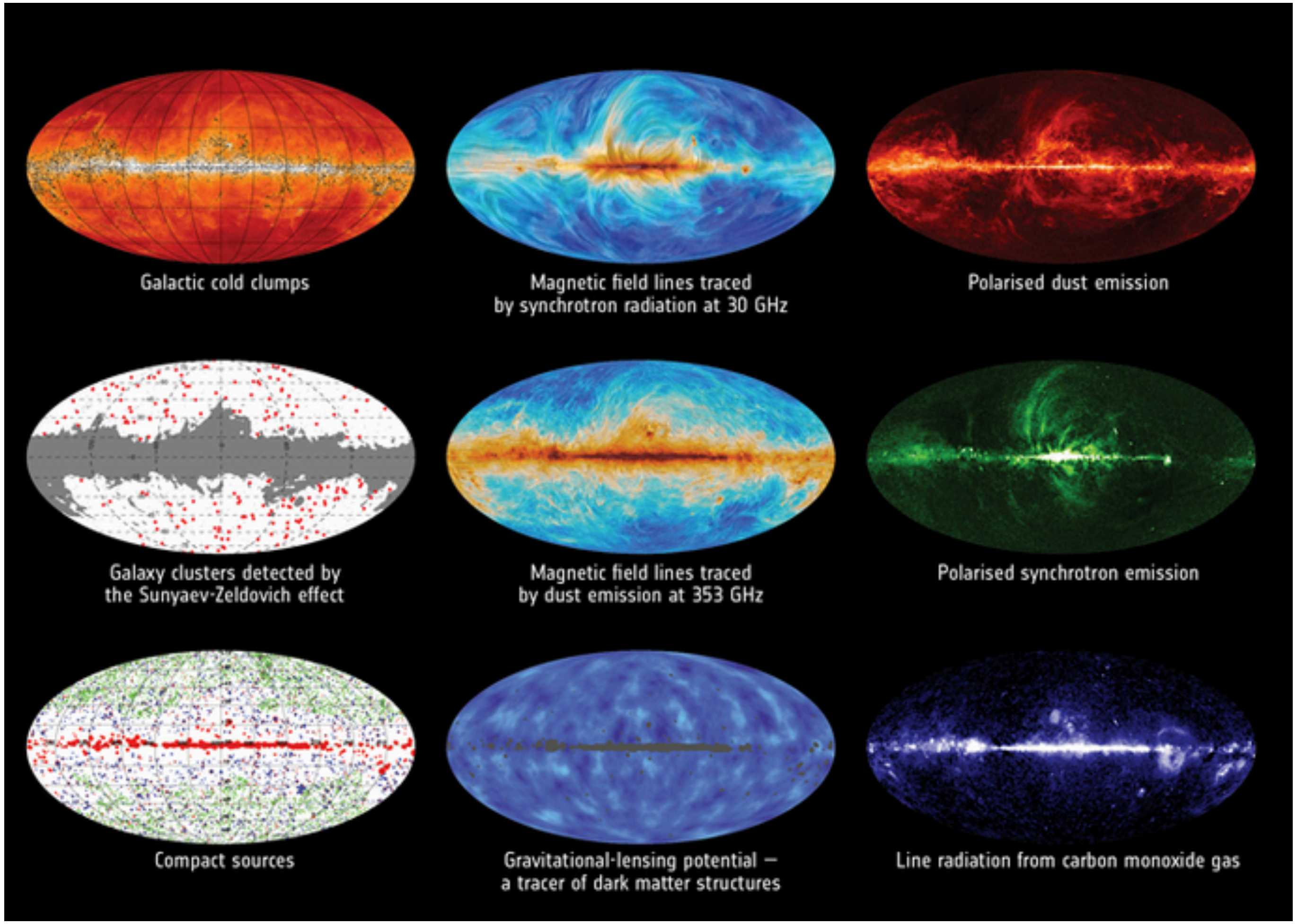
It depends on:

the C_l itself (**cosmic variance term**) → dominates on large scales

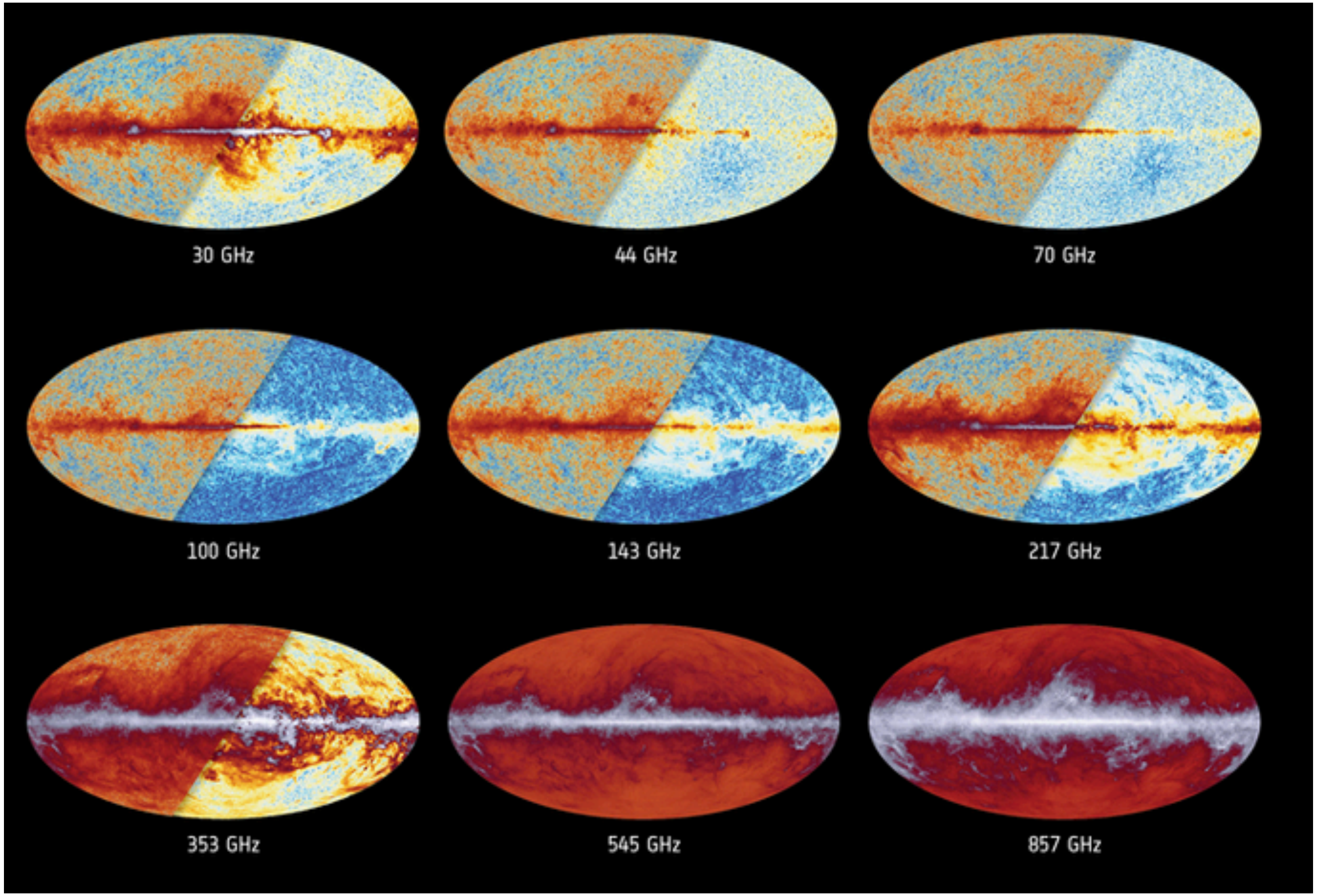
the noise amplified by the inverse beam (**noise term**) → dominates on small scales, because the transform of the beam goes to zero on small scales

All these operations are performed on **clean maps**, i.e :

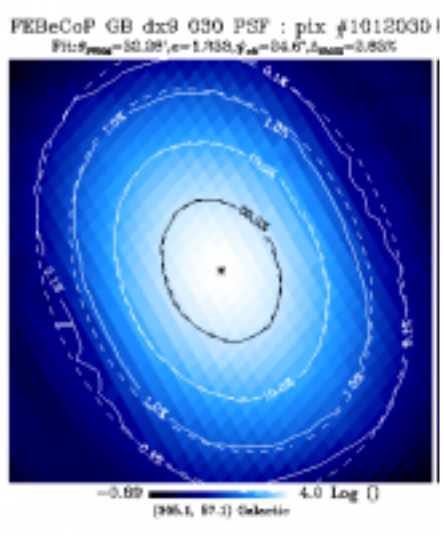
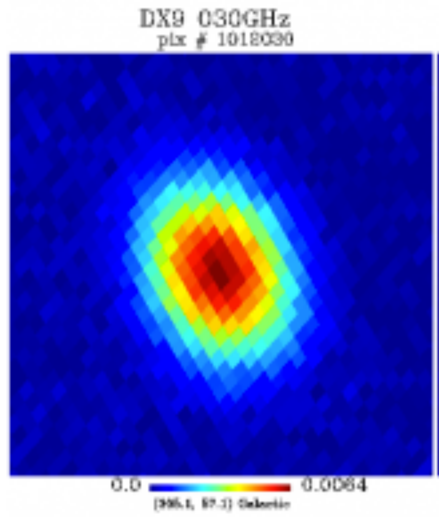
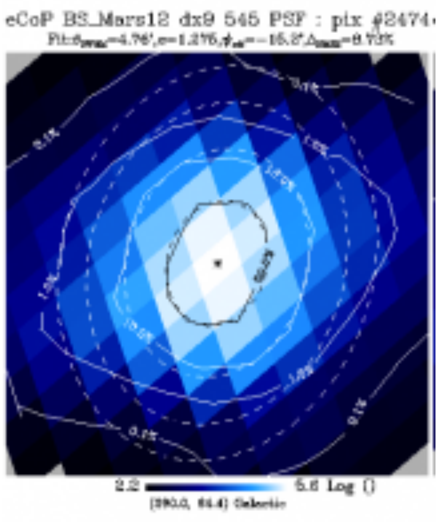
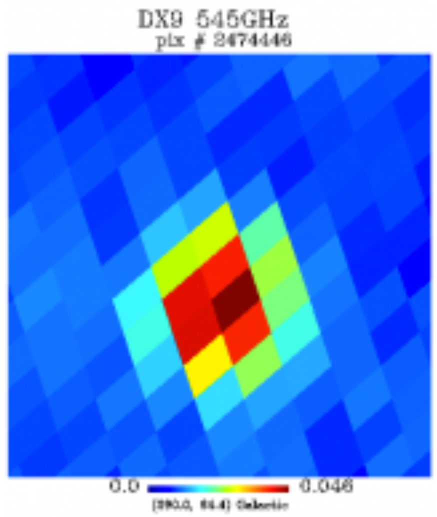
- after subtracting all **astrophysical contaminants** (biases) from each observed map



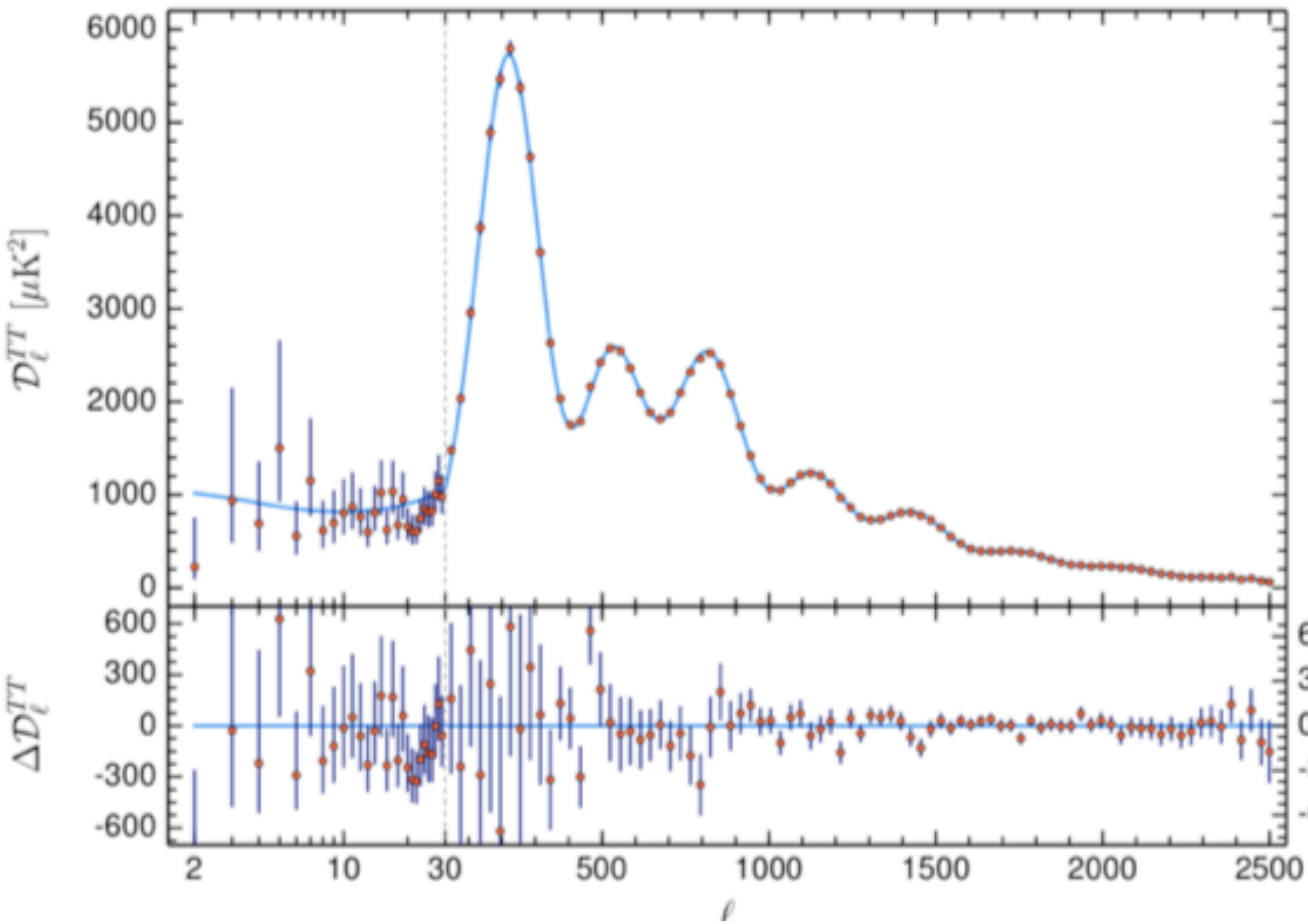
Then all the power spectrum is estimated in each of the observed maps (**observations at different frequencies**)



Beam and Noise are different for each map

	Beam profile	Compact source
30 GHz		
545 GHz		

Finally, the **Planck 2018** estimated dimensionless power spectrum of temperature anisotropies (combined from the measurements on the various maps) looks like this:



(datapoints, error bars, ΛCDM best-fit)