

# Universo Primitivo

## 2024-2025 (1º Semestre)

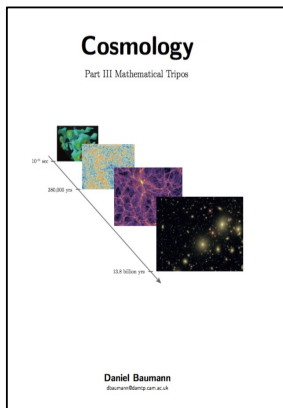
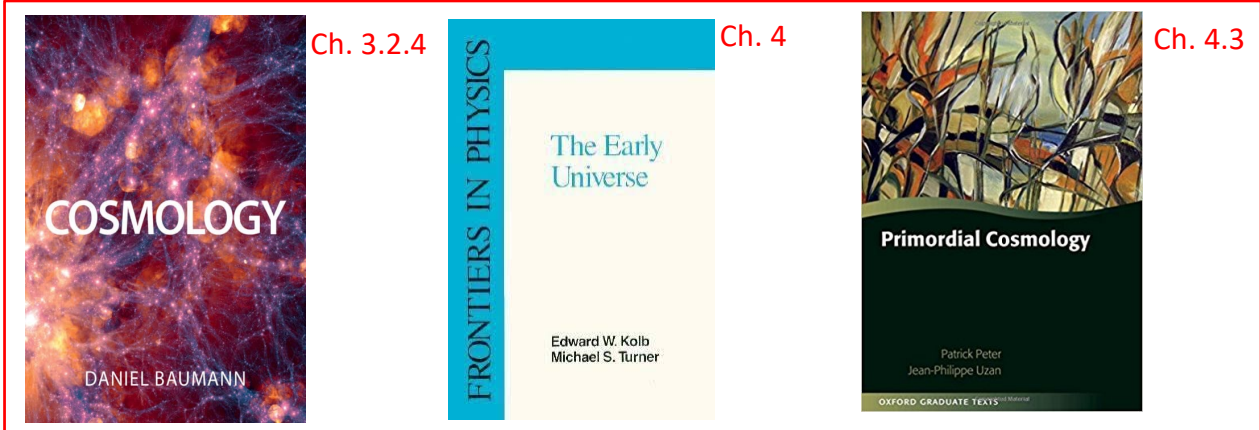
Mestrado em Física - Astronomia

### Chapter 6

#### 6 Big Bang Nucleosynthesis

- Initial Conditions;
- Nuclear statistical equilibrium;
- Neutron abundance;
- Helium abundance ;
- Comparison with observations
- BBN as a probe of cosmology and fundamental physics

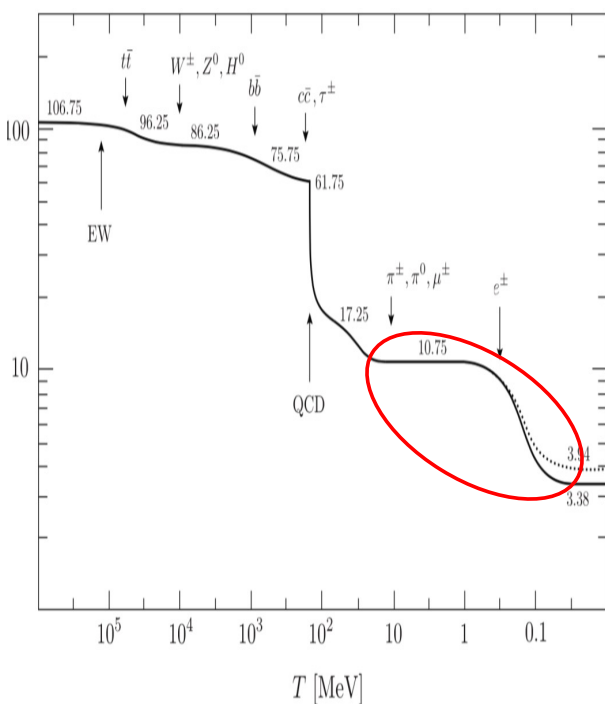
# References



Ch. 3.3

# Big-Bang Nucleosynthesis

## Initial conditions



After QCD phase transition and  $T \gtrsim 1$  MeV protons and neutrons...

- remain in equilibrium with the fluid due to weak interactions involving neutrinos.
- First atomic nuclei may form in equilibrium via 2-body nuclear reactions between protons + neutrons
- The proton to neutron ratio is  $n/p = n^{eq}/p^{eq}$

By  $T \sim 1 - 0.7$  MeV,

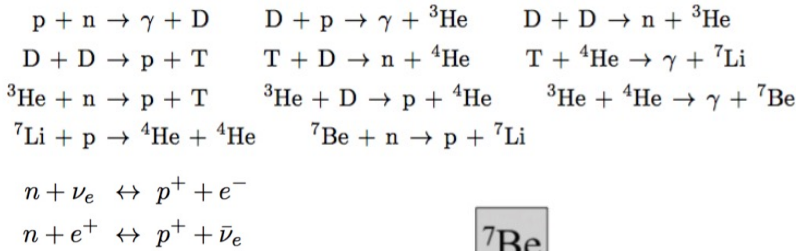
- weak interactions can no longer keep protons and neutrons in equilibrium
- Free neutrons tend to decay into protons by  $T/\text{MeV} \sim 0.8$ , while atomic nuclei remain in equilibrium (due to the electromagnetic interaction)
- Neutrinos also decouple;  $n/p$  starts to deviate from the equilibrium value,  $n/p \neq n^{eq}/p^{eq}$

By  $T \sim 0.7 - 0.5$  MeV,

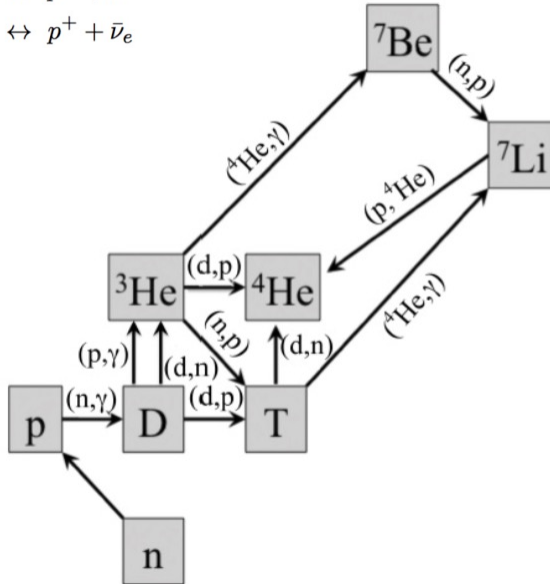
- The production of deuterium,  $n + p \rightarrow D + \gamma$ , ceases when the number of neutrons decrease.
- Light atomic nuclei are then consistently formed by 2-body reaction involving deuterium nuclei (3 body reactions are very unlikely).

# Big-Bang Nucleosynthesis

## Initial conditions



**Main nuclear reactions** that can be established during this phase



**Big-Bang Nucleosynthesis (BBN)** is able to predict the observed abundances of light elements!

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# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

Let us assume a given atomic nucleus  $A = n + p$  nucleons ( $A$  is the **nuclear mass number**,  $n$  is the number of neutrons, and  $p$  is the number of protons of the nucleus. The nuclear charge is given by the **atomic number**  $Z = p$ ).

The number density of this **nuclear species** at equilibrium is given by the non-relativistic expression derived in Series 2 (exercise 5.1):

$$n_A = g_A \left( \frac{m_A T}{2\pi} \right)^{3/2} \exp \left( \frac{\mu_A - m_A}{T} \right)$$

where the chemical potential needs to account for the number of protons and neutrons that make up the nucleus

$$\mu_A = Z\mu_p + (A - Z)\mu_n$$

We can also write similar equations to the free (non-relativistic) protons and neutrons, noticing that both these particles have 2 degrees of freedom (spin).

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# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

i.e., neutrons and protons have also non-relativistic equilibrium densities given by the previous expression ( $A = 1, g_A = 2$ ):

$$n_n = 2 \left( \frac{m_n T}{2\pi} \right)^{3/2} e^{-(m_n - \mu_n)/T};$$

$$n_p = 2 \left( \frac{m_p T}{2\pi} \right)^{3/2} e^{-(m_p - \mu_p)/T}.$$

The **nuclear binding energy**,  $B_A$ , of a nucleus with atomic mass,  $A$ , is defined as the difference between the total mass of free nucleons and the mass of the nucleus:

$$B_A = Zm_p + (A - Z)m_n - m_A$$

$$m_A = Zm_p + (A - Z)m_n - B_A.$$

Using these expressions in  $n_A$  (the previous slide) and approximating  $m_A = Am_B$  inside the (), with  $m_p \simeq m_n \simeq m_B$ , one obtains (Ex. Sheet 4's exercise):

$$n_A = \frac{g_A}{2^A} A^{3/2} \left( \frac{m_B T}{2\pi} \right)^{3(1-A)/2} n_p^Z n_n^{A-Z} e^{B_A/T}$$

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# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

This shows that **abundance of a nuclear species**, critically depends on :

- the abundance of protons and neutrons at a given  $T$  ;
- the binding energy to temperature ratio,  $B_A/T$

It is useful to write the nuclear abundances in terms of **mass fraction abundances**,  $X_A$ , defined as:

$$X_A \equiv \frac{n_A A}{n_B} \quad \text{where} \quad n_B = n_p + n_n + \sum_A A n_A$$

This definition allows on to write the following conservation equation of nuclear abundances

$$\sum_A X_A = 1$$

Using  $n_A = X_A n_B / A$  in the expression of  $n_A$  in the previous slide, one has:

$$X_A = \frac{g_A}{2^A} A^{5/2} \left( \frac{m_B T}{2\pi} \right)^{3(1-A)/2} \frac{n_p^Z n_n^{A-Z}}{n_B} e^{B_A/T};$$

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# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

The density ratio in the previous expression can be written as:

$$\frac{n_p^Z n_n^{A-Z}}{n_B} = \frac{n_p^Z n_n^{A-Z}}{n_B^Z n_B^{A-Z}} n_B^{A-1} = X_p^Z X_n^{A-Z} n_B^{A-1} = X_p^Z X_n^{A-Z} n_\gamma^{A-1} \eta^{A-1}$$

where we write  $n_B = n_\gamma \eta$  and define the **baryon to photon ratio** as:

$$\eta \equiv n_B/n_\gamma \quad \text{where,} \quad n_\gamma = \frac{2}{\pi^2} \zeta(3) T^3$$

$\eta$  is a central quantity in BBN. It can be calculated at present ( $T_0 = 2.7525$ ):

$$\eta = 2.74 \times 10^{-8} h^2 \Omega_B$$

Using these expressions in  $X_A$  (of the previous slide) one has (check all the steps!):

$$\begin{aligned} X_A &= \frac{g_A}{2^A} A^{5/2} \left( \frac{m_B T}{2\pi} \right)^{3(1-A)/2} X_p^Z X_n^{A-Z} \left( \frac{2}{\pi^2} \zeta(3) T^3 \right)^{A-1} \eta^{A-1} e^{B_A/T} \\ &= g_A A^{5/2} 2^{-A+A-1-3(1-A)/2} \pi^{-2A+2-3(1-A)/2} \zeta(3)^{A-1} T^{(1-A)(-3+3/2)} m_B^{3(1-A)/2} \eta^{A-1} X_p^Z X_n^{A-Z} e^{B_A/T} \\ &= g_A \zeta(3)^{A-1} 2^{-5/2+3/2A} \pi^{-1/2-1/2A} A^{5/2} T^{-3(1-A)/2} m_B^{3(1-A)/2} \eta^{A-1} X_p^Z X_n^{A-Z} e^{B_A/T}, \end{aligned}$$

# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

Which can be written in a nicer way...

$$X_A = F(A) \left( \frac{T}{m_B} \right)^{3(A-1)/2} \eta^{A-1} X_p^Z X_n^{A-Z} e^{B_A/T}$$

where

$$F(A) = g_A A^{5/2} \zeta(3)^{A-1} \pi^{(1-A)/2} 2^{(3A-5)/2}$$

This expression allows one to explicitly compute the mass fraction abundances of any nuclear species **assuming nuclear statistical equilibrium**. In particular one has:

D :	$X_2 = 16.3 \left( \frac{T}{m_B} \right)^{3/2} \eta e^{B_2/T} X_n X_p,$	$B_2 = 2.22 \text{ MeV}$
$^3\text{He}$ :	$X_3 = 57.4 \left( \frac{T}{m_B} \right)^3 \eta^2 e^{B_3/T} X_n X_p^2,$	$B_3 = 7.72 \text{ MeV} (^3\text{He})$
$^3\text{H}$ :		$B_3 = 6.92 \text{ MeV} (^3\text{H})$
$^4\text{He}$ :	$X_4 = 113 \left( \frac{T}{m_B} \right)^{9/2} \eta^3 e^{B_4/T} X_n^2 X_p^2,$	$B_4 = 28.3 \text{ MeV}$
$^{12}\text{C}$ :	$X_{12} = 3.22 \times 10^5 \left( \frac{T}{m_B} \right)^{33/2} \eta^{11} e^{B_{12}/T} X_n^6 X_p^6,$	$B_{12} = 92.2 \text{ MeV}$

# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

These abundances are constrained by the conservation equation which, if one neglects other elements, reads:

$$X_n + X_p + X_2 + X_3 + X_4 + X_{12} = 1$$

The neutron and proton fractions are related. Their mass fractions in equilibrium can be easily obtained. We know that

$$\left(\frac{n_n}{n_p}\right)_{\text{eq}} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(m_n - m_p)/T}$$

Dividing the numerator and the denominator on left hand side of this equation by  $n_b$  one obtains:

$$\left(\frac{n_n}{n_p}\right)_{\text{eq}} = \left(\frac{X_n}{X_p}\right)_{\text{eq}} \simeq e^{-Q/T} \quad (Q = m_n - m_p)$$

Note that expressions for  $X_n$  derived in the previous slides assume the approximation  $m_B = m_p = m_n$ . If this approximation is taken rigorously then  $X_n/X_p = 1$ . However, the mass difference ( $Q = m_n - m_p$ ) in the exponential should not be ignored whereas it is smaller impact on the ratio of masses of the right-hand side of  $n_n/n_p$ . <sup>11</sup>

# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

Also note that, since the mass fraction abundances add up to one, e.g.,

$$X_n + X_p + X_2 + X_3 + X_4 + X_{12} = 1$$

and **nuclear synthesis occurs via 2-body reactions** (such as those in slide 5):

- Heavier **nuclear species are only effectively produced after the lighter ones are produced**. This is the case of Helium-4 which is only produced via 2-body reaction involving Deuterium or Hydrogen-3
- If the abundance fraction of a given nuclear species increases, this happens at the expenses of some other species (which has its fraction reduced)

So, one can define an **estimate of the temperature at which a given nuclear species is effectively produced by setting  $X_A \sim 1$** . this can only happen for  $T \ll B_A$  so that the exponential term in  $X_n$  compensates  $\eta$  (the baryon to photon ratio) term, which is small.

$$X_A = F(A) \left(\frac{T}{m_B}\right)^{3(A-1)/2} \eta^{A-1} X_P^Z X_n^{A-Z} e^{B_A/T} \sim 1$$



# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

From this previous expression one can derive an **approximate equation to compute the temperature of effective production** of a given nuclear species,  $T_A$ . Setting  $X_A \sim X_n \sim X_p \sim 1$ , taking the logarithm of  $X_A$  and dropping  $\ln F(A)$ , which is also  $F_A \sim 1$ , gives:

$$0 = \frac{3}{2}(A-1) \ln \left( \frac{T_A}{m_B} \right) + (A-1) \ln \eta + \frac{B_A}{T_A}$$

This can be used with iterative numerical methods to estimate  $T_A$ ,

$$\begin{aligned} T_A &\approx -\frac{B_A}{\frac{3}{2}(A-1) \ln \left( \frac{T_A}{m_B} \right) + (A-1) \ln \eta} \\ &= \frac{B_A}{A-1} \frac{1}{\ln \eta^{-1} + \frac{3}{2} \ln \left( \frac{m_B}{T_A} \right)}. \end{aligned}$$

For example, using this expression for Deuterium, one obtains:

$$T_D = \frac{2.22}{1} \frac{1}{\ln(2 \times 10^{-8} \Omega_B h^2)^{-1} + \frac{3}{2} \ln \left( \frac{1 \text{ GeV}}{T_D} \right)}$$

Similar equations can be derived for other nuclear species.

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# Big-Bang Nucleosynthesis

## Nuclear Statistical equilibrium

Solving these type of equations, one obtains the following effective temperatures of production (in equilibrium) of the Deuterium, Tritium, and Helium-4:

$$T_D \approx 0.07 \text{ MeV} ; \quad T_{3\text{H}} \approx 0.11 \text{ MeV} ; \quad T_{4\text{He}} \approx 0.28 \text{ MeV}.$$

These temperatures can be converted to time using the Friedmann equation expressed in terms of temperature of the effective degrees of freedom in energy

$$\frac{\dot{a}}{a} = H = \sqrt{\frac{\hbar c}{3M_{pl}^2} \frac{\pi^2}{30} g_* T^4} = \frac{\pi}{3} \left( \frac{g_*}{10} \right)^{1/2} \frac{T^2}{M_{pl}}$$

where we assume radiation domination.

Taking  $g_* = 3.38$  one can derive the following expression for the beginning of the nucleosynthesis,

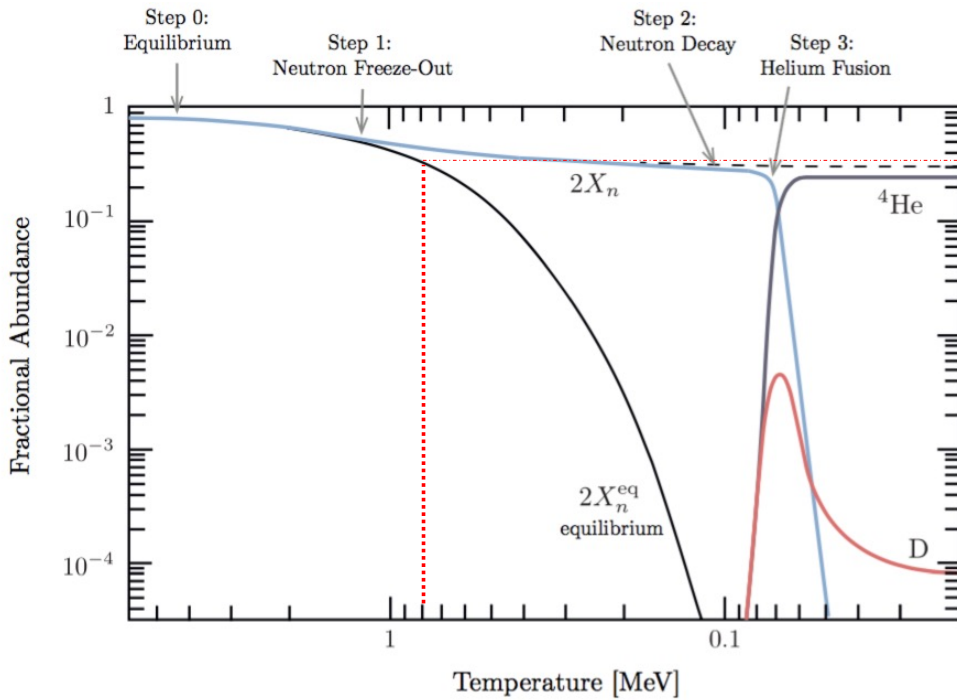
$$t_{\text{nuc}} = 132 \text{ s} \left( \frac{0.1 \text{ MeV}}{T_{\text{nuc}}} \right)^2$$

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# Big-Bang Nucleosynthesis

## Neutron's abundance

The production of nuclear elements within the mechanism of Big-Bang nucleosynthesis is directly related with the abundance of free neutrons, and the evolution of  $n_B$  or the baryon to photon ratio. One can tell the story of neutrons in a few steps:



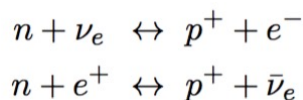
Neutrons decouple from the fluid and abandon equilibrium. They also decay into Protons.

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# Big-Bang Nucleosynthesis

## Neutron's abundance

**Step 0 (Equilibrium):** Above  $T \sim 1$  MeV protons and neutrons are in equilibrium via the nuclear reactions



The relative abundance of neutrinos to protons is then given by the equilibrium prediction (neglecting the chemical potential of the leptons,  $\mu_n \sim \mu_p$ ):

$$\left(\frac{n_n}{n_p}\right)_{\text{eq}} = \left(\frac{m_n}{m_p}\right)^{3/2} e^{-(m_n - m_p)/T}$$

where  $m_n - m_p = Q = 1.293$  MeV is the mass difference between neutrons and protons. So, the fraction of neutrons at equilibrium can be approximated by:

$$X_n^{\text{eq}} \simeq \frac{n_n^{\text{eq}}}{n_p^{\text{eq}} + n_n^{\text{eq}}} = \frac{n_n^{\text{eq}}/n_p^{\text{eq}}}{1 + n_n^{\text{eq}}/n_p^{\text{eq}}} \simeq \frac{e^{-Q/T}}{1 + e^{-Q/T}}$$

where  $n_B \simeq n_n + n_p$  is used in the first equality and  $m_n / m_p \simeq 1$  is used in the last equality. At  $T = 0.8$  MeV this gives,

$$X_n^{\text{eq}}(0.8 \text{ MeV}) = 0.17 \sim 1/6$$

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# Big-Bang Nucleosynthesis

## Neutron's abundance

**Step 1 (Decoupling):** As neutrinos decouple and positron-electron annihilation occurs, neutrons are forced to also decouple from the fluid. From the figure in a previous slide one expects that the **freeze out abundance of neutrons** should be close to:

$$X_n^\infty \sim X_n^{\text{eq}}(0.8 \text{ MeV}) \sim \frac{1}{6}$$

To confirm this expectation, one needs to integrate the Boltzmann equation for the interactions that keep neutrons and protons in contact with the plasma. As seen in Chapter 4, the **Boltzmann equation** for the 2-body interaction  $1 + 2 \rightleftharpoons 3 + 4$  is:

$$\frac{1}{a^3} \frac{d(n_1 a^3)}{dt} = -\langle \sigma v \rangle \left[ n_1 n_2 - \left( \frac{n_1 n_2}{n_3 n_4} \right)_{\text{eq}} n_3 n_4 \right]$$

For interactions of the form  $\mathbf{n} + \mathbf{l}_1 \rightleftharpoons \mathbf{p}^+ + \mathbf{l}_2$ , where  $l_i$  is a lepton **tightly bound to the plasma** one obtains:

$$\frac{1}{a^3} \frac{d(n_n a^3)}{dt} = -\Gamma_n \left[ n_n - \left( \frac{n_n}{n_p} \right)_{\text{eq}} n_p \right]$$

Since leptons are tightly bound to the fluid one has:  $n_{l_i} = n_{l_i}^{\text{eq}}$ , and  $\Gamma_n = \langle n_{l_1} \sigma v_{\rightarrow} \rangle$ .

# Big-Bang Nucleosynthesis

## Neutron's abundance

**Step 1 (Decoupling):** The solution of the Boltzmann equation is numerical. To compute the free neutron's fraction,  $X_n$ , one needs to use its definition (in slide 8) and compute the densities of all baryon species in the fluid at a given time.

However, one can simplify the calculation of  $X_n$  using the following approximations:

- before neutron decoupling  $n_b \simeq n_n + n_p \Leftrightarrow 1 \simeq X_n + X_p$
- the total number of baryons is conserved, i.e.,  $n_b a^3 = \text{constant}$ .

Using these assumptions the Boltzmann equation can be written as (see Black Board):

$$\begin{aligned} \frac{1}{a^3} \frac{d(n_n a^3)}{dt} \frac{1}{(n_B a^3)} &= -\frac{\Gamma_n}{(n_B a^3)} \left[ n_n - \left( \frac{n_n}{n_p} \right)_{\text{eq}} n_p \right] \Leftrightarrow \\ \Leftrightarrow \frac{d}{dt} \left( \frac{n_n a^3}{n_B a^3} \right) &= -\Gamma_n \left[ \frac{n_n}{n_B} - \left( \frac{n_n}{n_p} \right)_{\text{eq}} \frac{n_p}{n_B} \right] \Leftrightarrow \\ \Leftrightarrow \frac{dX_n}{dt} &= -\Gamma_n \left[ X_n - \left( \frac{n_n}{n_p} \right)_{\text{eq}} X_p \right] \Leftrightarrow \boxed{\frac{dX_n}{dt} = -\Gamma_n [X_n - (1 - X_n) e^{-Q/T}]} \end{aligned}$$

where we used  $X_p \simeq 1 - X_n$  and the fact that ( $Q = m_n - m_p$ ):

$$\left( \frac{n_n}{n_p} \right)_{\text{eq}} = \left( \frac{m_n}{m_p} \right)^{3/2} e^{-(m_n - m_p)/T} \simeq e^{-Q/T}$$

# Big-Bang Nucleosynthesis

## Neutron's abundance

### Step 1 (Decoupling):

To integrate this Boltzmann equation,

$$\frac{dX_n}{dt} = -\Gamma_n \left[ X_n - (1 - X_n)e^{-Q/T} \right]$$

it is useful to make a change of variable  $x = Q/T$ ,

$$\frac{dX_n}{dt} = \frac{dX_n}{dx} \frac{dx}{dt} = \frac{dX_n}{dx} \frac{d(Q T^{-1})}{dt} = \frac{dX_n}{dx} \left( -\frac{Q}{T^2} \frac{dT}{dt} \right) = -\frac{dX_n}{dx} \frac{x}{T} \frac{dT}{dt} = \frac{dX_n}{dx} x H$$

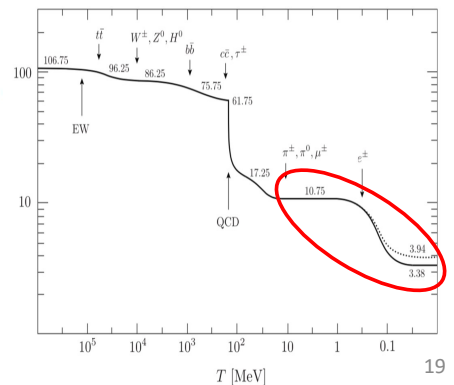
Where the last equality results from  $T = A_i g_{*S}^{-1/3}(T) a(t)^{-1}$  assuming  $g_{*S} = \text{constant}$  (see Riccati equation in Chap 5). This expression is therefore valid away from mass thresholds. Using this in the Boltzmann equation and writing

$$H = \sqrt{\frac{\rho}{3M_{\text{pl}}^2}} = \frac{\pi}{3} \sqrt{\frac{g_* Q^2}{10 M_{\text{pl}}}} \frac{1}{x^2}, \quad \text{with } g_* = 10.75 .$$

$\equiv H_1 \approx 1.13 \text{ s}^{-1}$

where  $H_1$  is the  $x$ -independent part of the Hubble function written as a function of  $x$  the Boltzmann equation reads:

$$\frac{dX_n}{dx} = \frac{\Gamma_n}{H_1} x \left[ e^{-x} - X_n(1 + e^{-x}) \right]$$



# Big-Bang Nucleosynthesis

## Neutron's abundance

### Step 1 (Decoupling):

The exact form of  $\Gamma_n$  depends on the lepton particles being considered. Its calculation can be done in Quantum Field Theory. It can be, generically, approximated by:

$$\Gamma_n(x) = \frac{255}{\tau_n} \cdot \frac{12 + 6x + x^2}{x^5}$$

where  $\tau_n = 886.7 \text{ s}$  is the neutron half-time decaying period.

With these expressions the **numerical integration of the Boltzmann equation** (blue curve) would give:

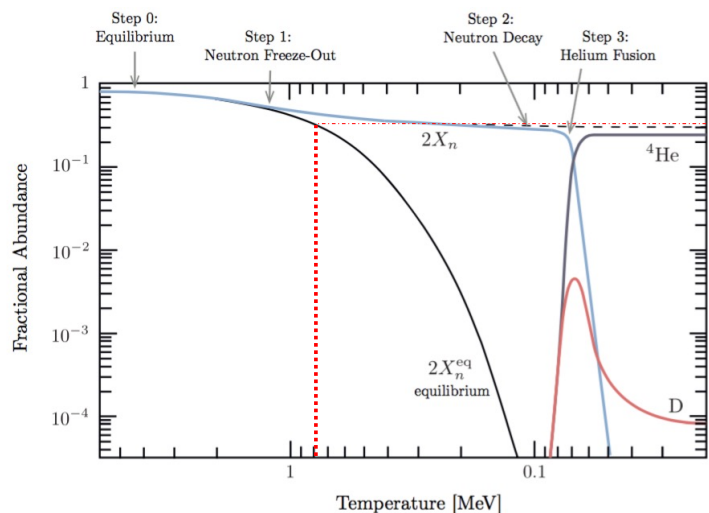
$$X_n^\infty \equiv X_n(x = \infty) = 0.15$$

only if neutrons don't decay! (Step 2)  
This is similar to the result in slide 17.  
So just **before Neutron decay** one has:

$$n_B \simeq n_p + n_n \iff 1 \simeq X_p + X_n$$

and

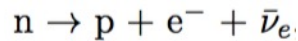
$$X_p \simeq 1 - X_n = 0.85 ; X_n/X_p \simeq 0.17$$



# Big-Bang Nucleosynthesis

## Neutron's abundance

**Step 2 (Neutron decay):** The decoupled neutrons also decay into protons via the process:



which has a half-time decaying period of  $\tau_n = 886.7 \pm 0.8$  sec.. **This decay can only become relevant when the universe is old enough (as old as the  $\tau_n$  decaying period).**

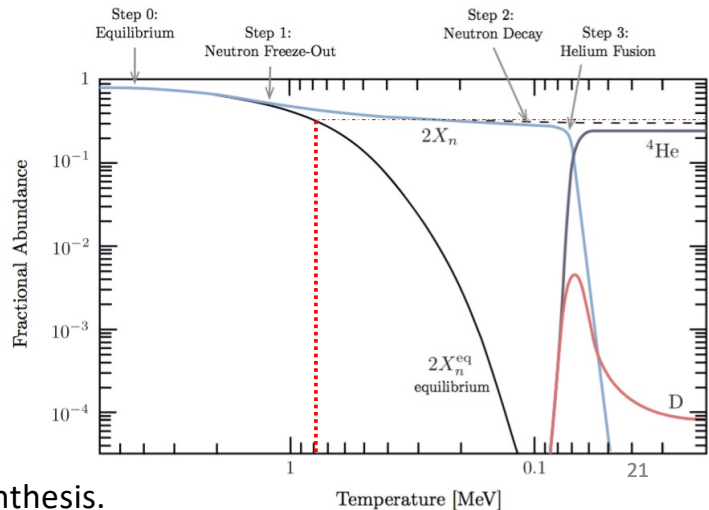
To include neutron decay into the calculation, we simply multiply the neutron abundance by an exponential term characteristic of nuclear decaying processes:

$$X_n(t) = X_n^{\text{no-decay}} e^{-t/\tau_n} \\ \simeq \frac{1}{6} e^{-t/\tau_n}$$

Where  $t$  is (radiation domination):

$$t = 132 \text{ s} \left( \frac{0.1 \text{ MeV}}{T} \right)^2$$

This decaying mechanism has strong implications for the nuclear species synthesis.



# Big-Bang Nucleosynthesis

## Neutron's abundance

### Step 2 (Neutron decay)

Using these assumption one can, for example, compute mass fraction abundances by  $t = t(T \sim 0.07 \text{ MeV}) = 269.4 \text{ s}$

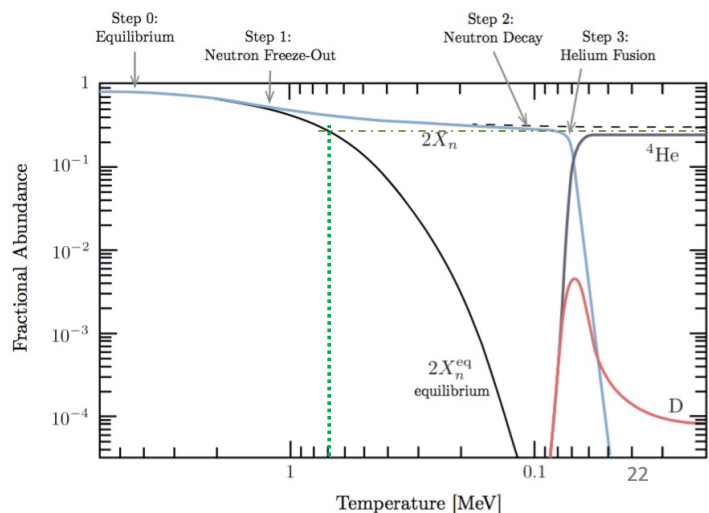
$$X_n(0.07 \text{ MeV}) = \frac{1}{6} \exp\left(-\frac{269.4 \text{ s}}{886.7 \text{ s}}\right) = 0.123$$

$$X_p(0.07 \text{ MeV}) = 1 - X_n = 0.877$$

and therefore:

$$\frac{X_n(0.07 \text{ MeV})}{X_p(0.07 \text{ MeV})} = 0.140 \sim \frac{1}{7}$$

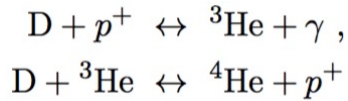
this is just before Helium-4 starts to be produced in an efficient way.



# Big-Bang Nucleosynthesis

## Helium abundance

**Step 3 (Helium fusion):** Helium is produced via the reactions:



that **require the existence of Deuterium**, which is produced via:  $n + p^+ \leftrightarrow D + \gamma$   
So, helium cannot be produced before a sufficient amount of deuterium is formed.

The **helium fraction abundance by the end of BBN** can be **estimated** as follows:

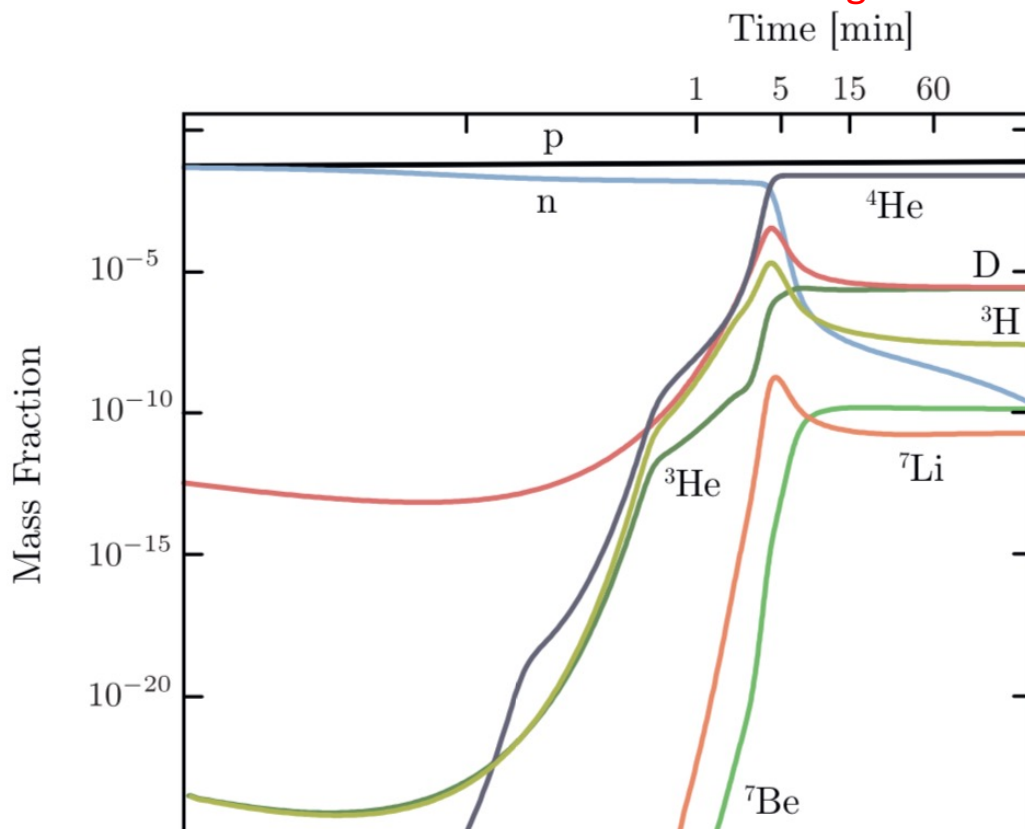
- Until before neutron decay ( $T \sim 0.07$ ) all baryons are in the form of free protons and neutrons:  $n_B^i \simeq n_p^i + n_n^i$
- By the end of BBN **hydrogen (p)** and **helium-4** nuclei are the 1<sup>st</sup> and 2<sup>nd</sup> most abundant elements (other nuclei are residual). So, baryon conservation allows to write:  $n_p^f + 4n_{4\text{He}}^f = n_p^i + n_n^i$
- By the end of BBN **about half of the initial neutrons are inside helium nuclei** (because each nucleus of helium contains 2 neutrons):  $n_{4\text{He}}^f = n_n^i/2$

Under these approximations, the **Helium mass fraction abundance** becomes:

$$X_{4\text{He}} = \frac{4n_{4\text{He}}^f}{n_p^f + 4n_{4\text{He}}^f} = \frac{4n_n^i/2}{n_p^i + n_n^i} = \frac{2n_n^i}{n_p^i + n_n^i} = \frac{2n_n^i/n_p^i}{1 + n_n^i/n_p^i} = \frac{2X_n^i/X_p^i}{1 + X_n^i/X_p^i} \simeq \frac{2/7}{1 + 1/7} \simeq \frac{1}{4}$$

# Big-Bang Nucleosynthesis

## Numerical evolution of mass fraction abundances of light elements:



# Big-Bang Nucleosynthesis

Comparison with observations:

Helium 4: constraints from ionized gas (metal poor) clouds

Deuterium: constraints from metal poor quasar absorption systems

Helium 3: is hard to constraint. Limits estimated from solar system and HII (metal abundant) regions in our galaxy

Lithium 7: constraints from low metallicity population II stars in our galaxy.

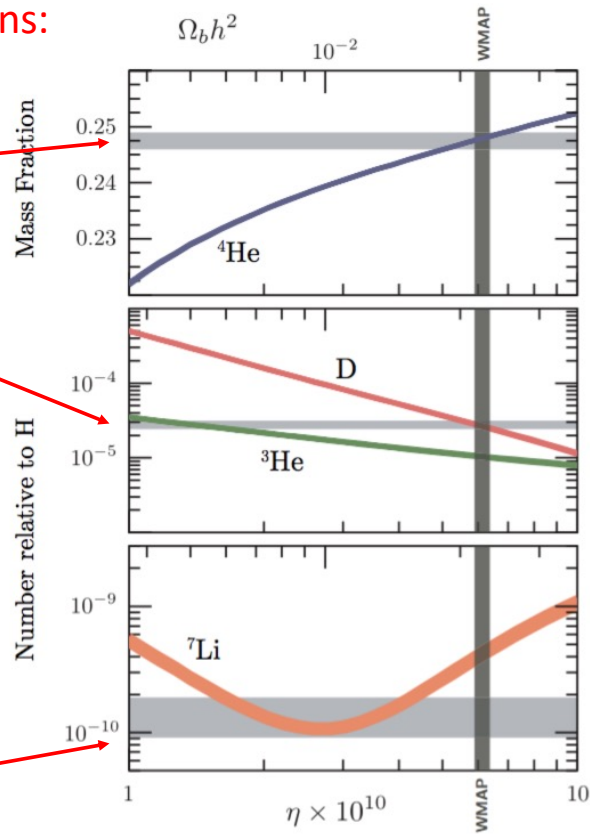


Figure 3.10: Theoretical predictions (colored bands) and observational constraints (grey bands).