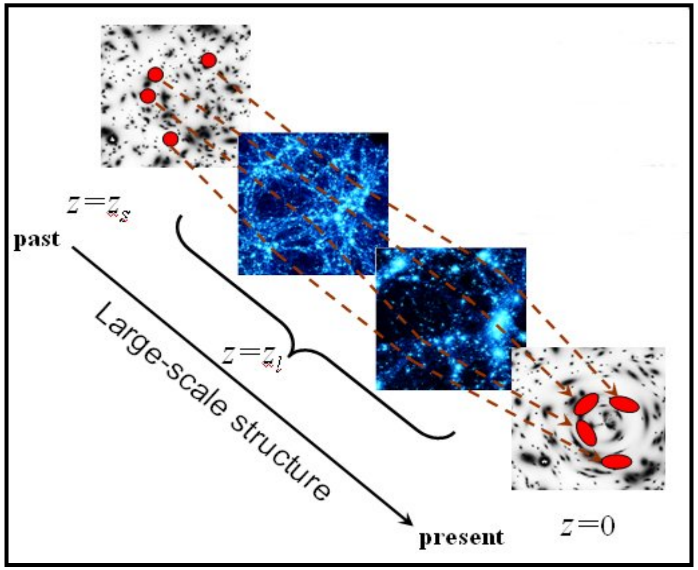
Probes of Structure Formation

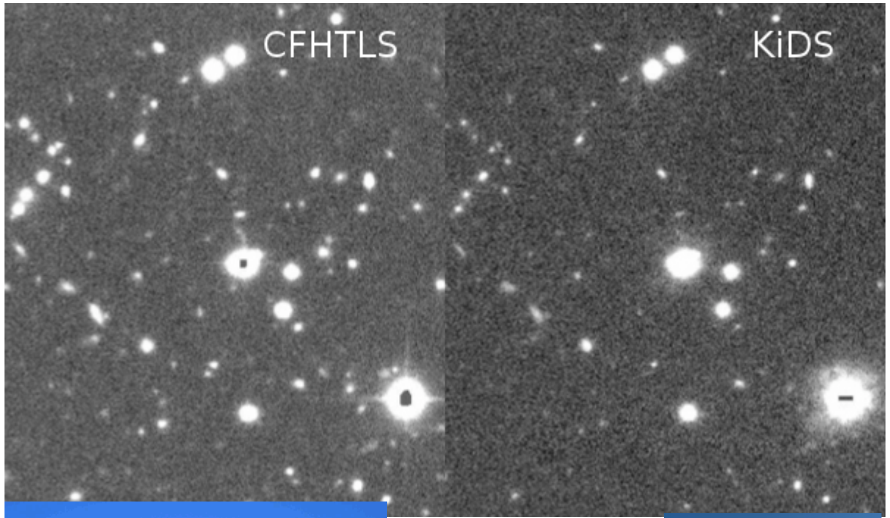
Weak Lensing

Weak Lensing

Observationally → the gravitational lensing distortions **shear power spectrum** (measured cosmological function) is estimated from the observed correlation function of galaxies ellipticities (observable)

Theoretically → need to compute the shear power spectrum from the gravitational potential power spectrum that in turn is related with the matter power spectrum





(CFHT)

(VST)

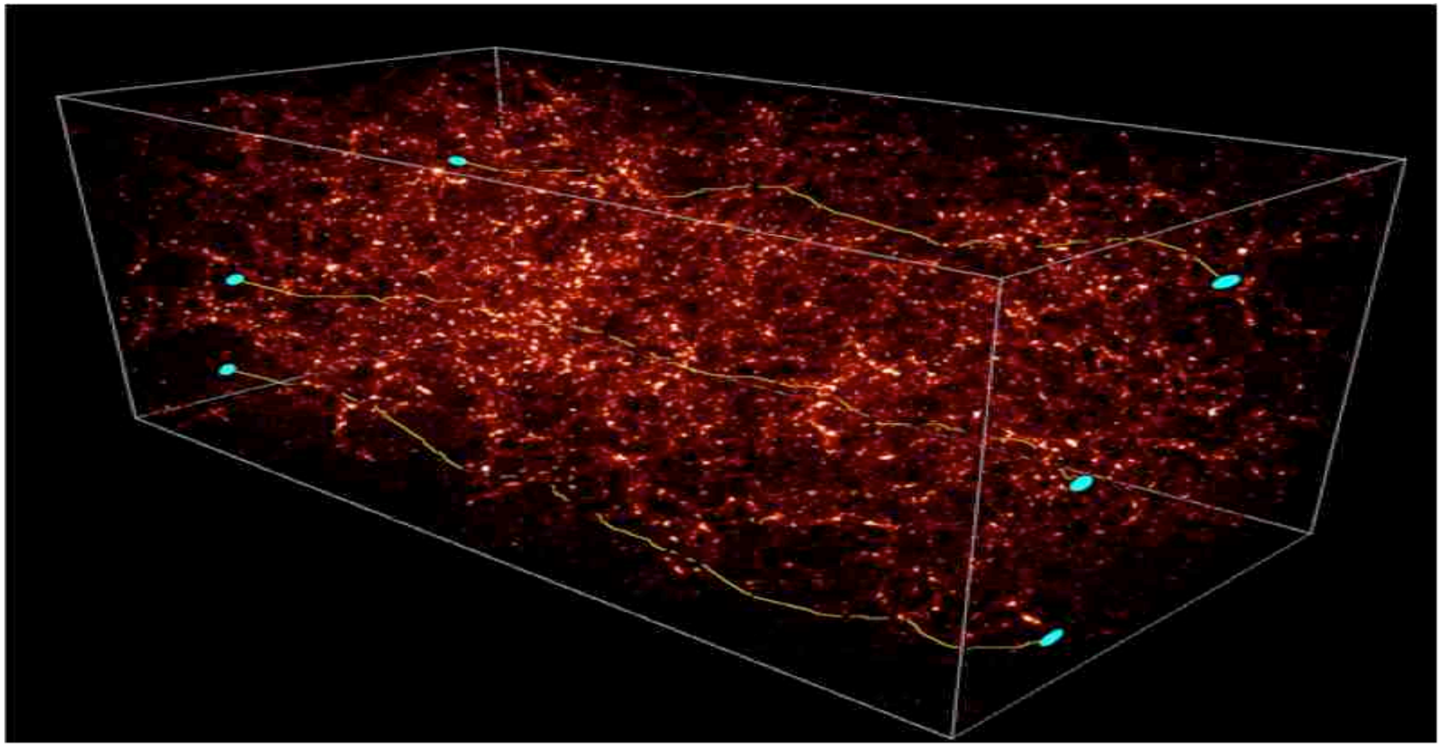
δ _dm → shear γ

It goes from 2D galaxy ellipticities → to 2D correlation function of shear → to 2D correlation function of dark matter

Weak lensing: theoretical predictions

We are interested in a specific gravitational lensing system: cosmological weak lensing, i.e.,

the weak gravitational lensing produced by the large-scale structure of dark matter (the lens) in the light emitted by distant galaxies (the sources)



It is a direct tracer of the dark matter distribution

Light propagation in the inhomogeneous universe

We want to derive the lens equation for this system.

For this we need to consider **propagation of light in the inhomogeneous universe**

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \left[dx_1^2 + dx_2^2 + dx_3^2\right]$$

We assume that a bundle of light rays emitted from a galaxy travels through the homogeneous spacetime and is deflected on a series of lens planes where the LSS are placed.

We already derived the deflection using the **principle of Fermat,** and found that **the deflection is the gradient of the lens potential in the plane orthogonal to the tangent to the path.**

$$\vec{\alpha} = -\frac{2}{c^2} \int_{\lambda_A}^{\lambda_B} \dot{\vec{u}}_x \, d\lambda = \frac{2}{c^2} \int_{\lambda_A}^{\lambda_B} \vec{\nabla}_\perp \Phi \, d\lambda,$$

On the other hand, when travelling through the homogeneous universe there is no deflection → the separation between two light rays of the bundle is just the trivial separation x between two light rays:

$$x_i = \theta_i \, f_K(w) \qquad f_K(w) \text{ is the comoving diameter angular distance,}$$

It is useful to write this simple result as the solution of a differential equation for the evolution of the comoving transverse separation:

$$\frac{d^2 x_i}{dw^2} + K x_i = 0.$$

We can now add the **local deflection solution** (caused by the gravitational potentials) to this equation, to get the complete equation for the **evolution of the comoving transverse separation**
(defined with respect to a reference light ray at x = 0):

$$\frac{d^2 \vec{x}}{dw^2} + K\vec{x} = -\frac{2}{c^2}\left[\vec{\nabla}_\perp \Phi(\vec{x}(\vec{\theta}, w), w) - \vec{\nabla}_\perp \Phi(0, w)\right]$$

note it has a homogenous and an inhomogenous term

Lens equation and the optical scalars

The **lens equation** is the solution of the **differential equation of the evolution of the comoving transverse separation.**

The general solution of an inhomogeneous differential equation is a linear combination of the homogeneous solution and the convolution of the equation Green's function with the inhomogeneous term.

So:

$$f(x) = f^{(0)}(x) + \int dx' \, g(x') \, G(x, x')$$

where the Green's function of our differential equation is $G(w, w') = f_K(w - w')$.

The solution is thus:

$$\vec{x}(\vec{\theta}, w) = f_K(w)\vec{\theta} - \frac{2}{c^2} \int_0^w dw' \, f_K(w - w') \left[\vec{\nabla}_\perp \Phi(\vec{x}(\vec{\theta}, w'), w') - \vec{\nabla}_\perp \Phi(0, w') \right] \Leftrightarrow$$

$$\Leftrightarrow \beta_i(\vec{\theta}, w) = \theta_i - \frac{2}{c^2} \int_0^w dw' \, \frac{f_K(w - w')}{f_K(w)} f_K(w') \left[\Phi_{,i} \left(\vec{x}(\vec{\theta}, w'), w' \right) - \Phi_{,i} \left(0, w' \right) \right].$$

Note that this is essentially a deviation to the usual triangle $x = D_A\,\theta$ (or $x = f_K\,\theta$) (valid for the homogeneous spacetime).

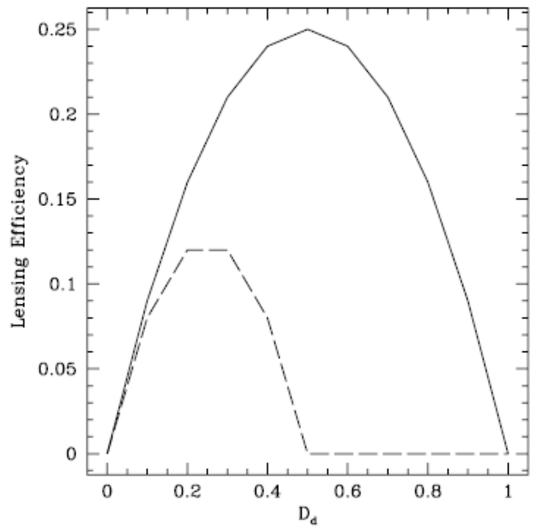
In the language of the lens equation this triangle would be $\beta = \theta$, i.e. a case with zero deflection.

When there are perturbations there is deflection and the 'triangle' changes to
$\beta = \theta + \alpha$

The second term of the equation is thus the solution for the deflection as function of the potential.

Note that the total deflection is the integral over all local deflections (the gradients of the various local potentials), each one multiplied by the 'weight' that appeared naturally in the solution (the ratio of distances i.e. of f_K functions) → this is known as the lensing efficiency factor:

$$\frac{D_{ds}D_d}{D_s}$$



This factor shows that **the lens at halfway of the trajectory is the one that contributes the most for the total deflection.**

Also note that the solution is recursive, because the separation x depends on the potential at the position x.

To get rid of this difficulty, we can Taylor expand the solution around the unperturbed trajectory (the one that lies on the positions $x = f_K\,\theta$).

$$\Phi_{,i}(x) = \Phi_{,i}(f_K\theta - f_K\alpha(x)) = \Phi_{,i}(f_K\theta) - f_K\alpha(x)\, f_K\Phi_{,ik}(f_K\theta) + \mathcal{O}(\alpha^2)$$

This results in:

$$\beta_i(\vec{\theta}, w) = \theta_i - \frac{2}{c^2}\int_0^w dw'\, \frac{f_K(w-w')}{f_K(w)} f_K(w')\Phi_{,i}(f_K\theta, w') +$$
$$+\frac{2}{c^2}\int_0^w dw'\, \frac{f_K(w-w')}{f_K(w)} f_K^2(w')\alpha(\vec{x})\Phi_{,ik}(f_K\theta, w') + f(w) + \mathcal{O}(\alpha^2),$$

Born approximation (the same that is done in quantum mechanics scattering)
+
higher-order terms

Keeping only the solution in the Born approximation, we can insert the amplification matrix definition $A_{ij}(\theta) = \frac{\partial \beta_i}{\partial \theta_j}$ to get the optical scalars:

$$A_{ij}(\vec{\theta}, w) = \delta_{ij} - \frac{2}{c^2} \int_0^w dw' \, \frac{f_K(w-w')}{f_K(w)} f_K(w') \Phi_{,ij} \, (f_K \theta, w').$$

$$A_{ij}(\vec{\theta}, w) = \delta_{ij} - \psi_{,ij} \, (\vec{\theta}, w).$$

where we defined the effective lensing potential:

$$\psi(\vec{\theta}, w) = \frac{2}{c^2} \int_0^w dw' \, \frac{f_K(w-w')}{f_K(w)} f_K(w') \Phi(f_K \theta, w')$$

We recover the result that **the optical scalar fields are second-order derivatives of the potential:**

convergence $$\kappa = \frac{1}{2}(\psi_{,11} + \psi_{,22}).$$

shear $$\gamma_1 = \frac{1}{2}(\psi_{,11} - \psi_{,22}) \quad , \quad \gamma_2 = \psi_{,12} \, .$$

rotation $$\omega = 0.$$

Lensing produces no **Rotation**. This is a consequence of the fact that a gravitational field is a gradient field (of a potential) → its rotational is zero.

Shear γ has two components, two terms in the optical matrix → it is a polar vector

Convergence k is a scalar and is the Laplacian of the potential → it is related with the mass of the lens through the Poisson equation:

$$\nabla_p^2 \Phi = 4\pi G\rho = 4\pi G\bar{\rho}\,\delta \Leftrightarrow \nabla^2\Phi = a^2\, 4\pi G\,\Omega_m \rho_c a^{-3}\,\delta = \frac{3H_0^2\,\Omega_m\,\delta}{2a},$$

$$\kappa(\vec{\theta}, w) = \frac{3}{2}\left(\frac{H_0}{c}\right)^2 \Omega_m \int_0^w dw' \, \frac{f_K(w-w')f_K(w')}{f_K(w)\,a(w')}\,\delta(f_K(w')\vec{\theta}, w').$$

We see that the convergence field is a weighted integral of the density contrast field.

This also means that **the power spectrum of the convergence can be related to the power spectrum of dark matter.**

Lensing signal

The convergence and shear amplitudes (i.e. the **lensing signal**) from the cosmological lensing effect over one galaxy are very small.

For example, consider a source galaxy at z_s = 0.8 and a lens at z_l = 0.4 with comoving size 8 Mpc (a cluster). For this system:

$$\kappa \approx \frac{3}{2}\Omega_{\rm m}\left(\frac{H_0}{c}\right)^2 \frac{D_{\rm LS}D_{\rm L}}{D_{\rm S}} \frac{R}{a^2(z_{\rm L})} \frac{\delta\rho}{\rho}$$

Inserting the distances D_L=1120 Mpc, D_S = 1500 Mpc, D_{LS} = 400 Mpc and r_H = 3000 Mpc/h, we get:

k ~ 0.0001

With these redshifts (which are typical of current surveys), a number N = D_S / R of lens planes fit along the trajectory. If a light ray typically crosses D_S / R ~100 planes, the signal increases to

k ~ 0.01

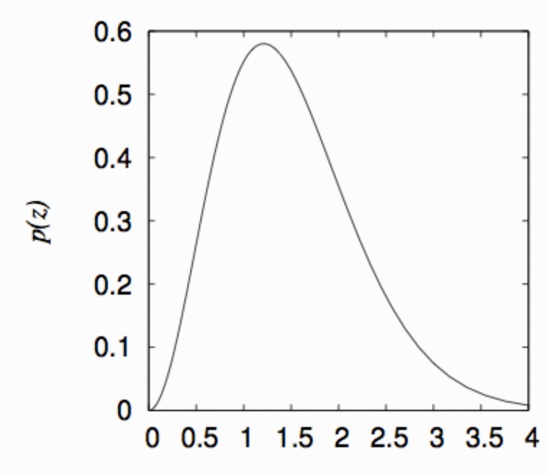
This is a small number, well inside the weak lensing regime.

Note that a shear of 0.01 corresponds to the difference in ellipticity between the ellipticities of Uranus and the Moon.

This shows that the cosmological weak lensing signal can only be detected **statistically**, measuring it over a large number of source galaxies.

For this, we **need to consider the convergence from a distribution of sources at various redshifts.**

The signal is integrated over the distribution: $\kappa(\vec{\theta}) = \int_0^{w_H} dw\, p(w)\, \kappa(\vec{\theta}, w)$.



with for example,

$$p(z) \propto \left(\frac{z}{z_0}\right)^{\alpha} \exp\left[-\left(\frac{z}{z_0}\right)^{\beta}\right]$$

For a distribution of sources, the convergence can be rewritten as

$$\kappa(\vec{\theta}) = \frac{3}{2}\left(\frac{H_0}{c}\right)^2 \Omega_m \int_0^{w_H} dw' \, \frac{f_K(w')}{a(w')} \, \delta(f_K(w')\vec{\theta}, w') \, g(w').$$

(integral along the line of sight, over the lenses at w')

where

$$g(w') = \int_w^{w_H} dw \, p(w) \, \frac{f_K(w - w')}{f_K(w)}.$$

(integral over the sources at w, for each lens at w')

Note that the optical scalars are perturbed quantities (as we say they do not arise in the homogeneous space-time).

They have zero mean, $< \kappa > = < \gamma > = 0.$

and **the cosmological information is on the moments, i.e., in the correlation function and power spectrum**.

The **power spectrum of the convergence** field is of course related with the power spectrum of dark matter:

$$P_\kappa(\ell) = \frac{9}{4}\left(\frac{H_0}{c}\right)^4 \Omega_m^2 \int_0^{w_H} dw \, \frac{g^2(w)}{a^2(w)} \, P_\delta\left(\frac{\ell}{f_K(w)}, w\right)$$

The convergence power spectrum is a weighted line-of-sight integral of the matter power spectrum

matter/gravity relation

diameter angular distances

redshift of sources

$$P(k,a) = A\, k^n T_k^2 D_+^2(a)$$

primordial power spectrum (inflation)

transfer function

linear growth

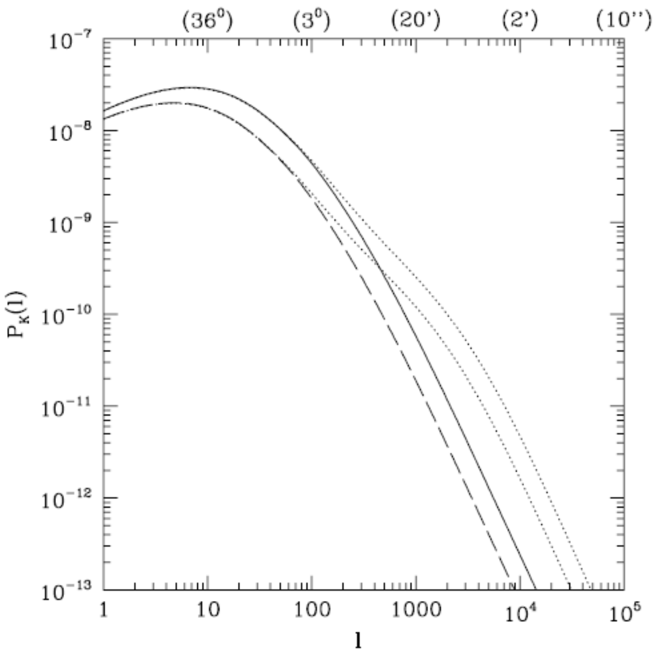
non-linear corrections

cosmological parameters

The convergence power spectrum is a projected power spectrum.

At each angular scale $l = 2\pi/\theta$, its amplitude has contributions from various k scales from the matter power spectrum at different redshifts:

$$k = l / f_K(w(z_{lens}))$$



Weak lensing cosmic shear surveys measure a lensing signal in the scale range from few arcmin to few degrees. A **typical scale of cosmic shear measurements** is:

θ=30 arcmin → l=1000
if z_s=1 → k=0.8 h/Mpc → r = 8 Mpc/h
(mildly non-linear scales)

Linear and non-linear convergence power spectrum for two different source redshift distributions (higher z_s has higher amplitude) → there is a **strong degeneracy between z_s and σ_8** → this shows it is crucial to know the redshifts in cosmic shear surveys.

We can also derive the **power spectrum of the shear**.

Since shear and convergence are both second-order derivatives of the cosmological lensing potential, their power spectra are related.

The Fourier transform of a function of the form $f(\vec{\theta}) = \psi_{,ij}$ is:

$$\tilde{f}(\vec{\ell}) = \int d^2\theta \, e^{i\vec{\ell}.\vec{\theta}} \frac{d^2}{d\theta_i d\theta_j} \int \frac{d^2\ell'}{(2\pi)^2} \tilde{\psi}(\vec{\ell'}) e^{-i\vec{\ell'}.\vec{\theta}} = \int \frac{d^2\ell'}{(2\pi)^2} (-\ell'_i \ell'_j) \tilde{\psi}(\vec{\ell'}) (2\pi)^2 \delta_D(\vec{\ell} - \vec{\ell'}) = -\ell_i \ell_j \, \tilde{\psi}(\vec{\ell}).$$

and so:

$$\tilde{\kappa} = -\frac{1}{2}\left(\ell_1^2 + \ell_2^2\right) \tilde{\psi} \quad , \quad \tilde{\gamma} = \left[-\frac{1}{2}\left(\ell_1^2 - \ell_2^2\right) - i\ell_1\ell_2\right] \tilde{\psi}.$$

Computing the shear power spectrum:

$$(2\pi)^2 \delta_D(\vec{\ell} - \vec{\ell'}) P_\gamma(\ell) = \langle \tilde{\gamma}(\ell) \tilde{\gamma}^*(\ell') \rangle$$

we get

$$\langle \tilde{\gamma}(\ell) \tilde{\gamma}^*(\ell') \rangle = \frac{(\ell_1^2 - \ell_2^2 + 2i\ell_1\ell_2)(\ell_1^2 - \ell_2^2 - 2i\ell_1\ell_2)}{\ell^4} \langle \tilde{\kappa}(\ell) \tilde{\kappa}^*(\ell') \rangle = \langle \tilde{\kappa}(\ell) \tilde{\kappa}^*(\ell') \rangle$$

i.e., **the shear and the convergence power spectra are identical.**

$$\boxed{P_\gamma(\ell) = P_\kappa(\ell)}$$

We can also derive the **correlation function of the shear**

$$\xi_\gamma(\vartheta) = \int \frac{d^2\ell}{(2\pi)^2} e^{-i\vec{\theta}.\vec{\ell}} \int \frac{d^2\ell'}{(2\pi)^2} e^{i\vec{\theta}'.\vec{\ell}'} \langle \tilde{\gamma}(\ell)\tilde{\gamma}^*(\ell') \rangle = \int \frac{d^2\ell}{(2\pi)^2} e^{-i\ell\,\vartheta\cos(\varphi)} P_\gamma(\ell)$$

Writing the scale vector (l_1, l_2) in polar coordinates, $\vec{\ell} = (\ell_1, \ell_2) = \ell(\cos\varphi, \sin\varphi)$.

$d^2 l = dl\, l\, d\varphi$, and the angular part $d\varphi$ can be integrated out, since from isotropy the power spectrum only depends on the modulus of l.

The integral of the angular part of the plane wave is given by a Bessel function:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi\, e^{i(n\varphi - x\sin(\varphi))}$$

Bessel function of the first kind, with order n

After integrating out the angular part, the correlation function is the following radial integral of the power spectrum:

$$\xi_\gamma(\vartheta) = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\ell\vartheta\cos(\varphi)} = 2\pi \int d\ell \, \ell P_\kappa(\ell) \, \frac{J_0(\ell\vartheta)}{(2\pi)^2}.$$

This shows that, as usual, the correlation function is a filtered version of the power spectrum, mixing the power of its scales, depending on the filter function.

We can also define a **power spectrum and correlation function for individual components of shear**:

$$\xi_{11}(\vartheta) = \left\langle \gamma_1(\vec{\theta})\gamma_1^*(\vec{\theta}') \right\rangle$$

$$\langle \tilde{\gamma}_1(\ell)\tilde{\gamma}_1^*(\ell) \rangle = \left(\frac{\ell_1^2 - \ell_2^2}{\ell^2} \right)^2 \langle \tilde{\kappa}(\ell)\tilde{\kappa}^*(\ell) \rangle = (\cos^2\varphi - \sin^2\varphi)^2 \langle \tilde{\kappa}(\ell)\tilde{\kappa}^*(\ell) \rangle$$

and so it relates with the convergence power spectrum as,

$$P_{11} = \cos^2(2\varphi)\, P_\kappa = \frac{1}{2}(1 + \cos 4\varphi)\, P_\kappa$$

The corresponding correlation function is the Fourier transform of P_{11}.

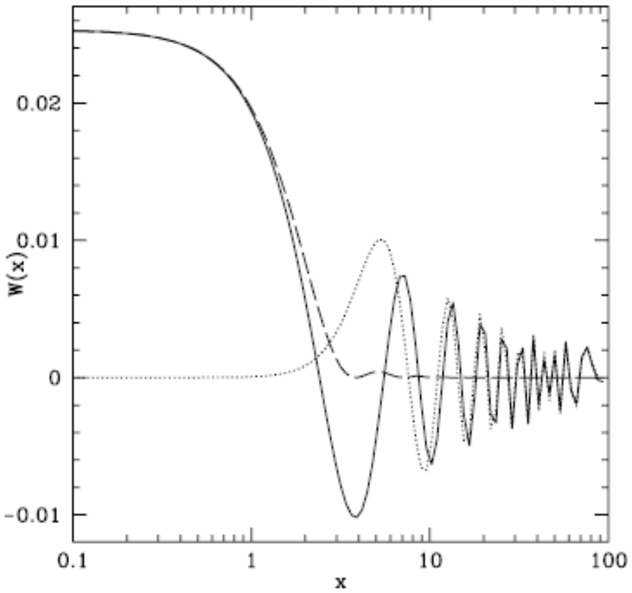
After integrating out the angular part, the 11 correlation function is:

$$\xi_{11}(\vartheta) = \frac{1}{2} \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \; [J_0(\ell\vartheta) + J_4(\ell\vartheta)]$$

Similarly for the other components:

$$\xi_{22}(\vartheta) = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\ell\,\vartheta\cos(\varphi)} \sin^2(2\varphi) = \frac{1}{2} \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \left[J_0(\ell\vartheta) - J_4(\ell\vartheta)\right]$$

$$\xi_{12}(\vartheta) = \frac{1}{2} \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-i\ell\,\vartheta\cos(\varphi)} \sin(4\varphi) = 0.$$



Solid: filter ξ+ (low-pass band)
Dotted: filter ξ- (narrow-band)

Usually the following linear combinations of shear correlation functions are defined:

$$\xi_+ = \xi_{11} + \xi_{22} = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell)\, J_0(\ell\theta) \;,\; \xi_- = \xi_{11} - \xi_{22} = \int \frac{d\ell}{2\pi} \ell P_\kappa(\ell)\, J_4(\ell\theta) \;,\; \xi_\times = \xi_{12} = 0$$

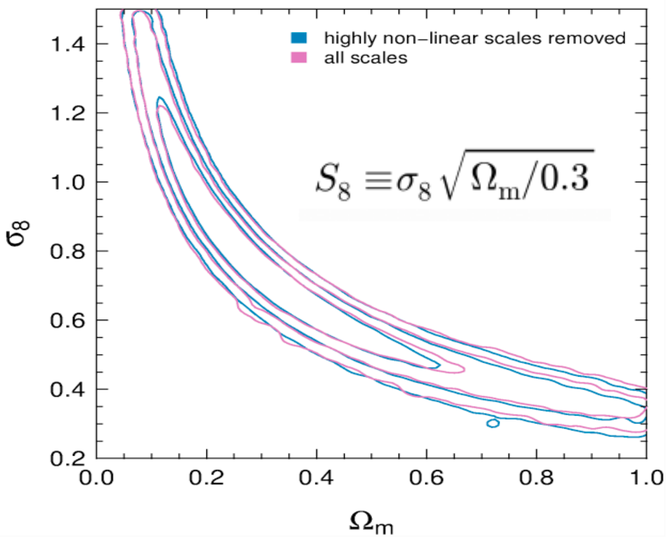
Cosmological content

The cosmological weak lensing power spectra define various filtered versions of the matter power spectrum P_δ

The cosmological weak lensing deflections are produced by LSS gravitational potentials → by the total mass in the structure (which is mostly dark matter) → lensing is sensitive to the total mass, **it is independent of the nature of matter (baryonic or dark) and of its dynamical state (relaxed or merging)**

It is sensitive to the cosmological parameters:

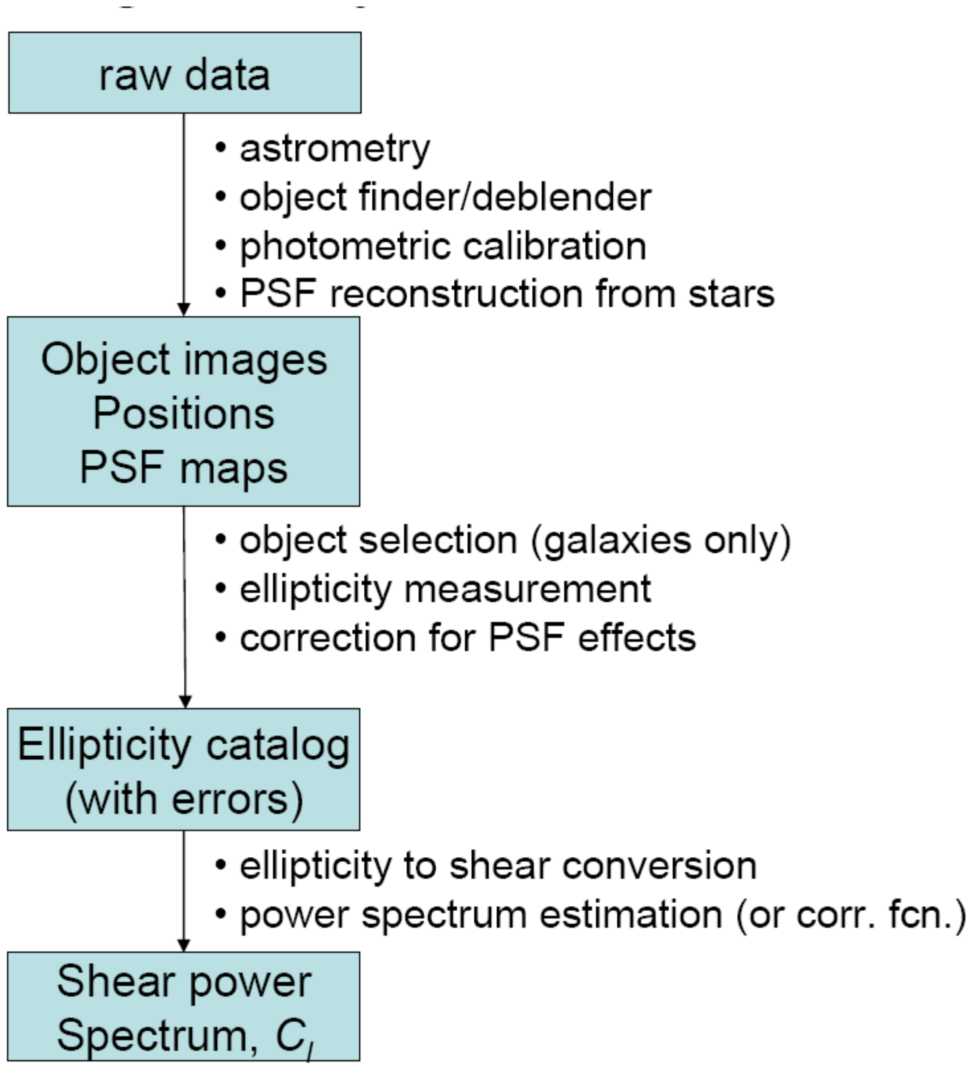
- through structure formation (P_δ)
- through direct dependences on H_0, Ω_m
- through the background evolution ($D_A(z)$ in the function g(w))



It is mainly sensitive to Ω_m and to the amplitude of P_δ (i.e., to σ_8) with a well defined degeneracy direction, and to the sources redshift distribution

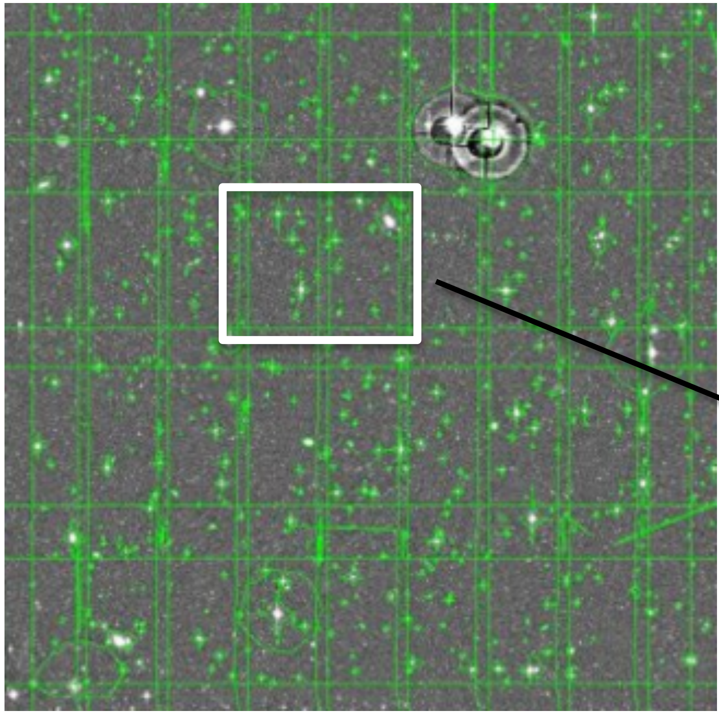
Weak lensing: estimator

Lensing Analysis Pipeline

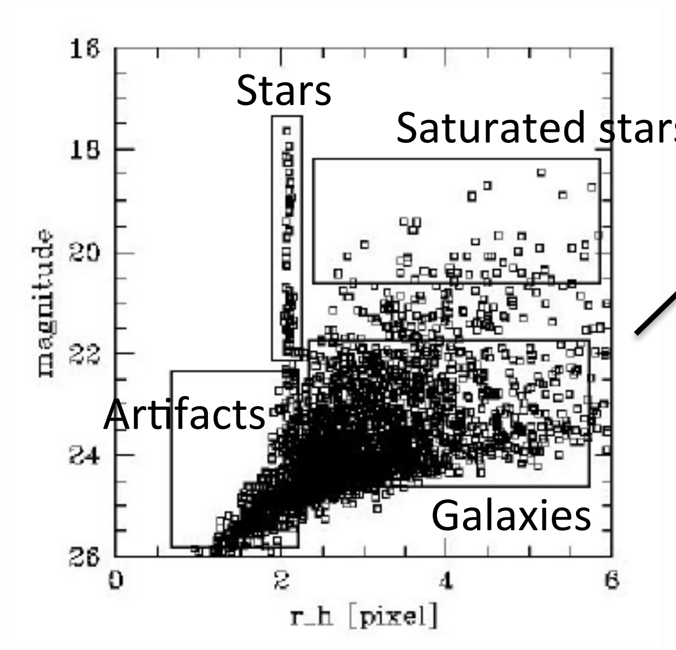


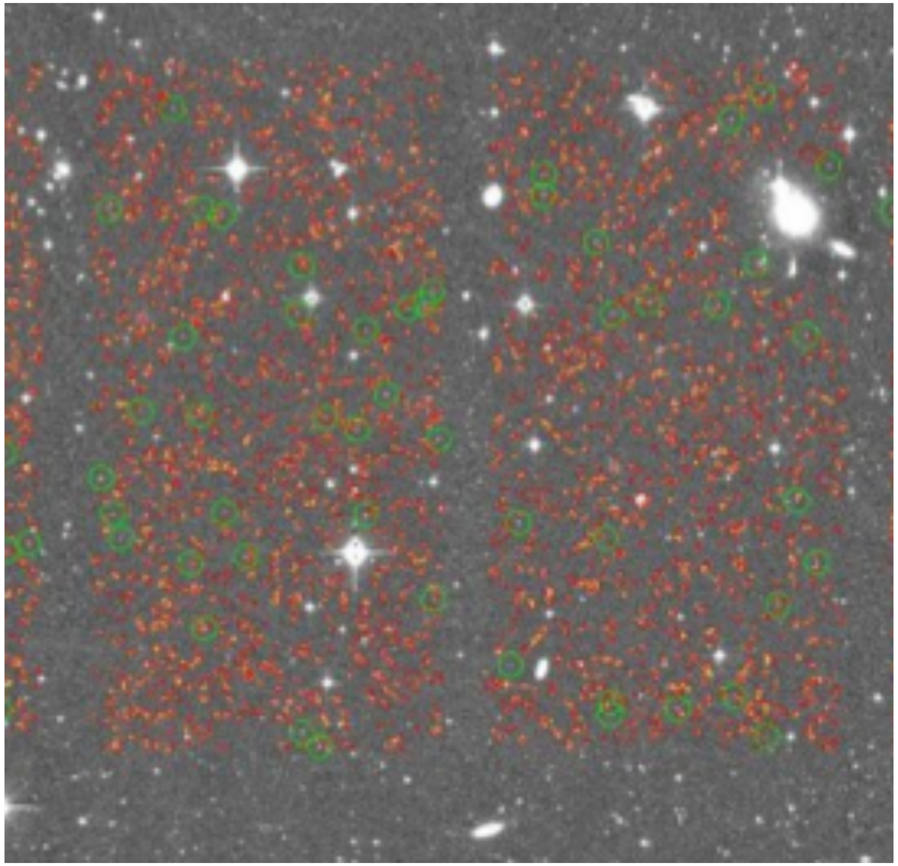
only stars and foreground galaxies
are visible in this image

background galaxies are visible now,
and also the “ghosts” from a
saturated star



Masking





Stars (green) / Galaxies (red) separation

Shear estimator

The estimator of shear is the **ellipticity**.

The shapes of distant galaxies in a 2D image are approximately ellipses (valid for both elliptical and spiral galaxies). They can be described by 2 parameters: eccentricity |e| (deviation from a circle) and orientation φ. These 2 parameters define the ellipticity, which is a traceless symmetric tensor.

Note that under a rotation of α, a traceless symmetric tensor transforms in the same way as a vector under a rotation of 2α.

$$\begin{bmatrix} e_1' & e_2' \\ e_2' & -e_1' \end{bmatrix} = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} e_1 & e_2 \\ e_2 & -e_1 \end{bmatrix} \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix},$$

which is equivalent to,

$$\begin{bmatrix} e_1' \\ e_2' \end{bmatrix} = \begin{bmatrix} cos2\alpha & sin2\alpha \\ -sin2\alpha & cos2\alpha \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

For this reason, traceless symmetric tensors are also called **pseudo-vectors**, which have π symmetry, instead of 2 π.
They are also called **spin-2** quantities and its components can be written in vector form:

$$e = |e| \exp(2i\varphi) = e_+ + ie_\times$$

The ellipticity of an object is computed from the second-order moments of brightness (with respect to the centroid of the image),

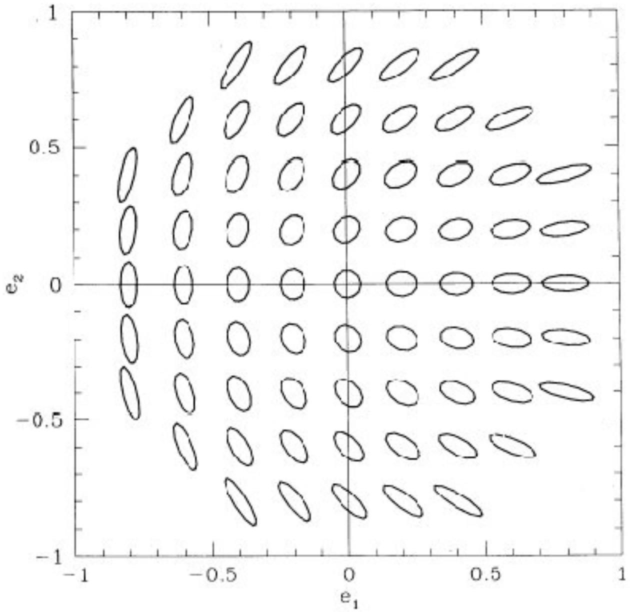
$$Q_{ij} = \int d^2\theta \, I(\theta)(\theta_i - \theta_i^0)(\theta_j - \theta_j^0).$$

as,

$$e = \left(\frac{Q_{xx} - Q_{yy}}{Q_{xx} + Q_{yy}}, \frac{2Q_{xy}}{Q_{xx} + Q_{yy}} \right)$$

So component e_+ measures the normalized excess of flux along the x-axis with respect to the flux along the y-axis

and component e_X measures the normalized excess of flux along the y = x line with respect to the flux along the y = −x line



The ellipticity ranges from 0 → the ellipticity of a circular object, to 1 → the limiting case of an extremely elliptical object that becomes one-dimensional.

It is dimensionless, not containing information about the size of the object, which is encoded in the trace $Q_{xx} + Q_{yy}$

To understand why the ellipticity is an estimator of the shear, let us consider a 2D image of a galaxy (the source shape) that is subject to weak gravitational lensing and will be transformed into a slightly different 2D (the image shape).

The moments of the source are transformed by the lens equation (the lensing transformation) into the moments of the image:

$$Q^s = A(\theta)\, Q\, A^t(\theta)$$

For example, for the trace of the moments we get,

$$Q^s_{11} + Q^s_{22} = Q_{11}\,[(1-\kappa-\gamma_1)^2 + \gamma_2^2] + Q_{22}\,[(1-\kappa+\gamma_1)^2 + \gamma_2^2] + Q_{12}\,4\gamma_2(1-\kappa)$$

Computing the transformation for all moments, and combining them to form the ellipticities, we get an expression for the transformation of the ellipticities.

We may neglect quadratic terms in the transformation, because we are in the weak lensing regime:

$$\kappa << 1\ ,\ |\gamma| << 1\ ,\ g \approx \gamma + \gamma\kappa.$$

where g is the reduced shear

$$g_i = \gamma_i/(1-\kappa)$$

In the weak lensing approximation the resulting transformation is:

$$\boxed{e_i^s = e_i - 2g_i}$$

and this is the **shear estimator.**

So the reduced shear produced by the lensing effect (which is g ~γ)
adds linearly to the intrinsic (source) ellipticity of the galaxy to produce the image galaxy ellipticity.

The estimator cannot give us the exact value of the shear acting on a galaxy because we do not know the source ellipticity e^s of a galaxy.

But it can be used to estimate the shear from the measured ellipticity, if we know the properties of the intrinsic ellipticity distribution.

$$2\gamma = e_{\rm obs} - < e_s > \pm \frac{\sigma_s^2}{N}$$

If the galaxies have intrinsically random ellipticities, which implies random orientations → $<e_s> = 0$ → the estimator is unbiased.

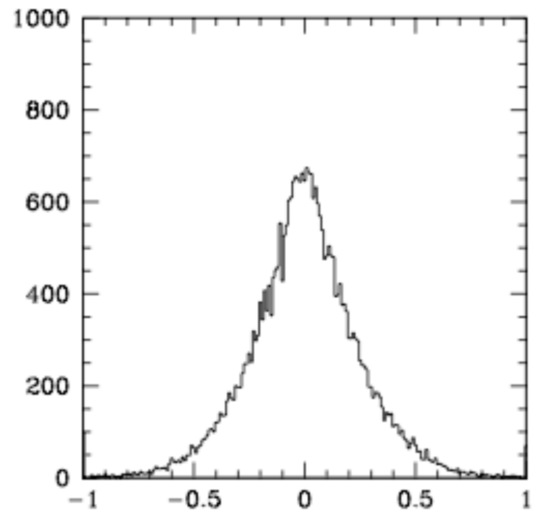
If the galaxies eccentricities and orientations are intrinsically correlated (for example for having been formed together in the same DM halo) → $<e_s> \neq 0$ → the estimator is biased.

In general it is always possible to find a sample of uncorrelated galaxies in the same 2D area of the sky, and have an **unbiased estimator**.

We already saw that the typical convergence (and shear) signal is 0.01.

The measured rms of ellipticity distributions is ~0.3 → it is much larger that the cosmic shear signal → it is due to the intrinsic ellipticities dispersion.

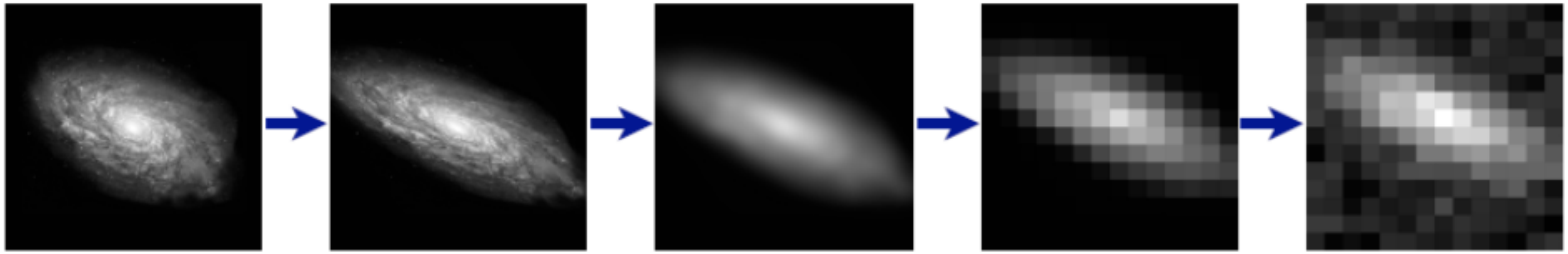
This means that the shear estimator is very noisy → a large number of galaxies is needed to be able to detect the cosmological lensing signal.



intrinsic ellipticities distribution

But the ellipticity of a galaxy image is not only induced by gravitational lensing → there are several other effects

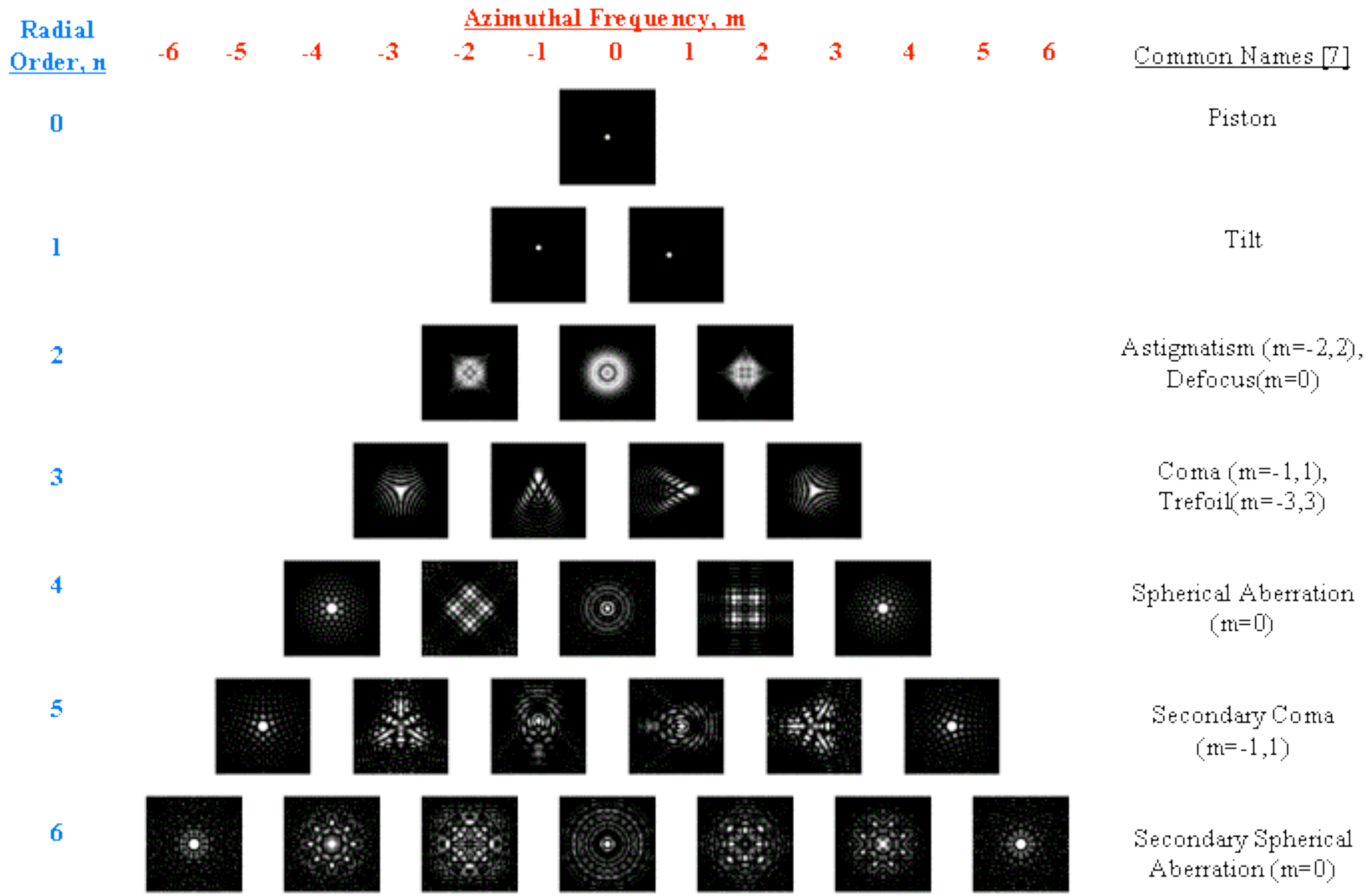
Galaxies: Intrinsic galaxy shapes to measured image:



So in reality **the estimator is biased**, and the non-cosmological distortions need to be corrected.

The dominating effect is the Point Spread Function (PSF) produced by the atmosphere and by the optical system of the telescope.

The PSF model convolves the image.
The amplitude of the PSF effect is much larger than the cosmological effect.



The types of PSF present in the optical system are characterized when building the telescope by simulating its wavefront.

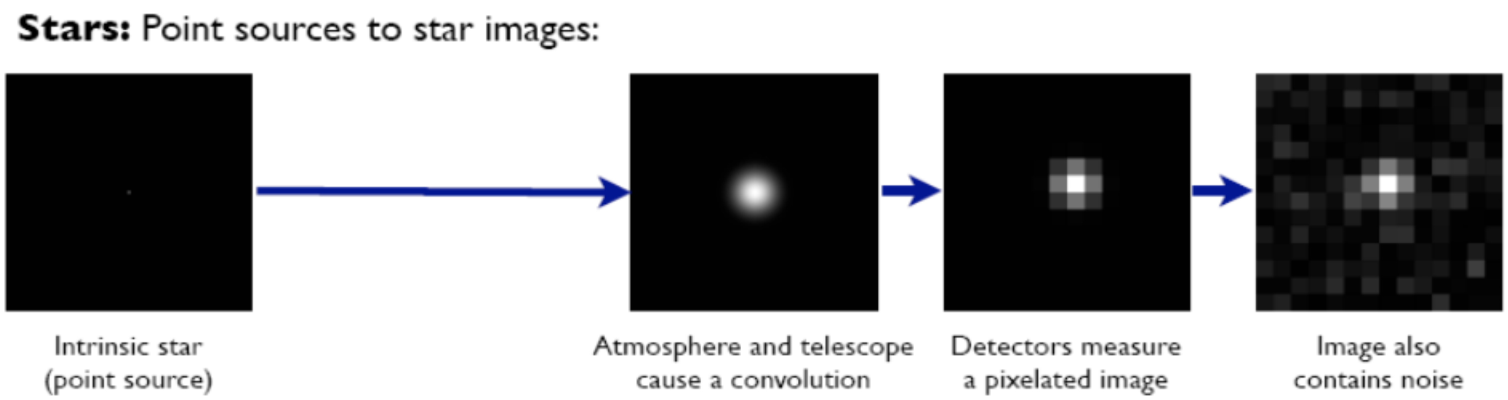
The biased shear estimator can be written as:

$$e_i^{s'} = e_i^{obs} - P_{ij}^{\gamma} g_j - P_{ij}^{sm} q_j$$

It includes the PSF anisotropy q → modeled as an **additive bias**
and the PSF isotropy, which decreases the response of the galaxies to shear, producing a change in the factor 2 in the original unbiased estimator → modeled as a **multiplicative bias**

The bias can be corrected because the PSF can be measured using stars. Stars are not affected by cosmological lensing → **any ellipticity detected in the stars in the image is produced by the PSF.**

In fact, stars are point-like and would not even be seen in an image if there was no isotropic PSF (like the seeing produced by the atmosphere).

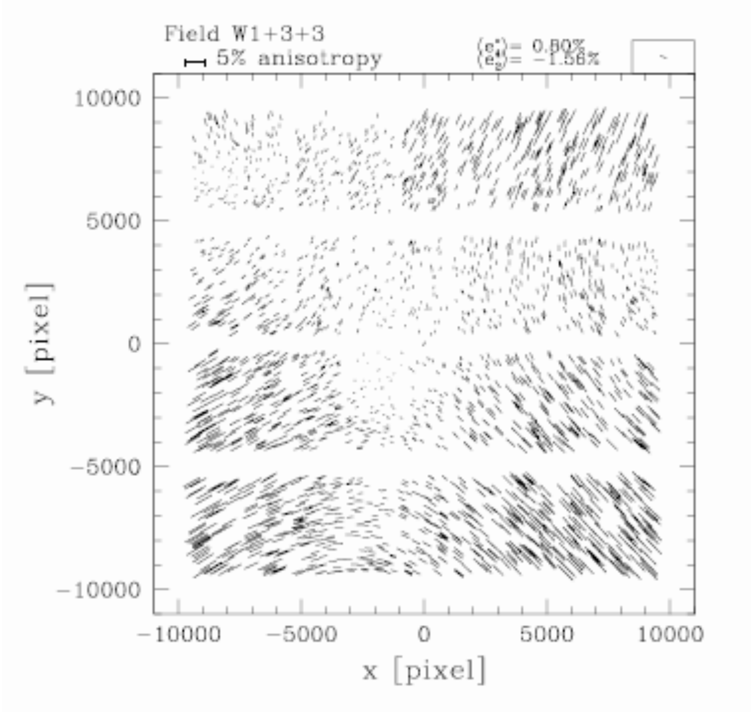


PSF is measured at stars positions → It is then interpolated across the FoV to find its values at the galaxies positions

PSF deconvolution:

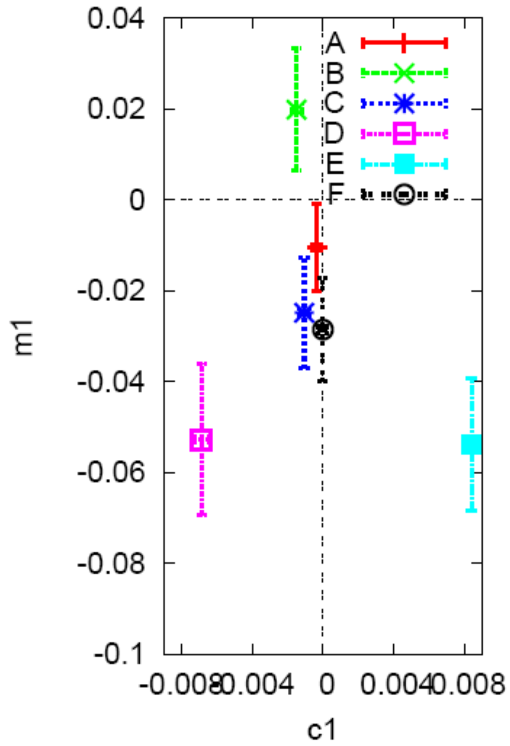
The PSF can then be subtracted (deconvolved) from the image.

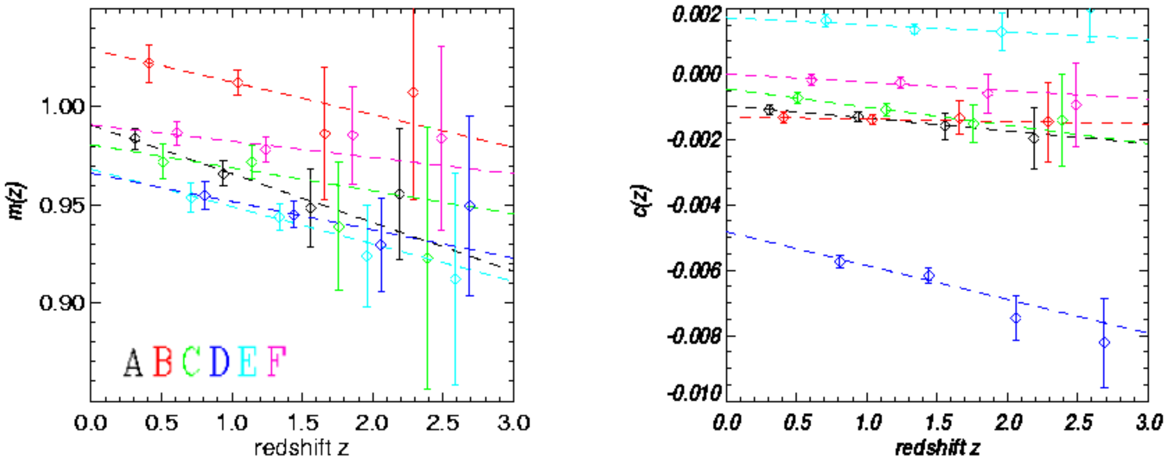
Simulations with known cosmic shear and PSF models may be used to check for residuals of the correction procedure → to calibrate the result:



Multiplicative and additive residuals
for 6 PSF simulations:

$$\langle \gamma \rangle - \gamma^{\text{input}} = m\, \gamma^{\text{input}} + c$$

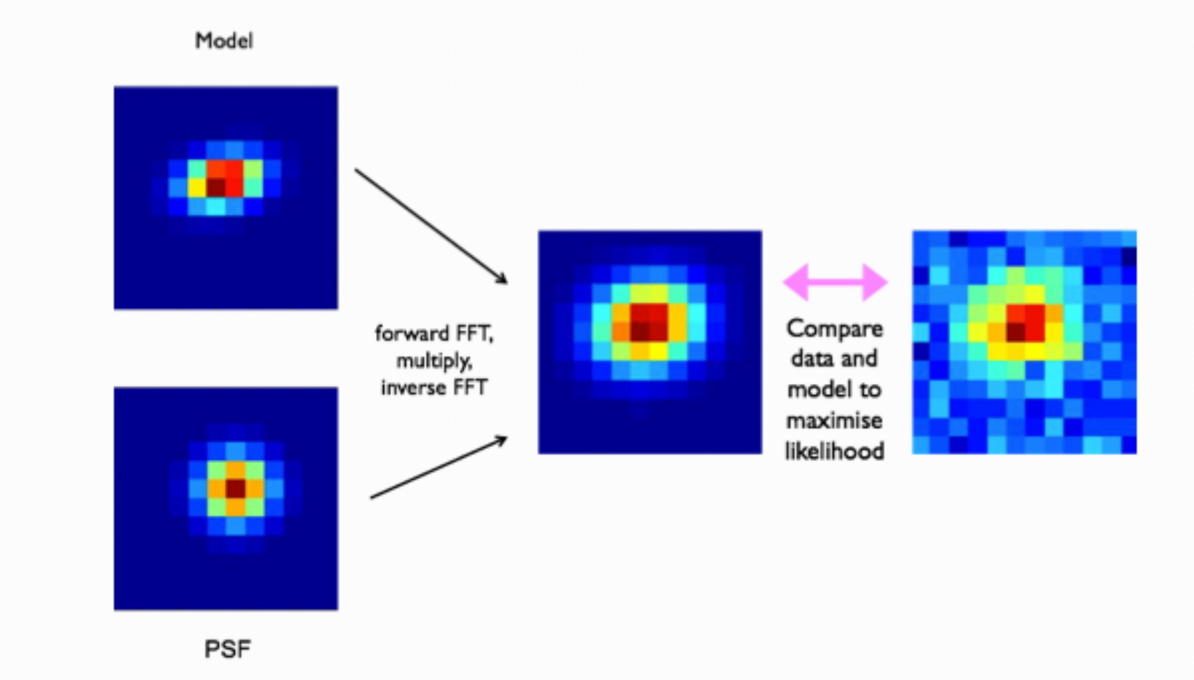




m and c evolution with redshift

If shear simulations are not used, the values of the residuals are not known. In that case they may be included in the estimator as nuisance parameters → **PSF calibration**

Alternatively, PSF may be corrected with **forward model fitting**:



The PSF (measured from stars) is convolved (multiplied in Fourier space) with models for the galaxy image.

Compare the results with the observed image → Bayesian analysis to find the best model.

In both cases calibration nuisance parameters are introduced, to ensure greater accuracy.

Shear correlation function estimator

The goal of weak lensing measurements is to go from ellipticity measurements → to 2D correlation function of shear → to 2D metric (potential) power spectrum or dark matter power spectrum → to compare with theoretical predictions

We are interested in the statistical properties of the ellipticity field and not on finding the individual shear of each galaxy → **we may estimate directly the shear correlation function instead of the shear.**

Roughly speaking, we saw that the shear is estimated from

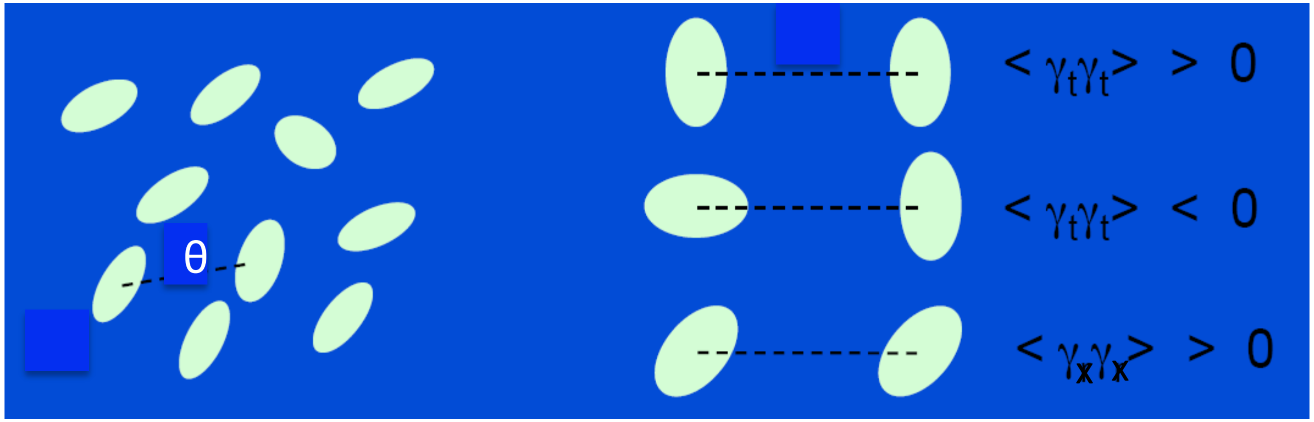
$e = e^s + \gamma$ (neglecting calibration factors)

So the ellipticity correlation function is an estimator of the shear correlation function:

$$\hat{\xi}_{ee} = \xi_{\gamma\gamma} + \xi_{e^s e^s} + \xi_{\gamma e^s}$$

The ellipticity correlation function of a discrete galaxy field is measured from the correlation of ellipticity pairs as function of separation:

$$\hat{\xi}_{tt}(\theta) = \frac{\sum_{i,j} w_i w_j \, e_t(\mathbf{x}_i) e_t(\mathbf{x}_j)}{\sum_{i,j} w_i w_j} \qquad \hat{\xi}_{\times\times}(\theta) = \frac{\sum_{i,j} w_i w_j \, e_\times(\mathbf{x}_i) e_\times(\mathbf{x}_j)}{\sum_{i,j} w_i w_j}.$$



Bias of the estimator

However, the ellipticity correlation function does not give us directly the shear correlation function. It is a biased estimator of it, due to the two additional effects that also contribute to the ellipticity correlation function:

- $\xi_{e^s e^s}$ **correlation function of the source ellipticities** (i.e., the intrinsic distribution of ellipticities, before the lensing effect).

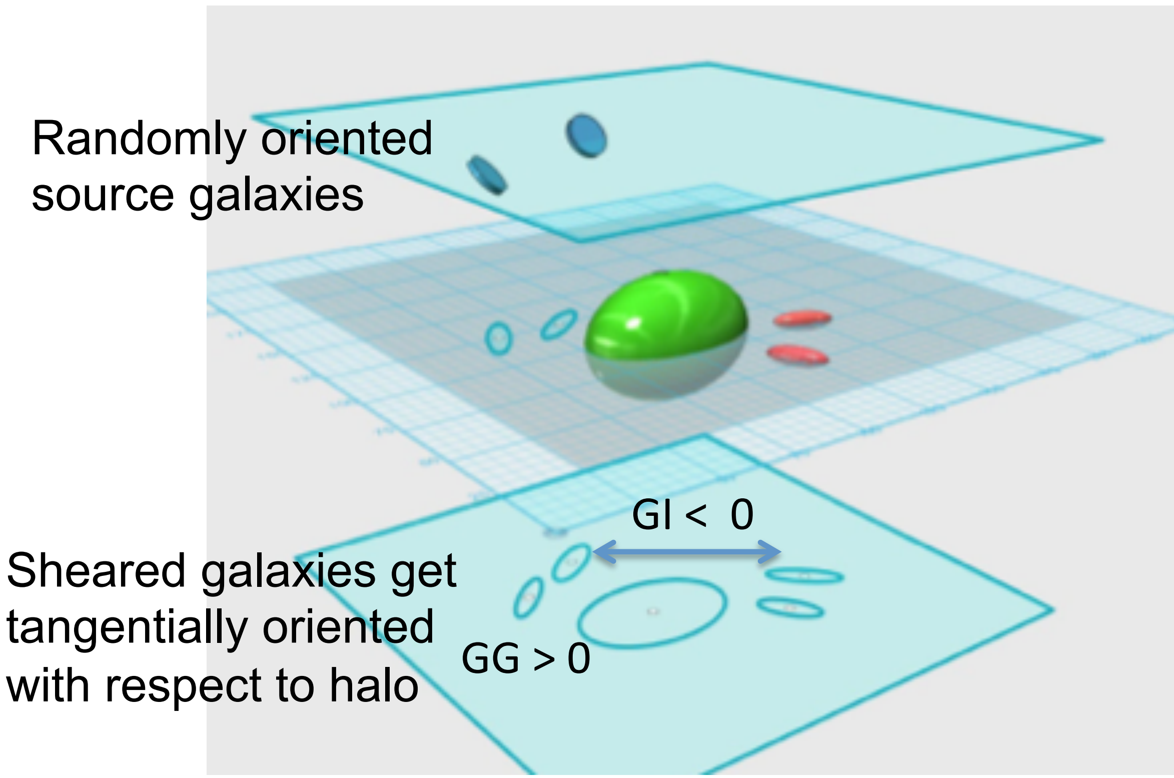
  It depends on the type of pairs involved:

  - for i=j it is a monopole constant term → a shot noise $\frac{\sigma_{e^s}^2}{n}$

  - for i ≠j it is the correlation of the intrinsic ellipticities between different galaxies → an intrinsic alignment (II)

- $\xi_{\gamma e^s}$ **shear-ellipticity cross-correlation**

  It is the correlation between the intrinsic shape of a galaxy and the shear produced in a second galaxy (its i=j contribution is zero, but i ≠j is not zero)→ another type of intrinsic alignment (GI)



The contamination from $\xi_{e^s e^s}$ (II) is zero if we do not consider galaxies at the same redshift bin

The contamination from $\xi_{\gamma e^s}$ (GI) depends on galaxy formation. It can be measured with <eδ> (galaxy-galaxy lensing)

Origin of the intrinsic alignments

Elliptical galaxies near halos are tidally streched → creates II

$$\varepsilon_+ \propto (\partial_y^2 - \partial_x^2)\phi$$

$$\varepsilon_X \propto 2\partial_x \partial_y \phi$$

Spiral galaxies orientation near halos determined by angular momentum L → do not correlate with halo orientation → no GI

$$\varepsilon_+ \propto (L_y^2 - L_x^2)$$

$$\varepsilon_x \propto 2L_x L_y$$

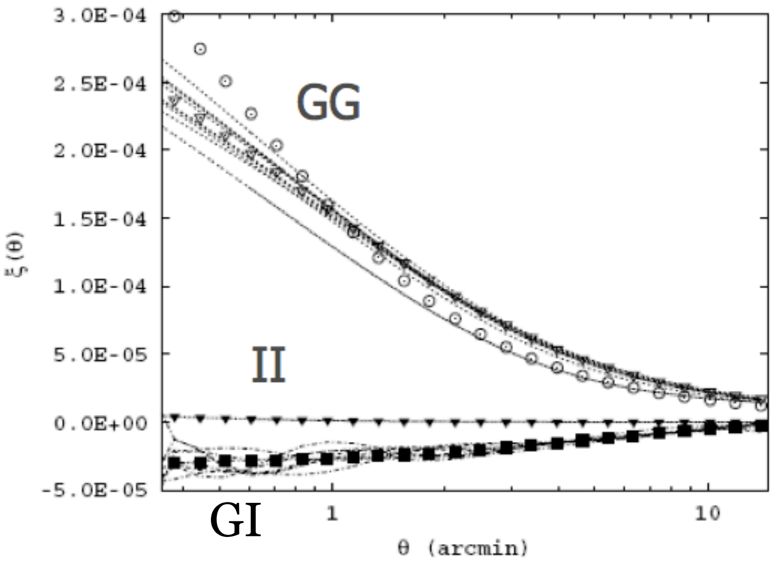
$$L_\alpha \propto \varepsilon_{\alpha\beta\gamma} J_{\beta\delta} \partial_\delta \partial_\gamma \phi$$

So **the shear correlation function estimator is biased** by construction, due to the presence of intrinsic alignments.

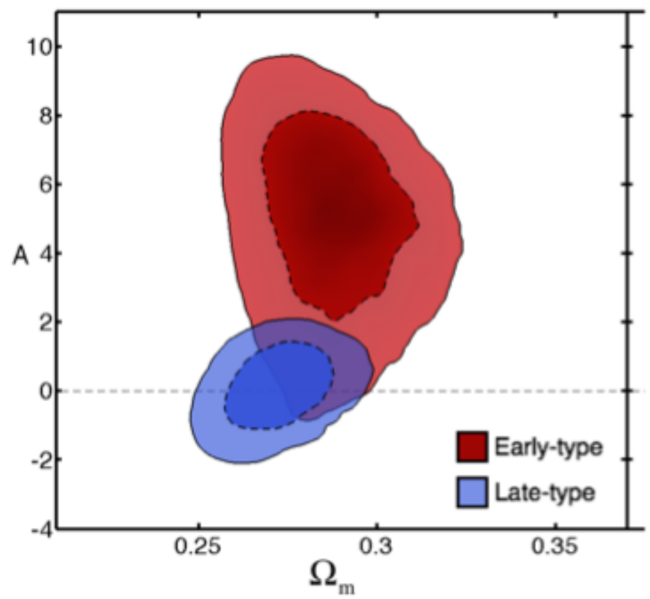
However, if the galaxy pairs in the correlation are at different redshifts, the dominant contribution for the ellipticity correlation is the shear correlation (GG):

II is zero (because the two galaxies are distant in redshift)

GI < 0 and ~ 10% GG



GI can be estimated from galaxy-galaxy lensing measurements using early-type (ellipticals) and late-type (spirals) galaxies



Besides the fundamental intrinsic alignment biases, there are **3 other main classes of systematics** that affect the measurement of the shear signal and impact the estimation of cosmological parameters.

They come from the **measurement of the ellipticities**, from the **determination of the source redshift distribution**, and from uncertainties on the **shear theoretical power spectrum.**

i) **Bias in the shear measurement :** there are many sources of bias in the measurement of shear, besides PSF residuals, that propagate into the correlation measurement. For example:

- Light-profile model bias: due to noise, the brightness moments need to be computed using a filter. This needs to correctly model the light profile, otherwise it will introduce a bias. It is easy to use a non-appropriate filter in cases of non-elliptical isophotes, or when there are color gradients (different profiles in different filters → bias broad-band measurements)

- Noise bias: in general, ellipticity is non-linear in pixel data → the simple fact that the flux values in the image pixels are noisy changes the shear-to-ellipticity linear relation → if we use it, we introduce a bias

- PSF residuals

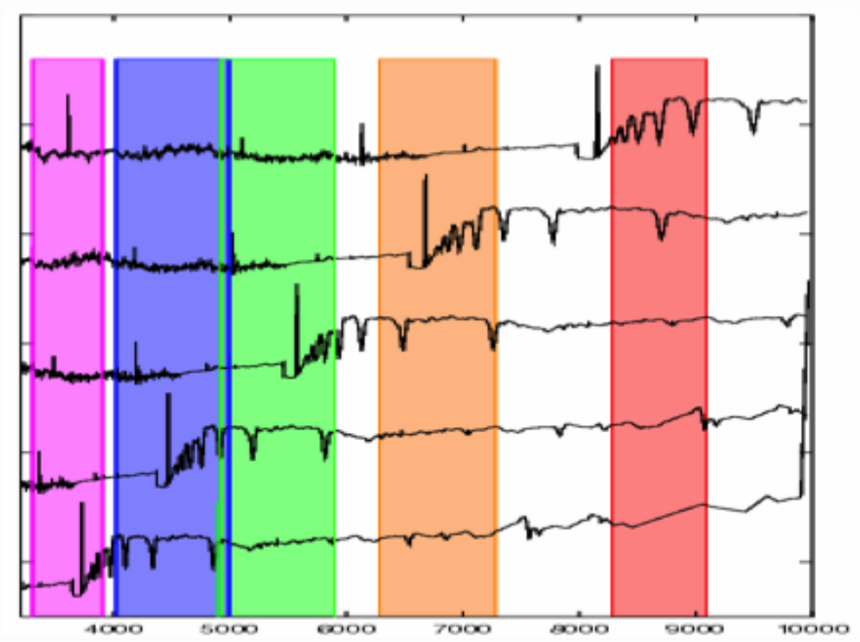
- Detector effects: charge transfer inefficiency (CTI)

ii) **Bias in the redshift distribution**:

- Wrongly identified photometric redshifts

Typical filters u ; g r i y z ; I J K

used to detect the strongest features, like the 4000 Angstrom-break for galaxies at various redshifts



Some **properties** of photometric redshift estimation:

In the redshift desert, z ~ 1.5 - 2.5 → neither 4000 A-break or Ly-break in visible range → very hard to access from ground.

Confusion between low-z dwarf ellipticals and high-z galaxies and confusion between Balmer and Lyman break → catastrophic outliers

UV band and IR needed for high redshifts → but UV is very inefficient and IR is absorbed by atmosphere → need space observations.

Need spectroscopic galaxy sample for comparison and calibration, or also for cross-correlation.

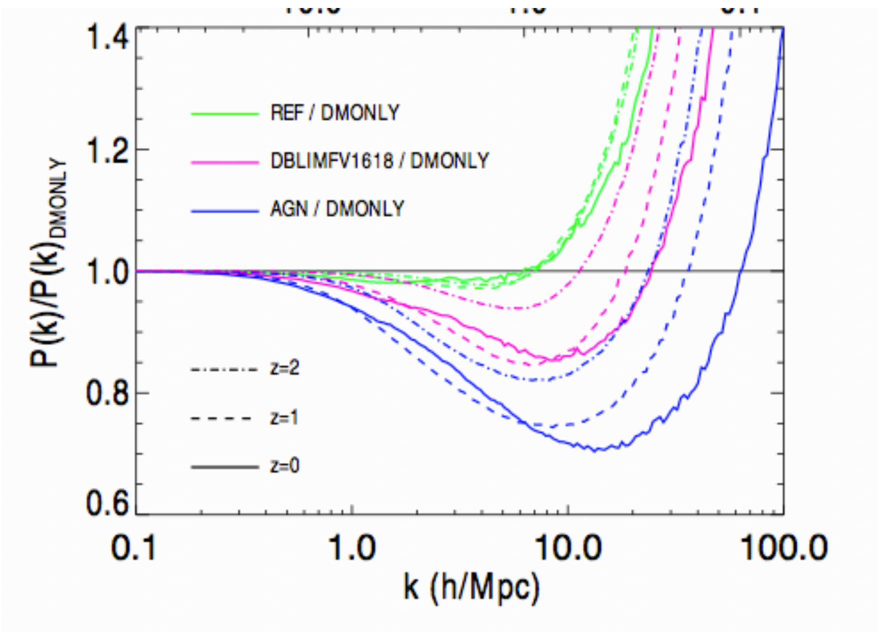
The typical **accuracy of photo-z** determination is: $\sigma \sim 0.05\,(1 + z)$

- Selection effects: for example blended galaxy images are discarded → under-representation of galaxies in crowded fields, which are high-density regions and have typically lower redshifts → biased n(z).

iii) **Bias in the shear power spectrum from baryonic effects**:

- **on small scales** 1 < k < 10 h/Mpc
gas pressure is important (baryonic matter is no longer dust) → suppression of structure formation, gas distribution is more diffuse than DM → less power in the total matter power spectrum

- **on very small scales** k > 10 Mpc
(~ R < 0.1 Mpc) there is baryonic cooling and AGN+SN feedback →



increase condensation of baryons → formation of stars and galaxies → increase of power spectrum amplitude

The shear correlation estimator can then be written with all the biases terms by including N = 4 + n_zbins **nuisance parameters**:

$$\xi_{ee}(\theta) = m\,\xi_{\gamma\gamma}(\theta) + c + A_{\rm I}\,f(\theta) + A_{\rm b}\,f(\theta, z_s) + A_{\rm phz}(z_s)$$

where the calibration parameters (m,c) account for all shear measurement biases.

Variance of the estimator

The full measurement of a cosmological quantity of interest (power spectrum, correlation function, etc) must include not only the estimate of the quantity but we also need to quantify the precision of the measurement (compute the error bars).

$\hat{\xi}$ is the estimator of the correlation function → it is the measurement.

The measurement $\hat{\xi}$ is interpreted as one possible realization of the true value of ξ.

$< \xi >$ is the true value of the correlation function → it is the theoretical computation of ξ, computed from the model (structure formation).

Even for direct measurements in the real space, $\hat{\xi}$ and $< \xi >$ are different because of noise (variance of the estimator, and also intrinsic 'cosmological noise') and bias (the estimator may have systematic errors that need to be corrected or taken into account in nuisance parameters).

The **variance of the estimator is**:

$$C_{ij} = \left\langle (\hat{\xi} - < \xi >)_i \, (\hat{\xi} - < \xi >)_j \right\rangle$$

In the case of the cosmic shear correlation functions, we can already see that, since ξ depends on <e e> → <YY> → <δδ>, its variance will depend on 4-pt functions <δδδδ> → **the full computation of the error bars of a power spectrum requires the theoretical computation of the trispectrum**. (It is the variance of a variance)

Let us consider the estimator for ξ+

$$\hat{\xi}_+(\vartheta) = \frac{\sum_{ij} w_i\, w_j\, (\epsilon_{it}\epsilon_{jt} + \epsilon_{i\times}\epsilon_{j\times})\, \Delta_\vartheta\left(|\boldsymbol{\theta}_i - \boldsymbol{\theta}_j|\right)}{N_\mathrm{p}(\vartheta)}$$

(assuming all external biases are accounted for)

ξ+ combines the two components (t,X) of the ellipticity.

The weights are needed to distinguish the quality of the measurements of different galaxies.

So the correlation for each separation ϑ is the sum of all contributions (e_i e_j) from the N_p galaxy pairs in the bin ϑ :

$$\Delta_\vartheta(\phi) = 1 \text{ for } \vartheta - \Delta\vartheta/2 < \phi \leq \vartheta + \Delta\vartheta/2,$$

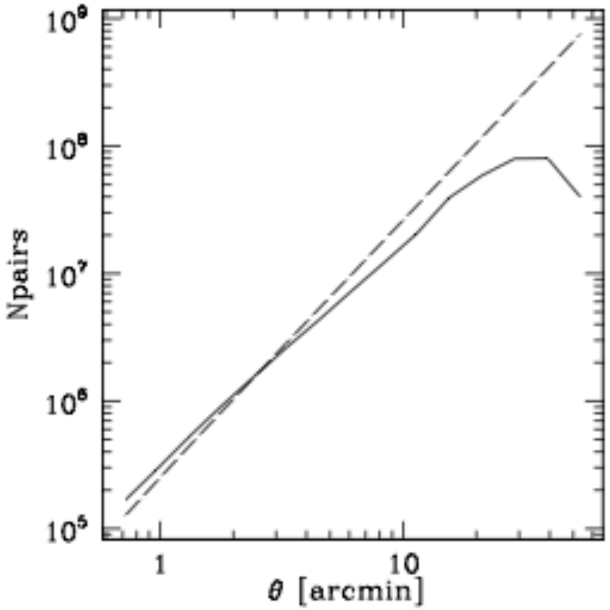
$$= 0 \text{ otherwise}$$

and is divided by the number of contributing pairs (i.e., it is an average).

The number of pairs **increases with separation** and depends on the survey area and density:

$$N_{\mathrm{p}}(\vartheta) = A\, n\, 2\pi\vartheta\, \Delta\vartheta\, n.$$

This approximate expression comes from considering that the survey is a single connected field of area A with N galaxies (density n) → the galaxies on a circular shell of radius ϑ around a central galaxy, form pairs of separation ϑ. Then, consider shells around all galaxies to get the total number of pairs for that separation.



Npairs

dashed : formula

solid: measured

The measured number of pairs on large scales is smaller due to the edge of the field-of-view.

Assuming no intrinsic alignments, this estimator is unbiased:

$$\left\langle \hat{\xi}_+(\vartheta) \right\rangle = \xi_+(\vartheta),$$

using e = $e^s + \gamma$ we can write:

$$\left\langle \epsilon_{it}\epsilon_{jt} + \epsilon_{i\times}\epsilon_{j\times} \right\rangle = \sigma_\epsilon^2 \delta_{ij} + \xi_+ \left(\left| \boldsymbol{\theta}_i - \boldsymbol{\theta}_j \right| \right)$$

(where σ_e^2 is the shot noise term, i.e., the auto-correlation term)

Now, let us **compute the variance** of the unbiased estimator:

$$\mathrm{Cov}(\hat{\xi}_\pm, \theta_1; \hat{\xi}_\pm, \theta_2) = \left\langle \left(\hat{\xi}_\pm(\theta_1) - \xi_\pm(\theta_1) \right) \left(\hat{\xi}_\pm(\theta_2) - \xi_\pm(\theta_2) \right) \right\rangle$$

To compute it, **we need to compute the cross-correlation between the correlation function at two separations**:

$$\left\langle \hat{\xi}_+(\theta_1)\hat{\xi}_+(\theta_2)\right\rangle = \frac{1}{N_p(\theta_1)N_p(\theta_2)}\sum_{ijkl} w_i w_j w_k w_l \left\langle (e_{i1}e_{j1} + e_{i2}e_{j2})(e_{k1}e_{l1} + e_{k2}e_{l2})\right\rangle$$

Notice that, since the correlation function separations are not independent (contrary to linear power spectrum scales), we have to consider all cases $e_i\, e_j$ and cannot simplify them to e_i^2

The calculation is involved because of this, and also due to the presence of the extra term of e_s, and also because the ellipticity and shear fields have two components.

Inserting $e = e^s + \gamma$, the quantities $\langle (e_i\, e_j)\, (e_k\, e_l) \rangle$ become,

$$\langle e_{i\alpha}e_{j\beta}e_{k\mu}e_{l\nu}\rangle = \frac{\sigma_e^2}{2}\left(\delta_{jl}\delta_{\beta\nu}\langle\gamma_{i\alpha}\gamma_{k\mu}\rangle + \delta_{jk}\delta_{\beta\mu}\langle\gamma_{i\alpha}\gamma_{l\nu}\rangle + \delta_{il}\delta_{\alpha\nu}\langle\gamma_{j\beta}\gamma_{k\mu}\rangle + \delta_{ik}\delta_{\alpha\mu}\langle\gamma_{j\beta}\gamma_{l\nu}\rangle\right) +$$

$$+ \langle\gamma_{i\alpha}\gamma_{j\beta}\gamma_{k\mu}\gamma_{l\nu}\rangle + \left\langle e^s_{i\alpha}e^s_{j\beta}e^s_{k\mu}e^s_{l\nu}\right\rangle$$

(greek indexes account for the 2 components 1,2)

Notice that even though none of the correlation functions $\xi(\theta_1)$ and $\xi(\theta_2)$ are computed at separation zero, their variance depends on the shot noise, because it includes terms $\theta_1 = \theta_2$ → **the covariance of a quantity that is itself a pure covariance, also depends on the variance of the covariance (and not just on the covariance of the covariance).**

(in other words, a 2-pt signal at non-zero separations is not affected by shot noise, but its covariance is).

Now, using Wick's theorem and **assuming Gaussianity** (no connected 4-pt), we can write all 4-pt quantities as products of 2-pt quantities. In this Gaussian approximation, we get:

$$\left\langle \epsilon_{i\alpha}\epsilon_{j\beta}\epsilon_{k\mu}\epsilon_{l\nu}\right\rangle = \frac{\sigma_\epsilon^2}{2}\Big(\delta_{jl}\delta_{\beta\nu}\left\langle\gamma_{i\alpha}\gamma_{k\mu}\right\rangle + \delta_{jk}\delta_{\beta\mu}\left\langle\gamma_{i\alpha}\gamma_{l\nu}\right\rangle + \delta_{il}\delta_{\alpha\nu}\left\langle\gamma_{j\beta}\gamma_{k\mu}\right\rangle + \delta_{ik}\delta_{\alpha\mu}\left\langle\gamma_{j\beta}\gamma_{l\nu}\right\rangle\Big)$$
$$+\left\langle\gamma_{i\alpha}\gamma_{j\beta}\right\rangle\left\langle\gamma_{k\mu}\gamma_{l\nu}\right\rangle + \left\langle\gamma_{i\alpha}\gamma_{k\mu}\right\rangle\left\langle\gamma_{j\beta}\gamma_{l\nu}\right\rangle + \left\langle\gamma_{i\alpha}\gamma_{l\nu}\right\rangle\left\langle\gamma_{j\beta}\gamma_{k\mu}\right\rangle$$
$$+\left(\frac{\sigma_\epsilon^2}{2}\right)^2\left(\delta_{ik}\delta_{jl}\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{il}\delta_{jk}\delta_{\alpha\nu}\delta_{\beta\mu}\right)$$

Inserting this in the variance of the estimator, we obtain 3 different contributions for the **error budget**.

The third term is diagonal, it only affects the diagonal of the covariance matrix.

$$\mathrm{Cov}(\hat{\xi}_+, \vartheta_1; \hat{\xi}_+, \vartheta_2) = \frac{1}{N_\mathrm{p}(\vartheta_1) N_\mathrm{p}(\vartheta_2)} \left[\sigma_\epsilon^4 \bar{\delta}(\vartheta_1 - \vartheta_2) \sum_{ij} w_i^2 w_j^2 \Delta_{\vartheta_1}(ij) \right.$$

It is the shot noise contribution to the error budget.
It depends only on the intrinsic ellipticity dispersion, i.e., on the shape noise.

The second term contains only shear correlations. It is a purely cosmological term, coming from the shear 4-pt function.

$$\mathrm{Cov}(\hat{\xi}_+, \vartheta_1; \hat{\xi}_+, \vartheta_2) = \frac{1}{N_\mathrm{p}(\vartheta_1) N_\mathrm{p}(\vartheta_2)} \times$$

$$\times \sum_{ijkl} w_i w_j w_k w_l \Delta_{\vartheta_1}(ij) \Delta_{\vartheta_2}(kl) \left(\xi_+(il) \xi_+(jk) + \cos\left[4\left(\varphi_{il} - \varphi_{jk}\right)\right] \xi_-(il) \xi_-(jk) \right) \Big]$$

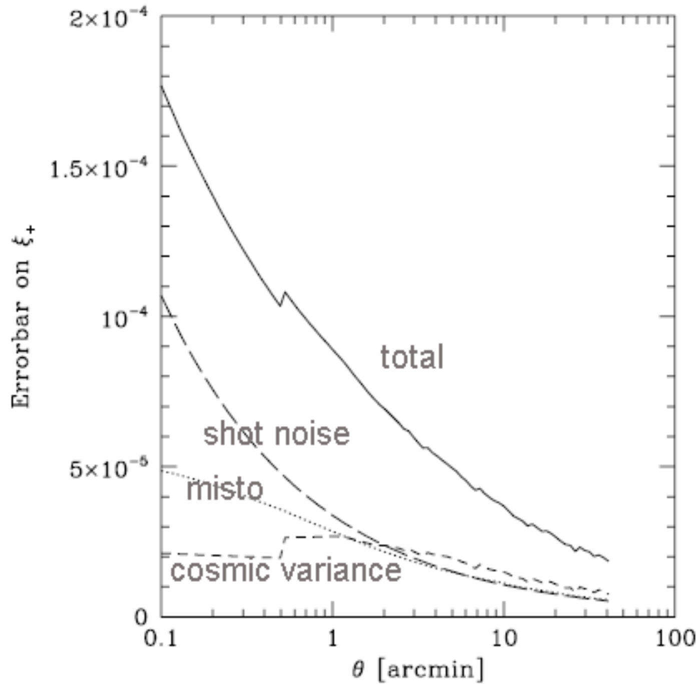
It is the only source of noise remaining in the absence of shape noise.
It is the cosmic variance contribution to the error budget.

The first term correlates shot noise with cosmic variance:

$$\mathrm{Cov}(\hat{\xi}_+, \vartheta_1; \hat{\xi}_+, \vartheta_2) = \frac{1}{N_\mathrm{p}(\vartheta_1) N_\mathrm{p}(\vartheta_2)} 2\sigma_\epsilon^2 \sum_{ijk} w_i^2 w_j w_k \Delta_{\vartheta_1}(ij) \Delta_{\vartheta_2}(ik) \xi_+(jk)$$

It is a mixed term.

The error bars are the square root of the diagonal of the covariance matrix (or noise matrix). Their relative contribution to the error budget is:



The variance is larger on small scales and:

- Shot noise dominates on small scales
- Cosmic variance dominates on large scales

Notice that the amplitude of the error bars depends essentially on the number of pairs, (divides all error terms) i.e., the uncertainty of cosmic shear surveys depends mainly on:

- Area of the survey
- Density of source galaxies

This analytical result is valid in the Gaussian fields approximation.

To compute the covariance matrix without this approximation we need to **consider the trispectrum**, or measure the **dispersion of the correlation function on the data** or on **numerical simulations** of the shear field.

The **observed shear field** follows a non-Gaussian distribution, not only due to the non-linear regime of structure formation, but also because in practice a complex survey geometry introduces couplings in the measured modes and modifies the distribution → non-Gaussian covariance matrix is really needed.

Numerical simulations of the lensing field consist on N-body simulations + Ray-tracing. They are anyway needed in cosmic shear analysis for various reasons, besides computing the non-Gaussian covariance matrix:

- To compute the theoretical non-linear power spectrum (analytical extensions of the linear theory are only valid up to k ~ 0.5h/Mpc)

- To include baryonic physics, which further modify dark-matter halo properties → hydrodynamic simulations needed.

- To model systematic effects that correlate to astrophysics or the LSS, like intrinsic alignments that may also be included in the N-body simulation.

- To test the mathematical approximations made : Born approximation, neglecting of lens-lens coupling (second-order terms in the light-propagation equation), replacement of reduced shear by shear.

Numerical shear maps are produced by ray-tracing through N-body output snapshot boxes: light-rays are sent to every direction from the observer to a source at high redshift → they travel on straight lines between lens planes → N-body particles are projected onto lens planes and their surface mass density and gravitational potential computed → the induced deflection angle α is computed → the ray changes direction → this is repeated until reaching a source galaxy.

From multiple rays, the shear at each observing direction of the image is obtained.