



Ciências
ULisboa

Modelação Numérica

Aula 15

Equação de advecção-difusão. Condições fronteira cíclicas, de von Neumann e de Dirichlet. Condução de calor num sólido.

$$\frac{\partial T}{\partial t} = -u \frac{\partial T}{\partial x} + K_D \frac{\partial^2 T}{\partial x^2}$$

Adveção-difusão, implícita: $M\vec{T}^{n+1} = \vec{b}^n$

$$\begin{bmatrix} \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(\frac{u\Delta t}{2\Delta x} - \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \dots & \left(-\frac{u\Delta t}{2\Delta x} - \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) \\ \left(-\frac{u\Delta t}{2\Delta x} - \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(\frac{u\Delta t}{2\Delta x} - \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{u\Delta t}{2\Delta x} - \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \dots & \left(-\frac{u\Delta t}{2\Delta x} - \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right) \end{bmatrix} \begin{bmatrix} T_0^{n+1} \\ T_1^{n+1} \\ \vdots \\ T_{N_x-1}^{n+1} \end{bmatrix} = \vec{b}^n$$

fronteira
cíclica

$$b_k = T_k^n - \Delta t u(1 - \alpha) \left(\frac{T_{k+1}^n - T_{k-1}^n}{2\Delta x} \right) + \Delta t K_D(1 - \alpha) \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta x^2}$$

$u = 0$ equação da **difusão/condução**: fronteira cíclica

$$\begin{bmatrix}
 \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \dots & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) \\
 \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \\
 \vdots & & \ddots & \\
 \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right)
 \end{bmatrix}
 \begin{bmatrix}
 T_0^{n+1} \\
 T_1^{n+1} \\
 \vdots \\
 T_{N_x-1}^{n+1}
 \end{bmatrix}
 = \vec{b}$$

$$b_0 = T_0^n + \Delta t K_D (1 - \alpha) \frac{T_{N_x-1}^n + T_1^n - 2T_0^n}{\Delta x^2}$$

$$b_{N_x-1} = T_{N_x-1}^n + \Delta t K_D (1 - \alpha) \frac{T_{N_x-2}^n + T_0^n - 2T_{N_x-1}^n}{\Delta x^2}$$

$$b_{k(k=1\dots N_x-2)} = T_k^n + \Delta t K_D (1 - \alpha) \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta x^2}$$

Condição de von Neumann

$u = 0$ equação da difusão/condução: fronteira $\left(\frac{\partial T}{\partial x}\right)_{0, N_x-1} = 0$

$$\begin{bmatrix}
 \left(1 + \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \dots & 0 \\
 \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \\
 \vdots & \vdots & \vdots & \\
 0 & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{\alpha\lambda\Delta t}{\Delta x^2}\right) &
 \end{bmatrix}
 \begin{bmatrix}
 T_0^{n+1} \\
 T_1^{n+1} \\
 \vdots \\
 T_{N_x-1}^{n+1}
 \end{bmatrix}
 = \vec{b}$$

$$\begin{aligned}
 b_0 &= T_0^n \\
 &+ \Delta t K_D (1 - \alpha) \frac{T_0^n + T_{0+1}^n - 2T_0^n}{\Delta x^2} \\
 b_k &= T_k^n + \Delta t K_D (1 - \alpha) \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta x^2} \\
 b_{N_x-1} &= T_{N_x-1}^n \\
 &+ \Delta t K_D (1 - \alpha) \frac{T_{N_x-2}^n + T_{N_x-1}^n - 2T_{N_x-1}^n}{\Delta x^2}
 \end{aligned}$$

Com o mesmo processo poderia impor-se qualquer condição na fronteira para a derivada, i.e. $\left(\frac{\partial T}{\partial x}\right)_{0, N_x-1} = F_{0, N_x-1}$

Condição de Dirichlet

$u = 0$ equação da difusão/condução: fronteira $T_0 = T_b(t)$

$$\begin{bmatrix}
 1 & 0 & \dots & 0 \\
 \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \\
 \vdots & & \ddots & \vdots \\
 0 & & \left(-\frac{\alpha\lambda\Delta t}{\Delta x^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta x^2}\right)
 \end{bmatrix}
 \begin{bmatrix}
 T_0^{n+1} \\
 T_1^{n+1} \\
 \\ \\
 T_{N_x-1}^{n+1}
 \end{bmatrix}
 = \vec{b}$$

$b_0 = T_b^{n+1}$
 $b_k = T_k^n + \Delta t K_D (1 - \alpha) \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta x^2}$

Problema da condução de calor no solo

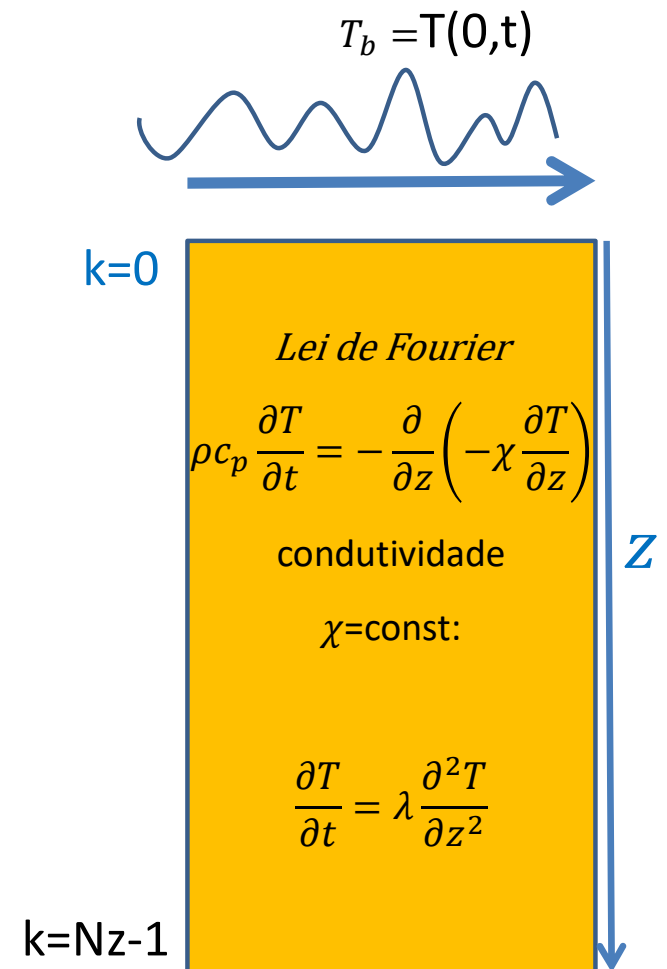
Em $k = 0$: condição Dirichlet (T forçada)

$$T_k^{n+1} = T_0((n + 1) \times \Delta t)$$

Em $k = N_z - 1$: condição de von Neumann (gradiente nulo)

$$\frac{\partial T}{\partial z} = 0 \Leftrightarrow T_{Nz} = T_{Nz}$$

Como $-\chi \nabla T$ é o fluxo de calor, em $k = N_z - 1$ o fluxo de calor é nulo.



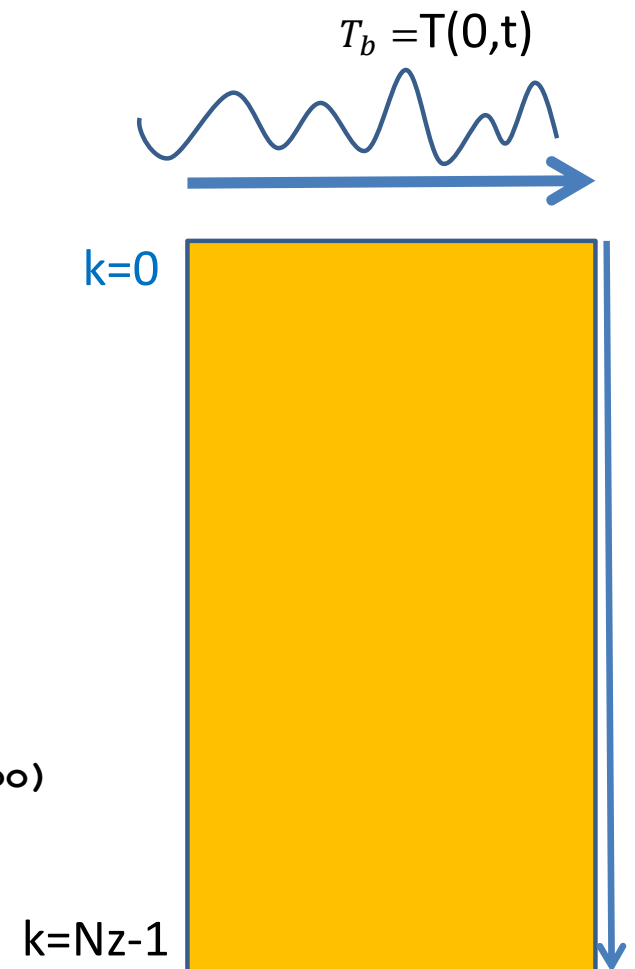
Condução de calor

$$\begin{bmatrix}
 1 & 0 & \dots & 0 \\
 \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \\
 \vdots & & \ddots & \\
 0 & & \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(1 + \frac{\alpha\lambda\Delta t}{\Delta z^2}\right)
 \end{bmatrix}
 \begin{bmatrix}
 T_0^{n+1} \\
 T_1^{n+1} \\
 \vdots \\
 T_{z-1}^{n+1}
 \end{bmatrix}
 = \vec{b}
 \quad
 \begin{aligned}
 b_0 &= T_b^{n+1} \\
 b_k &= T_k^n + \Delta t \lambda (1 - \alpha) \frac{T_{k-1}^n + T_{k+1}^n - 2T_k^n}{\Delta z^2} \\
 b_{N_z-1} &= T_{N_z-1}^n + \Delta t \lambda (1 - \alpha) \frac{T_{N_z-2}^n + T_{N_z-1}^n - 2T_{N_z-1}^n}{\Delta z^2}
 \end{aligned}$$

```

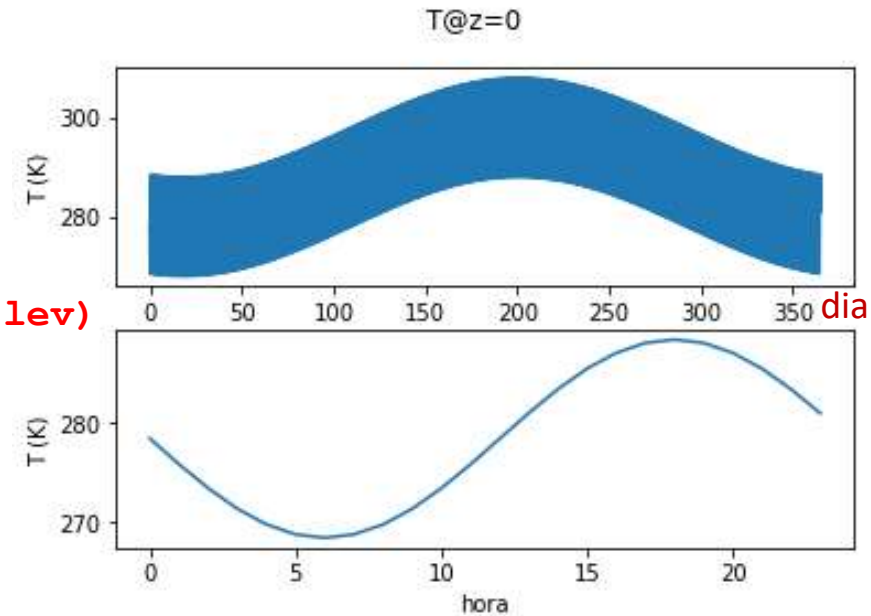
import numpy as np
import matplotlib.pyplot as plt
import time
from scipy.sparse import lil_matrix
from scipy.sparse.linalg import spsolve
alpha=0.5 #Crank-Nicholson
Nz=500;Lz=5.;dz=Lz/Nz; #dz=1cm
z=np.arange(dz,Lz,dz)
TimeSpan=365*24*3600. #1 ano
dt=3600.;tempo=np.arange(0.,TimeSpan,dt);nt=len(tempo)
ddia=24*3600;dano=365*ddia
lam=0.25/1600/890 #difusividade térmica do solo

```



Condição fronteira superior (z=0)

```
Tmed=288;AmpD=10;AmpA=10
lev=np.array([0,2,4,9,19,29,39,59],int);nlev=len(lev)
Tz=np.ones((nt,nlev))*Tmed #condição inicial
T0=Tmed+AmpD*np.sin(2*np.pi*tempo/ddia+np.pi)\
    +AmpA*np.sin(2*np.pi*tempo/dano-np.pi*3./5);
plt.subplot(2,1,1);plt.plot(tempo/3600/24,T0)
plt.ylabel('T (K)');plt.xlabel('dia juliano')
plt.subplot(2,1,2);plt.plot(tempo[:24]/3600,T0[:24])
plt.ylabel('T (K)');plt.xlabel('hora')
plt.suptitle('T@z=0')
```



M

```
plt.figure()
Tmin=np.min(T0);Tmax=np.max(T0);
zmin=-np.max(z);zmax=0;
T=Tmed*np.ones((Nz)) #perfil inicial de T
beta=alpha*lam*dt/dz**2
zeta=(1-alpha)*lam*dt/dz**2
M=lil_matrix((Nz,Nz)) #M=np.zeros((Nz,Nz))
b=np.zeros(Nz)
M[0,0]=1
for k in range(1,Nz-1):
    M[k,k-1]=-beta
    M[k,k]=1+2*beta
    M[k,k+1]=-beta
M[Nz-1,Nz-2]=-beta
M[Nz-1,Nz-1]=1+beta
M=M.tocsr()
timestart=time.process_time()
```

Matriz tridiagonal

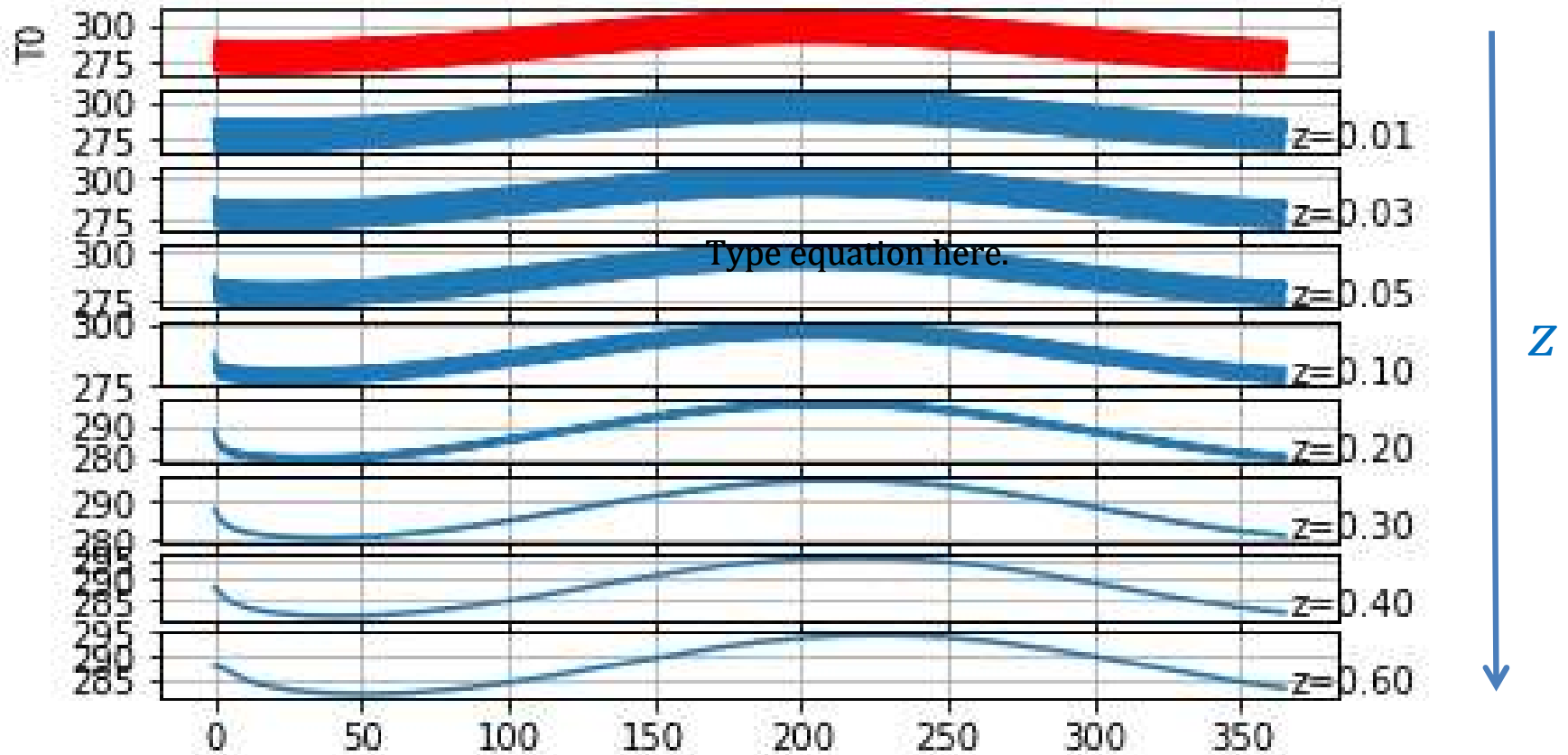
$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(1 + \frac{2\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \\ \vdots & & \ddots & \vdots \\ 0 & & \left(-\frac{\alpha\lambda\Delta t}{\Delta z^2}\right) & \left(1 + \frac{\alpha\lambda\Delta t}{\Delta z^2}\right) \end{bmatrix}$$

```

for it in range(1,nt):
    b[0]=T0[it]
    for iz in range(1,Nz-1):
        b[iz]=T[iz]+zeta*(T[iz-1]-2*T[iz]+T[iz+1])
    b[Nz-1]=T[Nz-1]+zeta*(T[Nz-2]-T[Nz-1])
    T=spsolve(M,b)
    for klev in range(nlev):
        Tz[it,klev]=T[lev[klev]]
timespent=time.process_time()-timestart
plt.subplot(nlev+2,1,1)
tempoh=tempo/3600/24;plt.plot(tempoh,T0,color='red')
plt.grid();plt.ylabel('T0')
for klev in range(nlev):
    ax=plt.subplot(nlev+2,1,klev+2)
    plt.plot(tempoh,Tz[:,klev]); plt.grid()
    ax2=ax.twinx(); ax2.set_yticks([])
    ax2.set_ylabel('z=%3.2f' % (z[lev[klev]]),rotation=0)
plt.suptitle(r'$\partial T / \partial t = \lambda \nabla^2 T, \backslash$
            lambda=%4.2e, ts=%3.1f$ s' %(lam,timespent)

```

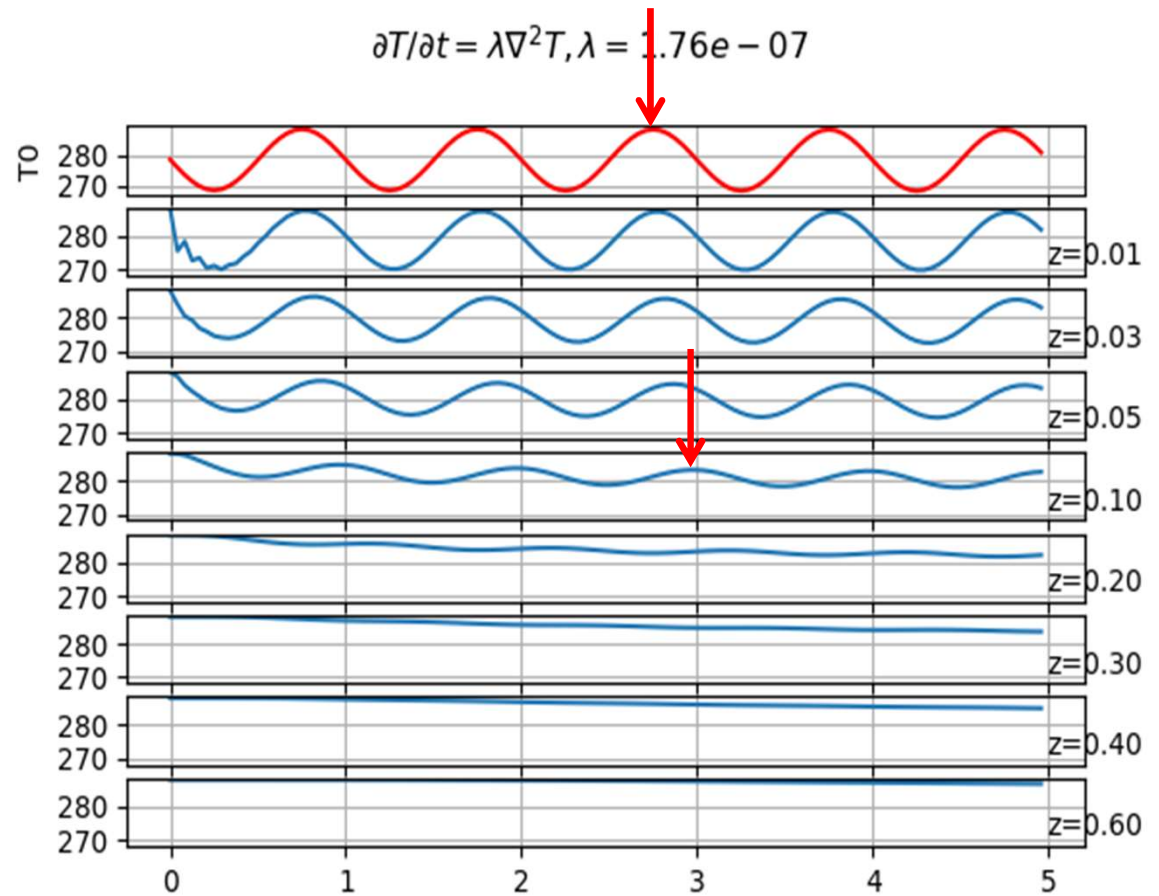
$$\partial T / \partial t = \lambda \nabla^2 T, \text{ lambda} = 1.76e - 07, \text{ ts} = 6.3 \text{ s}$$



À superfície (z=1cm) o solo acompanha o ar.

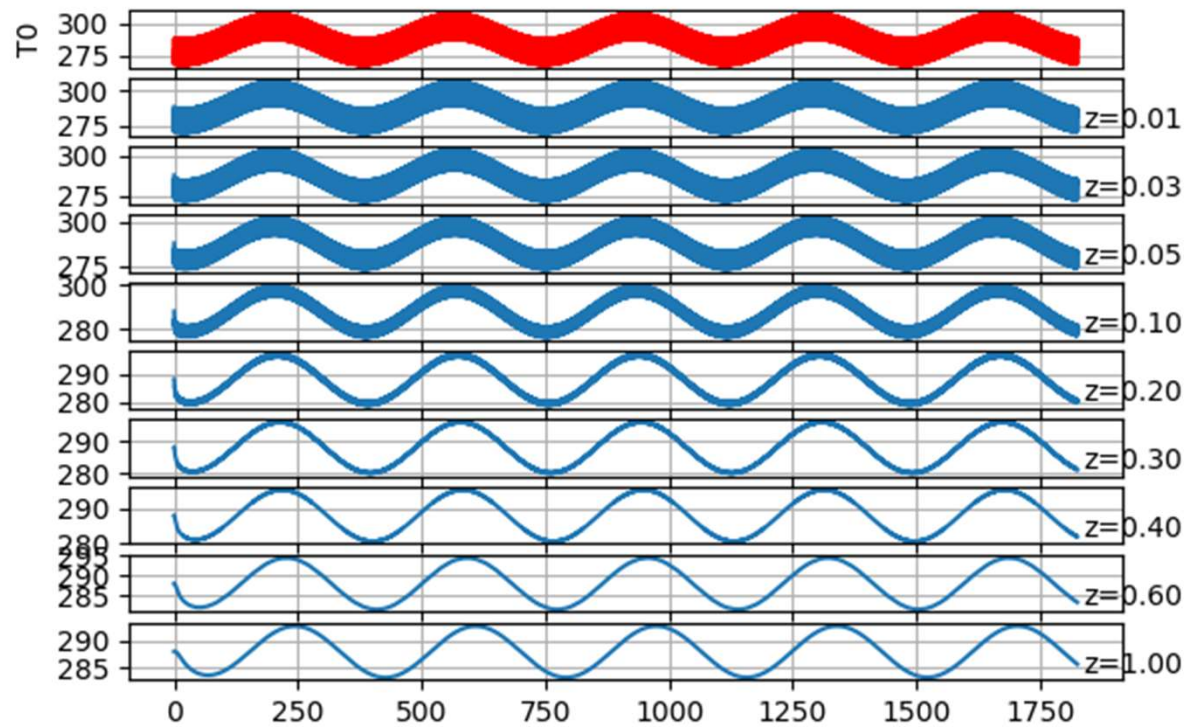
Em profundidade segue a evolução da superfície com atenuação e atraso (**desfasamento**).

Nesta fase arrefece lentamente em profundidade pois o estado inicial foi T=288, e a superfície está mais fria (inverno)



Integração a 5 anos

$$\partial T / \partial t = \lambda \nabla^2 T, \lambda = 1.76e - 07$$



O ciclo diurno é completamente atenuado nas primeiras dezenas de cm.

Abaixo só há ciclo anual com **desfasamento progressivo**.

O comportamento depende é claro da **difusividade térmica**, i.e., das propriedades físicas do meio.

Em cada nível a temperatura oscila em torno da sua temperatura média (igual em todos os níveis)

