Problem Set: Interfaces, Capillary Waves, and Membranes

(Doi Ch. 4; van Saarloos–Vitelli–Zeravcic Ch. 7)

Instructions. Unless stated otherwise, work in $d = 2$ for height fields $h(x, y)$ and in $d = 3$ for bulk fields $\phi(\boldsymbol{r})$. State assumptions clearly (small-slope, incompressibility, isothermal, etc.). Where appropriate, give scaling arguments *and* (when feasible) explicit derivations.

Notation. For a surface of height $h(x, y)$ above a reference plane, ∇ denotes the in-plane gradient, and Fourier modes are $h(\boldsymbol{r}) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\, h_{\boldsymbol{q}}\, \mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{r}}$.

Part I. Capillary waves and interfacial roughness

1. **Capillary-wave Hamiltonian from area expansion.** A nearly flat interface is described by $z = h(x, y)$ with $|\nabla h| \ll 1$.

   (a) Expand the area element $\mathrm{d}A = \sqrt{1 + (\nabla h)^2}\, \mathrm{d}^2 r$ to quadratic order and show that the excess free energy is
   $$F[h] \simeq \frac{\gamma}{2} \int \mathrm{d}^2 r\, (\nabla h)^2.$$

   (b) Write F in Fourier space as $F = \frac{\gamma}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\, q^2 |h_{\boldsymbol{q}}|^2$.

2. **Capillary-wave spectrum and roughness divergence.** Using equipartition at temperature T,

   (a) Derive the capillary-wave spectrum
   $$\left\langle |h_{\boldsymbol{q}}|^2 \right\rangle = \frac{k_\mathrm{B} T}{\gamma q^2}.$$

   (b) Compute the mean-square height fluctuations $\left\langle h^2 \right\rangle$ in a finite system of lateral size L with microscopic cutoff a. Show the logarithmic divergence $\left\langle h^2 \right\rangle \sim (k_\mathrm{B} T / 2\pi\gamma) \ln(L/a)$.

   (c) Explain what physical mechanisms (gravity, finite thickness, pinning, etc.) can regularize this divergence.

Part II. Diffuse interfaces and the Cahn–Hilliard framework

3. **Chemical potential from a square-gradient free energy.** Consider
   $$F[\phi] = \int \mathrm{d}^3 r \left[f(\phi) + \frac{\kappa}{2} (\nabla\phi)^2 \right].$$

   (a) Compute the functional derivative $\mu = \delta F / \delta\phi$ and show
   $$\mu = f'(\phi) - \kappa \nabla^2 \phi.$$

   (b) Interpret the physical meaning of the $-\kappa\nabla^2\phi$ contribution.

4. **Derivation of the Cahn–Hilliard equation.** Assume conserved dynamics with flux $\boldsymbol{J} = -M\nabla\mu$ (constant mobility $M > 0$).

(a) Starting from $\partial_t \phi + \nabla \cdot \boldsymbol{J} = 0$, derive

$$\partial_t \phi = M \nabla^2 \big(f'(\phi) - \kappa \nabla^2 \phi \big) \,.$$

(b) Explain why the dynamics is fourth order in space and why this is consistent with conservation.

5. **Equilibrium interface profile for a ϕ^4 model.** Take

$$f(\phi) = \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4, \qquad b > 0,\ a < 0.$$

(a) Consider a planar interface $\phi(y)$ between $\phi = \pm\phi_0$. Show that the Euler–Lagrange equation reduces to

$$\kappa \phi''(y) = f'(\phi).$$

(b) Solve for $\phi(y)$ and show that it has the form $\phi(y) = \phi_0 \tanh(y/\xi)$ for appropriate ϕ_0 and ξ.

(c) Sketch $\phi(y)$ and identify the interfacial width ξ.

6. **Surface tension from the diffuse-interface solution.** Using the interface profile from the previous problem,

(a) Show that the surface tension can be written as

$$\gamma = \int_{-\infty}^{\infty} \mathrm{d}y \, \kappa \left(\frac{\mathrm{d}\phi}{\mathrm{d}y} \right)^2 .$$

(b) Evaluate γ (up to numerical prefactors is fine if the algebra is heavy) and give its scaling with a, b, κ.

Part III. Membranes: Helfrich energy and fluctuations

7. **Helfrich Hamiltonian and curvature invariants.** A fluid membrane has elastic energy

$$F = \int \mathrm{d}A \left[\frac{\kappa_b}{2}(2H)^2 + \bar{\kappa}\, K \right],$$

where H is mean curvature and K Gaussian curvature.

(a) Define H and K (geometrically) and state their values for a sphere of radius R.

(b) State the Gauss–Bonnet theorem and explain why $\int \mathrm{d}A\, K$ depends only on topology (for closed surfaces).

8. **Bending energy of a sphere.** Compute F for a closed spherical vesicle of radius R.

(a) Show that the bending contribution is $F_{\mathrm{bend}} = 8\pi\kappa_b$.

(b) Discuss why it does not depend on R (in this simplified model).

9. **Membrane fluctuation spectrum (Monge gauge).** For a nearly flat tensionless membrane with height field $h(x, y)$,

$$F[h] = \frac{\kappa_b}{2} \int \mathrm{d}^2 r \, (\nabla^2 h)^2.$$

(a) Write F in Fourier space and derive the thermal spectrum

$$\langle |h_{\boldsymbol{q}}|^2 \rangle = \frac{k_{\mathrm{B}} T}{\kappa_b q^4}.$$

(b) Compare with capillary waves and discuss which modes dominate the roughness at large scales.

10. **Adding tension to a membrane.** Now consider

$$F[h] = \int \mathrm{d}^2 r \left[\frac{\gamma}{2} (\nabla h)^2 + \frac{\kappa_b}{2} (\nabla^2 h)^2 \right].$$

(a) Derive $\langle |h_{\boldsymbol{q}}|^2 \rangle$ and identify the crossover wavenumber $q_\times$ between tension- and bending-dominated regimes.

(b) Provide a scaling sketch (log–log) of $\langle |h_q|^2 \rangle$ vs q and mark $q_\times$.

Part IV. Optional computational mini-projects

Choose *one* (or more) of the following.

1. **Cahn–Hilliard simulation (1D or 2D).** Simulate

$$\partial_t \phi = M \nabla^2 \big(\phi^3 - \phi - \kappa \nabla^2 \phi \big)$$

(periodic boundaries). Measure a characteristic domain size $R(t)$ (e.g. from the structure factor peak) and test the coarsening law $R(t) \sim t^{1/3}$.

2. **Capillary-wave sampling.** Generate interface configurations $h_{\boldsymbol{q}}$ drawn from the Gaussian measure with $\langle |h_{\boldsymbol{q}}|^2 \rangle = k_{\mathrm{B}} T / (\gamma q^2)$. Reconstruct $h(\boldsymbol{r})$ and measure $\langle h^2 \rangle$ versus system size L to verify the logarithmic divergence.

3. **Membrane spectrum and crossover.** Repeat the previous project for the mixed Hamiltonian $\frac{\gamma}{2} (\nabla h)^2 + \frac{\kappa_b}{2} (\nabla^2 h)^2$ and verify the crossover at $q_\times \simeq \sqrt{\gamma / \kappa_b}$.