AULA 14

SUMÁRIO. Fórmula canónica de Jordan: matrizes nilpotentes.

▷ Um BLOCO DE JORDAN $\mathbf{J}_k(\lambda) \in \mathbb{C}^{k\times k}$ é uma matriz da forma

$$\mathbf{J}_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

onde o escalar $\lambda \in \mathbb{C}$ ocorre k vezes na diagonal, 1 ocorre $k-1$ vezes na "sobre-diagonal" e todas as restantes entradas são iguais a 0. Em particular, temos

$$\mathbf{J}_1(\lambda) = \begin{bmatrix} \lambda \end{bmatrix}, \quad \mathbf{J}_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{e} \quad \mathbf{J}_3(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

No caso em que $\lambda = 0$, dizemos que $\mathbf{J}_k(0)$ é um BLOCO DE JORDAN NILPOTENTE; nesta situação, temos $\mathbf{J}_k(0)^k = 0$.

Uma MATRIZ DE JORDAN $\mathbf{J} \in \mathbb{C}^{n\times n}$ é uma matriz diagonal por blocos

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{n_1}(\lambda_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{n_2}(\lambda_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{n_r}(\lambda_r) \end{bmatrix}$$

onde $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ e $n_1, \ldots, n_r \in \mathbb{N}$ são tais que $n_1 + n_2 + \cdots + n_r = n$.

PROPOSIÇÃO 14.1. *Seja $k \in \mathbb{N}$, $k \geqslant 2$ e seja $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ a base canónica de $\mathbb{C}^{k\times k}$. Então,*

(a) $\mathbf{J}_k(0)\mathbf{e}_1 = \mathbf{0}$ *e* $\mathbf{J}_k(0)\mathbf{e}_{i+1} = \mathbf{e}_i$ *para qualquer* $1 \leqslant i \leqslant k-1$.

(b) $\mathbf{J}_k(0)^t = \mathbf{0}$ *para qualquer* $t \in \mathbb{N}$, $t \geqslant k$.

(c) $\mathbf{J}_k(0)^{\mathrm{T}}\mathbf{J}_k(0) = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{k-1} \end{bmatrix}$.

(d) $\big(\mathbf{I}_k - \mathbf{J}_k(0)^{\mathrm{T}}\mathbf{J}_k(0)\big)\mathbf{v} = (\mathbf{v}^{\mathrm{T}}\mathbf{e}_1)\mathbf{e}_1$ *para qualquer* $\mathbf{v} \in \mathbb{C}^{k\times 1}$.

Demonstração. (a) Por definição, temos $\mathbf{J}_k(0) = \begin{bmatrix}\mathbf{0} & \mathbf{e}_1 & \cdots & \mathbf{e}_{k-1}\end{bmatrix}$, logo $\mathbf{J}_k(0)\mathbf{e}_{i+1} = \mathbf{e}_i$ para qualquer $1 \leqslant i \leqslant k-1$ (porque $\mathbf{J}_k(0)\mathbf{e}_1, \ldots, \mathbf{J}_k(0)\mathbf{e}_k$ são as colunas de $\mathbf{J}_k(\lambda)$).

(b) Temos

$$\mathbf{J}_k(0)^i\mathbf{e}_i = \mathbf{J}_k(0)^{i-1}(\mathbf{J}_k(0)\mathbf{e}_i) = \mathbf{J}_k(0)^{i-1}\mathbf{e}_{i-1}, \qquad 1 < i \leqslant n,$$

de onde resulta (por indução) que $\mathbf{J}_k(0)^i e_i = 0$ para qualquer $1 \leqslant i \leqslant n$. Em particular, obtemos $\mathbf{J}_k(0)^k = \mathbf{0}$ e, portanto, $\mathbf{J}_k(0)^t = \mathbf{0}$ para todo $t \in \mathbb{N}$, $t \geqslant k$.

(c) Temos

$$\mathbf{J}_k(0)^{\mathsf{T}}\mathbf{J}_k(0) = \begin{bmatrix}\mathbf{0}\\ \mathbf{e}_1^{\mathsf{T}}\\ \vdots \\ \mathbf{e}_{k-1}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{e}_1^{\mathsf{T}} & \cdots & \mathbf{e}_{k-1}\end{bmatrix} = \begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & \mathbf{e}_1^{\mathsf{T}}\mathbf{e}_1 & \cdots & \mathbf{e}_1^{\mathsf{T}}\mathbf{e}_{k-1}\\ \vdots & \vdots & & \vdots\\ 0 & \mathbf{e}_{k-1}^{\mathsf{T}}\mathbf{e}_1 & \cdots & \mathbf{e}_{k-1}^{\mathsf{T}}\mathbf{e}_{k-1}\end{bmatrix} = \begin{bmatrix}0 & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{k-1}\end{bmatrix}$$

(d) Pela alínea anterior,

$$\big(\mathbf{I}_k - \mathbf{J}_k(0)^{\mathsf{T}}\mathbf{J}_k(0)\big)\mathbf{v} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix}\mathbf{v} = v_1\mathbf{e}_1 = (\mathbf{v}^{\mathsf{T}}\mathbf{e}_1)\mathbf{e}_1$$

onde $\mathbf{v} = \begin{bmatrix}v_1 & \cdots & v_k\end{bmatrix}^{\mathsf{T}}$. □

Teorema 14.2. *Seja $\mathbf{T} \in \mathbb{C}^{n\times n}$ uma matriz estritamente triangular superior (isto é, uma matriz triangular superior com diagonal nula). Então, existe uma matriz invertível $\mathbf{P} \in \mathbb{C}^{n\times n}$ tal que*

$$\mathbf{P}^{-1}\mathbf{T}\mathbf{P} = \begin{bmatrix}\mathbf{J}_{n_1}(0) & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_{n_2}(0) & \cdots & \mathbf{0}\\ \vdots & \vdots & & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{n_r}(0)\end{bmatrix}$$

para alguns $n_1, \ldots, n_r \in \mathbb{N}$ tais que $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ e $n_1 + n_2 + \cdots + n_r = n^{(*)}$.

Demonstração. Procedemos por indução sobre n. Se $n = 1$, então $\mathbf{A} = \begin{bmatrix}0\end{bmatrix}$ e o resultado é trivial. Assim, suponhamos que $n \geqslant 2$ e que o resultado é verdadeiro para qualquer $n' \in \mathbb{N}$ com $n' < n$ e qualquer matriz $\mathcal{T}_0 \in \mathbb{C}^{n'\times n'}$ que seja estritamente triangular superior. Ora,

$$\mathbf{T} = \begin{bmatrix}0 & \mathbf{u}^{\mathsf{T}}\\ \mathbf{0} & \mathbf{T}_0\end{bmatrix}, \qquad \mathbf{u} \in \mathbb{C}^{(n-1)\times 1}, \quad \mathbf{T}_0 \in \mathbb{C}^{(n-1)\times(n-1)}.$$

Por hipótese de indução, existe uma matriz invertível $\mathbf{P}_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ tal que

$$\mathbf{P}_0^{-1}\mathbf{T}_0\mathbf{P}_0 = \begin{bmatrix}\mathbf{J}_{m_1}(0) & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_{m_2}(0) & \cdots & \mathbf{0}\\ \vdots & \vdots & & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{m_s}(0)\end{bmatrix}$$

(*) Nesta situação, dizemos que $(n_1, \ldots, n_r)$ é uma partição inteira de n.

para alguns $m_1, \ldots, m_s \in \mathbb{N}$ tais que $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$ e $m_1 + m_2 + \cdots + m_s = n - 1$. Sendo assim, temos

$$\begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0^{-1} \end{bmatrix} \mathbf{T} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0^{-1} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{u}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{T}_0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{u}^{\mathsf{T}}\mathbf{P}_0 \\ \mathbf{0} & \mathbf{P}_0^{-1}\mathbf{T}_0\mathbf{P} \end{bmatrix}.$$

Pondo $\mathbf{u}^{\mathsf{T}}\mathbf{P}_0 = \begin{bmatrix} \mathbf{v}^{\mathsf{T}} & \mathbf{w}^{\mathsf{T}} \end{bmatrix}$ onde $\mathbf{v} \in \mathbb{C}^{m_1 \times 1}$ e $\mathbf{w} \in \mathbb{C}^{(n-m_1-1)\times 1}$, obtemos

$$\begin{aligned} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0^{-1} \end{bmatrix} \mathbf{T} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0 \end{bmatrix} &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0^{-1} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{u}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{T}_0 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \mathbf{u}^{\mathsf{T}}\mathbf{P}_0 \\ \mathbf{0} & \mathbf{P}_0^{-1}\mathbf{T}_0\mathbf{P} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{v}^{\mathsf{T}} & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} \end{aligned}$$

onde $\mathbf{J} = \begin{bmatrix} \mathbf{J}_{m_2}(0) & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{J}_{m_s}(0) \end{bmatrix} \in \mathbb{C}^{(n-m_1-1)\times(n-m_1-1)}$.

Agora, consideremos a matriz $\mathbf{Q} = \begin{bmatrix} 1 & \mathbf{v}^{\mathsf{T}}\mathbf{J}_{m_1}^{\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_1} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix} \in \mathbb{C}^{n\times n}$. É claro que $\mathbf{Q}$ é invertível com $\mathbf{Q}^{-1} = \begin{bmatrix} 1 & -\mathbf{v}^{\mathsf{T}}\mathbf{J}_{m_1}^{\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_1} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix}$; além disso, temos

$$\mathbf{Q}^{-1} \begin{bmatrix} 0 & \mathbf{v}^{\mathsf{T}} & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} \mathbf{Q} = \begin{bmatrix} 0 & \mathbf{v}^{\mathsf{T}}\big(\mathbf{I}_{m_1} - \mathbf{J}_{m_1}(0)^{\mathsf{T}}\mathbf{J}_{m_1}(0)\big) & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} = \begin{bmatrix} 0 & v_1\mathbf{e}_1^{\mathsf{T}} & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix}$$

onde $\{\mathbf{e}_1, \ldots, \mathbf{e}_{m_1}\}$ é a base canónica de $\mathbb{C}^{m_1\times 1}$ e $\mathbf{v} = \begin{bmatrix} v_1 & \cdots & v_{m_1} \end{bmatrix}^{\mathsf{T}}$; notemos que $v_1 = \mathbf{v}^{\mathsf{T}}\mathbf{e}_1$ e que $\big(\mathbf{I}_{m_1} - \mathbf{J}_{m_1}(0)^{\mathsf{T}}\mathbf{J}_{m_1}(0)\big)\mathbf{v} = v_1\mathbf{e}_1$ (pela proposição anterior). Neste ponto, temos duas situações a considerar: ou $v_1 \neq \mathbf{0}$, ou $v_1 = 0$.

Em primeiro lugar, suponhamos que $v_1 \neq \mathbf{0}$. Então,

$$\begin{bmatrix} v_1 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{I}_{m_1} & 0 \\ \mathbf{0} & \mathbf{0} & v_1\mathbf{I}_{n-m_1-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 & v_1\mathbf{e}_1^{\mathsf{T}} & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} v_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_1} & 0 \\ \mathbf{0} & \mathbf{0} & v_1\mathbf{I}_{n-m_1-1} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{e}_1^{\mathsf{T}} & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix}.$$

Ora, temos $\begin{bmatrix} 0 & \mathbf{e}_1^{\mathsf T} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) \end{bmatrix} = \mathbf{J}_{m_1+1}(0)$ e, portanto,

$$\begin{bmatrix} 0 & \mathbf{e}_1^{\mathsf T} & \mathbf{w}^{\mathsf T} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{e}_1\mathbf{w}^{\mathsf T} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}.$$

Como $\mathbf{J}_{m_1+1}(0)e_{i+1} = e_i$ para $1 \leqslant i \leqslant m_1$, concluímos que

$$\begin{bmatrix} \mathbf{I}_{m_1+1} & \mathbf{e}_2\mathbf{w}^{\mathsf T} \\ \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix} \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{e}_1\mathbf{w}^{\mathsf T} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m_1+1} & \mathbf{e}_2\mathbf{w}^{\mathsf T} \\ \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{e}_2\mathbf{w}^{\mathsf T}\mathbf{J} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}.$$

Mais geralmente, para qualquer $1 \leqslant i \leqslant m_1$, obtemos

$$\begin{bmatrix} \mathbf{I}_{m_1+1} & \mathbf{e}_{i+1}\mathbf{w}^{\mathsf T}\mathbf{J}^{i-1} \\ \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix} \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{e}_i\mathbf{w}^{\mathsf T}\mathbf{J}^{i-1} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m_1+1} & \mathbf{e}_{i+1}\mathbf{w}^{\mathsf T}\mathbf{J}^{i-1} \\ \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{e}_{i+1}\mathbf{w}^{\mathsf T}\mathbf{J}^{i} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}.$$

Em particular, para $i = m_1$, vem

$$\begin{aligned} &\begin{bmatrix} \mathbf{I}_{m_1+1} & \mathbf{e}_{m_1}\mathbf{w}^{\mathsf T}\mathbf{J}^{m_1-1} \\ \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix} \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{e}_{m_1}\mathbf{w}^{\mathsf T}\mathbf{J}^{m_1-1} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m_1+1} & \mathbf{e}_{m_1}\mathbf{w}^{\mathsf T}\mathbf{J}^{m_1-1} \\ \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{e}_{m_1+1}\mathbf{w}^{\mathsf T}\mathbf{J}^{m_1} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}, \end{aligned}$$

uma vez que $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$, logo

$$\mathbf{J}^{m_1} = \begin{bmatrix} \mathbf{J}_{m_2}(0)^{m_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m_3}(0)^{m_1} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{m_s}(0)^{m_1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

Em conclusão, no caso em que $v_1 \neq 0$, provámos que existe uma matriz invertível $\mathbf{P} \in \mathbb{C}^{n\times n}$ tal que

$$\mathbf{P}^{-1}\mathbf{T}\mathbf{P} = \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{m_1+1}(0) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m_2}(0) & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{m_s}(0) \end{bmatrix},$$

como se queria.

Para concluir a demonstração, supomos que $\mathbf{v}_1 = 0$, de modo que

$$\mathbf{Q}^{-1} \begin{bmatrix} 0 & \mathbf{v}^{\mathsf T} & \mathbf{w}^{\mathsf T} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} \mathbf{Q} = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{w}^{\mathsf T} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix}.$$

Ora, temos

$$\begin{bmatrix} \mathbf{0} & \mathbf{I}_{m_1} & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix} \begin{bmatrix} 0 & \mathbf{0} & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J}_{m_1}(0) & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{m_1} & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-m_1-1} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{J}_{m_1}(0) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{0} & \mathbf{J} \end{bmatrix}.$$

Voltando a usar a hipótese de indução, existe uma matriz invertível $Q_0 \in \mathbb{C}^{(n-m_1)\times(n-m_1)}$ tal que

$$\mathbf{Q}_0^{-1} \begin{bmatrix} 0 & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \mathbf{Q}_0 = \begin{bmatrix} \mathbf{J}_{m_2'}(0) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m_3'}(0) & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{m_t'}(0) \end{bmatrix}$$

onde $m_2', \ldots, m_t' \in \mathbb{N}$ tais que $m_2' \geqslant \cdots \geqslant m_t'$ e $m_2' + \cdots + m_t' = n - m_1$. Sendo assim,

$$\begin{bmatrix} \mathbf{I}_{m_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \mathbf{w}^{\mathsf{T}} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{m_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{m_1}(0) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{m_2'}(0) & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{m_t'}(0) \end{bmatrix}.$$

Para obter a forma desejada, basta fazer uma permutação das linhas (por blocos) e a permutação inversa das colunas (o que corresponde a multiplicar à direita por uma *matriz de permutação* e à esquerda e pela matriz inversa). □