Cosmologia Física

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Structure Formation

Newtonian perturbed fluid equations

Newtonian Treatment

Gravity is well described by Newtonian physics if

- scales are sub-Hubble,
  i.e., much smaller than the scale of curvature of the spacetime
- the fluid is non-relativistic (in terms of velocity) $v \ll c$
- the fluid is non-relativistic (in terms of matter) $p \ll \rho$

Remember: H(a) is a decreasing function.

In a decelerating Universe, the comoving Hubble radius grows (and a comoving scale does not grow) **all scales gradually enter in the Hubble radius (becoming 'Newtonian')** → the universe becomes 'less relativistic' with time → the Newtonian description is more accurate in the matter-dominated epoch

and less accurate in the radiation (early Universe) and dark energy (late Universe) epochs.

Under these conditions, the Newtonian treatment of a cosmological fluid accurately describes structure formation.

In this description the evolution of the density contrast (and also of the potential and the peculiar velocity) is fully described by 3 equations (2 conservation equations + 1 constraint equation):

- Continuity equation
- Euler equation
- Poisson equation

This set of equations has similar information than the set of 3 equations used to describe the evolution of the homogeneous universe:

- Energy density conservation → a continuity equation

- Friedmann eq. → the zero-order equivalent to Poisson equation.

- Raychaudhuri eq. → a second-order equation of motion (like Euler eq.).

  Note that even though it is originally a constraint equation (Einstein eq) and not a conservation equation, they are related in the homogeneous case where only 2 independent equations are needed.

Newtonian perturbed fluid equations

Continuity equation

The continuity equation is the equation of conservation of mass.
Extending it to the relativistic framework, it becomes an equation for the conservation of energy.

This equation tells us there is no creation of energy → at a certain location of a fluid within a fixed volume, energy (or mass) may only change in time because it may flow to another location (within the volume).

So, **the local fluid flow, determined by its velocity, is responsible for a change of density.**

So, besides **expansion** (that automatically induces a change of density), there is another way to make the density change: the **peculiar velocity** → **it is an inhomogeneous contribution to the evolution of density, which was not present in the energy conservation equation in the homogeneous universe.**

In physical coordinates (r) the equation is:

$$\frac{d\rho}{dt} + \vec{\nabla}_r .(\rho\vec{u}) = 0$$

u is the flow velocity of the fluid

Now, if the fluid flows in an evolving background (as it is the case in cosmology), the volume changes in time → the physical coordinate 'r' changes with time,

and it has a dependence on both space and time. It is a "coordinate that evolves".

So, the **total time derivative** dρ/dt also includes spatial derivatives:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r_x}\frac{\partial r_x}{\partial t} + \frac{\partial f}{\partial r_y}\frac{\partial r_y}{\partial t} + \frac{\partial f}{\partial r_z}\frac{\partial r_z}{\partial t} = \frac{\partial f}{\partial t} + \vec{u}.\vec{\nabla}_r f$$

This is called a convective derivative or a material derivative.

Notice that it is also similar to a covariant derivative in General Relativity, where the covariant derivative is equal to a partial derivative plus the terms that involves the connections (that represents the change of the reference frame - in that case not due to expansion but due to curvature -).

Now, in the expanding universe, **the velocity in the expression of the total derivative, concerning the change of the physical position vector with time, is the velocity of the expansion (the background velocity)**

$$\vec{\bar{u}} = \dot{a}\vec{x}$$

(no peculiar velocity involved here in the coordinates change),

where x represents the (x,y,z) comoving coordinates, the “truly spatial” coordinates, i.e., independent of time

$x=r_x/a$, $y=r_y/a$, $z=r_z/a$

Notice also that since the expansion makes the physical sizes to increase, while the comoving sizes are constant, **this is equivalent to consider that the physical orthonormal reference frame shrinks** (such that the corresponding sizes increase), i.e.,

$$\nabla_r = \text{-}\nabla_x / a$$

So, the total time derivative of the density is,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} - \frac{\dot{a}}{a} x \, . \, \vec{\nabla}_x \, \rho$$

includes **local contribution** (perturbation, peculiar, inhomogeneous)

only **comoving contribution** (background, homogeneous)

and the equation becomes

$$\frac{\partial\rho}{\partial t} - \frac{\dot{a}}{a} x \, . \, \vec{\nabla}_x \, \rho + \vec{\nabla}_r \, .(\rho\vec{u}) = 0$$

Considering now the flow velocity of the fluid,

this term may also be decomposed in local and comoving contributions → the sum of the expansion velocity with the peculiar velocity v.

$$u = \dot{r} = \dot{a}x + v$$

Remember: peculiar velocity is the inhomogeneous contribution to the velocity → it means that the scale (the perturbation) does not follow exactly the expansion → the perturbed region may expand slower than the mean universe.

Inserting the total velocity u and the comoving coordinates in the last term, and also writing the density as

$$\rho = \bar{\rho}(1+\delta)$$

we get

$$\dot{\bar{\rho}}(1+\delta) + \bar{\rho}\dot{\delta} - \frac{\dot{a}}{a}\bar{\rho}\,\cancel{\vec{x}\cdot\vec{\nabla}\delta} + \frac{\dot{a}}{a}\bar{\rho}\,\cancel{\vec{x}\cdot\vec{\nabla}\delta} + \frac{1}{a}\bar{\rho}\,\vec{v}\cdot\vec{\nabla}\delta + \frac{1}{a}\bar{\rho}\,3\dot{a} + \frac{1}{a}\bar{\rho}\,\vec{\nabla}\cdot\vec{v} +$$
$$+ 3\frac{\dot{a}}{a}\bar{\rho}\delta + \frac{1}{a}\bar{\rho}\delta\,\vec{\nabla}\cdot\vec{v} = 0 \quad (=)$$

There are terms in the equation involving only background quantities and other terms involving perturbations. **We can separate the equation in zero-order and first-order "sub-equations".**

Let us also neglect the terms that are not linear in the perturbed quantities (they are small <<1, when δ < 1). The two sub-equations are then:

$$\dot{\bar{\rho}} + 3\frac{\dot{a}}{a}\bar{\rho} = 0$$

the zero-order equation - the **background counterpart** - is the well-known **continuity equation of the homogeneous Universe** (this is the reason why the energy conservation equation was called the continuity equation, even though apparently it did not look like the continuity equation of a fluid)

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla.\mathbf{v} = 0$$

the first-order equation - the **comoving, perturbed and linearized continuity equation** - this one looks like the continuity equation of a fluid

Note that after all these steps the comoving continuity equation looks like:

the original equation (classical, non-expanding, with partial time derivative and local peculiar velocity flow)

plus an extra term that shows the contribution of the expansion to the evolution of density (i.e. the relativistic correction).

Poisson equation

Poisson eq. connects the gravitational potential with the matter density.

In physical coordinates (r) the equation is: $\Delta_{\mathbf{r}}\phi = 4\pi G\rho.$

In comoving coordinates (x=r/a) it is: $\Delta\phi = 4\pi G a^2 \rho.$

Once again, we can insert the density contrast and separate the equation in zero and first-order parts. $\rho = \bar{\rho}(1+\delta)$

We may also define a homogeneous and a perturbed potential:

$$\Phi = \bar{\Phi} + \delta\Phi$$

With this definition, the equation is immediately separated.

The **first order equation** is the **comoving, perturbed and linearized Poisson equation:**

$$\Delta\delta\Phi = 4\pi G a^2 \bar{\rho}\delta$$

and the **zero-order equation**,
the **background counterpart** is just:

$$\Delta\bar{\Phi} = 4\pi G a^2 \bar{\rho}$$

We do not see immediately that this is the Friedmann equation (which is the zero-order counterpart of Poisson equation).

But the Friedmann equation tells us that

$$4\pi G a^2 \bar{\rho} = \frac{3}{2}\dot{a}^2$$

and so

$$\Delta\bar{\Phi} = 3\frac{\partial^2\bar{\Phi}}{\partial x^2} = 3\frac{\dot{a}^2}{2}$$

(isotropy, x, y, z, equivalent)

Integrating it, we get

$$\bar{\Phi} = \frac{1}{4}\dot{a}^2 x^2 = \left(\frac{\bar{u}}{2}\right)^2$$

i.e., the "potential of the universe" is given by the square of the "velocity of the universe", which is consistent with the dimensional result of a potential being a velocity square (e.g: virial theorem).

Euler equation

The Euler eq. tells us how the velocity field changes in time given a gravitational potential.

It is the equation of motion of the fluid. In physical coordinates (r) the equation is:

$$\left(\frac{\partial u}{\partial t}\right)_r + (u.\nabla_r)u = -\nabla_r\Phi$$

→ source

Again, like in the continuity equation, this is a total time derivative, i.e., the acceleration of the fluid is not just the change of the velocity field at a given position, but it also depends on the expansion.

Inserting the total derivative (i.e., the **comoving coordinates**), and the **velocity perturbations** through

$$u = \dot{r} = \dot{a}x + v$$

we get,

$$\frac{\partial v}{\partial t} + \ddot{a}x + \frac{\dot{a}}{a}v + \frac{1}{a}v.\nabla_x v = -\frac{1}{a}\nabla_x\Phi$$

It is a vectorial equation → there is one equation per component of v

The second term is the only zero-order one.

The **zero-order equation** is then: $\ddot{a}x = -\frac{1}{a}\nabla_x\bar{\Phi}$

Note that inserting here the background Poisson equation (integrated once with respect to x, to get the gradient and not the Laplacian), this equation becomes:

$$2a\ddot{a} = -\dot{a}^2$$

Inserting here the Friedmann equation, to replace $\dot{a}^2$, we recover the Raychadhuri equation (or the other way around substituting $\ddot{a}$ with the Raychadhuri equation, we recover the Friedmann equation)

→ the **background counterpart** of the the Euler equation is then a combination of the Friedmann and the Raychaudhuri equations.

Indeed, in the homogeneous universe, we saw that it is possible to combine those two equations to obtain an energy conservation equation.

The **first order Euler equation** is written directly from the remaining terms:

$$\frac{\partial v}{\partial t} + \frac{\dot{a}}{a} v + \frac{1}{a} v . \nabla_x v = -\frac{1}{a} \nabla_x \delta\Phi$$

Considering only linear terms in the perturbations, we get the **comoving, perturbed and linearized Euler equation:**

$$\frac{\partial v}{\partial t} + \frac{\dot{a}}{a} v = -\frac{1}{a} \nabla_x \delta\Phi$$

Density contrast evolution equation

In summary, the 3 resulting (first-order) perturbed linearized equations are:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla.\mathrm{v} = 0 \qquad \frac{\partial v}{\partial t} + \frac{\dot{a}}{a}v = -\frac{1}{a}\nabla\Phi \qquad \nabla^2\Phi = 4\pi\, G\, a^2\bar{\rho}\,\delta$$

(we will follow the standard notation of using ϕ for the potential perturbation instead of δϕ)

For the matter component $\bar{\rho} = \Omega_m a^{-3}\rho_c$

and the Poisson equation can also be written as

$$\nabla^2\phi = \frac{3H_0^2}{2a}\Omega_m\delta$$

Now, combining the:

time derivative of the continuity equation with the
divergence of the Euler equation and
inserting Poisson equation,

we get the evolution equation for the density contrast:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G \bar{\rho} \delta = 0$$

(linearized)

Let us now look at some properties of this equation.

2nd term

The second term is a **friction term** → it works against the growth.

Its coefficient is 2H(a),

showing that the **background expansion works against structure formation** → it is called the Hubble drag

3rd term

The third term is the **potential term** (comes from Poisson's equation)

It has a fixed sign (it is always negative), so the solutions need to be monotonous → no oscillating solutions → **the density contrast may grow**

Note that in principle all monotonic solutions are possible: δ can increase, decrease or even stay constant, depending on the relation between the coefficients of the second and third terms.

The cosmological parameters (and in particular the parameters of the homogeneous universe) are important for the efficiency of structure formation.

For example:

A larger amount of matter Ω_m → larger coefficient of the 3rd term → solution with faster growth → **more matter favours structure formation**
(the physical reason behind it is that a higher Ω_m creates a 'deeper potential well' through Poisson equation, which favours clustering).

Notice that the mean matter density Ω_m and the density contrast δ are independent quantities. Unlike the density, there can be other cosmological inhomogeneous fields where the amplitude of clustering is lower if Ω_m is larger (it is the case of the temperature anisotropies of CMB).

Now, the density contrast is also a function of space.

The equation we derived only tells us about the time evolution of the density contrast.

Can its spatial variation be separated from the time evolution?

We need to **Fourier transform δ(x,t) and study the time evolution of all δ_k scales.**

Fourier transforming the 3 perturbed quantities that appear in the equations **(density contrast, peculiar velocity and potential)** the 3 (non-linearized) equations become:

$$\frac{d\delta_k}{dt} + \frac{i}{a} k.v_k + \sum_{k'} i\delta_{k'}(k.v_{k-k'}) = 0$$

Continuity

$$\frac{dv_k}{dt} + 3\frac{\dot{a}}{a} v_k + \sum_{k'} i[v_{k'}.(k-k')v_{k-k'}] = -\frac{i}{a} k\Phi_k$$

Euler

$$-|k|^2 \Phi_k = \frac{3H_0^2}{2a} \Omega_m \delta$$

Poisson

Note that the evolution of a scale δ_k depends on a sum over all other scales k' → there is mode coupling.

But we also see that mode coupling is higher-order in the perturbed quantities.
It goes to zero if we linearize equations:

$$\frac{d\delta_k}{dt} + \frac{i}{a} v_k \,.\, k = 0$$ Continuity

$$\frac{dv_k}{dt} + \frac{\dot{a}}{a} v_k = -\frac{i}{a} k\, \phi_k$$ Euler

$$-|k|^2 \phi_k = \frac{3H_0^2}{2a} \Omega_m \delta_k$$ Poisson

These equations are almost identical to the linear equations in real space.

The only difference is that spatial derivatives are now replaced by multiplications with the scale k.

This means that in the linear regime, the evolution of the modes is independent.

Each scale has its own evolving equation, with no influence from the other scales.

This simplifies a lot the problem → the equations are the same for all scales (in the Newtonian approach, i.e., not valid for relativistic large super-Hubble scales).

What about other vector and tensor perturbations - are they relevant for the evolution of the density contrast?

Remember that the wavenumber k is a 3D vector. However, from isotropy (in the statistical cosmological principle), **we just need to consider the modulus of k. In other words, the only relevant direction is the one set by the flow of the clustering matter (the 1D longitudinal k mode).** The **transversal modes** (i.e. the ones orthogonal to the direction of the flow v) are suppressed

So, the internal product v.k in the continuity equation implies that only the longitudinal component of the velocity field contributes to the growth, i.e., if v.k = 0 → δ would not grow → **the orthogonal component of the velocity field produces no growth of perturbations**.
(this just means that if the matter flow does not go to the center of the potential, there is no growth of density contrast).

In addition, if we impose irrotational and adiabatic initial conditions for the fluid perturbations in the energy-momentum tensor → the velocity perturbation is a pure potential flow (representing radial collapsing directions, i.e., parallel with the flow) → the velocity field is a gradient field, it is described by a scalar

→ Any of these two arguments show that the vector part of v does not impact the growth of δ.

Moreover, vector metric perturbations decay → also no vector contributions from the metric perturbations to the growth of structure.

Finally, notice no tensor contribution appear in the perturbed fluid equations.

All these reasons show that:

metric tensor perturbations,

metric vector perturbations

matter vector perturbations

are not relevant here → **structure formation is a scalar process.**

Now, **combining the three linearized equations in Fourier space**, we get the single equation that describes the evolution of all modes in linear sub-Hubble regimes.

This equation looks exactly the same as the one in real space (and later on we will drop the k index) :

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\,\dot{\delta}_k - 4\pi G\bar{\rho}\,\delta_k = 0$$

Note that even though the equation is the same for all scales, this does not mean that all scales will have the exact same evolution, since there will be differences because of:

- domain of validity of the Newtonian approach (size of the scale)

- cosmological component considered (dark matter, baryonic matter, radiation)

- epoch of the Universe (matter, radiation or dark energy dominated regimes)

We can now start computing the evolution of the density contrast of the various components of the cosmological fluid, and in the various epochs.

The various epochs and components in the Universe

Remember that the mean densities (zero order) of the various components of the cosmological fluid evolve differently with time → **the Universe is dominated by one component at a time, defining different epochs.**

This is easily seen by computing the time-evolution of the mean density, from the zero-order continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0$$

Matter (dark matter and baryonic matter)

is defined by having pressure p=0, i.e., **w=0** (equation-of-state)

Inserting p=0 in the continuity equation, the solution is easy to find:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3}$$

ρ_0 is an integration constant, i.e., an initial condition, the density at a_0, and as we know, the initial conditions define the model parameters.

If we choose to define the initial condition today, at $a_0 = 1$, then ρ_0 is the density today, which defines the well-known matter density cosmological parameter,

$$\Omega_m = \frac{\rho_0}{\rho_c} = \frac{8\pi G\rho_0}{3H_0^2}$$

Note that the parameter Ω_m, not only parameterizes the matter density today, but also depends on the Hubble constant. **So the values of Ω_m in two Universes with the same matter densities but different expansion rates, would be different.**

For this reason, it is useful to introduce the physical matter density parameter, defined as $\omega_m = \Omega_m h^2$

Radiation

is a cosmological component consisting on relativistic particles, with v=c, and so eventual pressure perturbations propagate with speed of sound equal to the speed of light.

However this does not imply $p = \rho$, but $p=\rho/3$, i.e., **w=1/3**, because the flux of photons hitting a given surface may be spread over the 3D space, and so the net pressure over a surface is on average 1/3.

Note that in a 5D space-time (with 4 spatial directions), the radiation equation of state would be w=1/4

Note that, even in 4D space-time, if the Universe was not isotropic, the pressure would not be isotropically distributed, and the mean pressure could be different from 1/3.

Anyway, inserting $p=\rho/3$ in the continuity equation, the solution is easy to find:

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-4}$$

where ρ_0 defines the radiation density parameter

$$\Omega_r = \frac{\rho_0}{\rho_c} = \frac{8\pi G \rho_0}{3H_0^2}$$

Note that the energy density of a radiation fluid dilutes faster than a matter fluid, the extra a^{-1} factor explains the extra dilution due to the redshift of the photon frequency.

Notice also that if we consider that the Universe is a black-body with temperature T and energy density ρ_r, the energy is related to the temperature as $\rho_r \sim T^4$ (Stefan-Boltzmann law) → **the temperature of the Universe decreases linearly with a,** and this is why the temperature is used as a "time scale" in the early universe (as the redshift is also used).

Dark energy

is a component that produces accelerated expansion.

From the second Friedmann equation (Raychadhuri equation)

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p)$$

we see that the acceleration is positive if **w < -1/3**

Notice that in GR pressure is a source of gravity (just like mass or energy density), a negative pressure contributes to a repulsion, and may accelerate the expansion.

There are many models of dark energy. The simplest and most used one is the cosmological constant, where **w = -1**

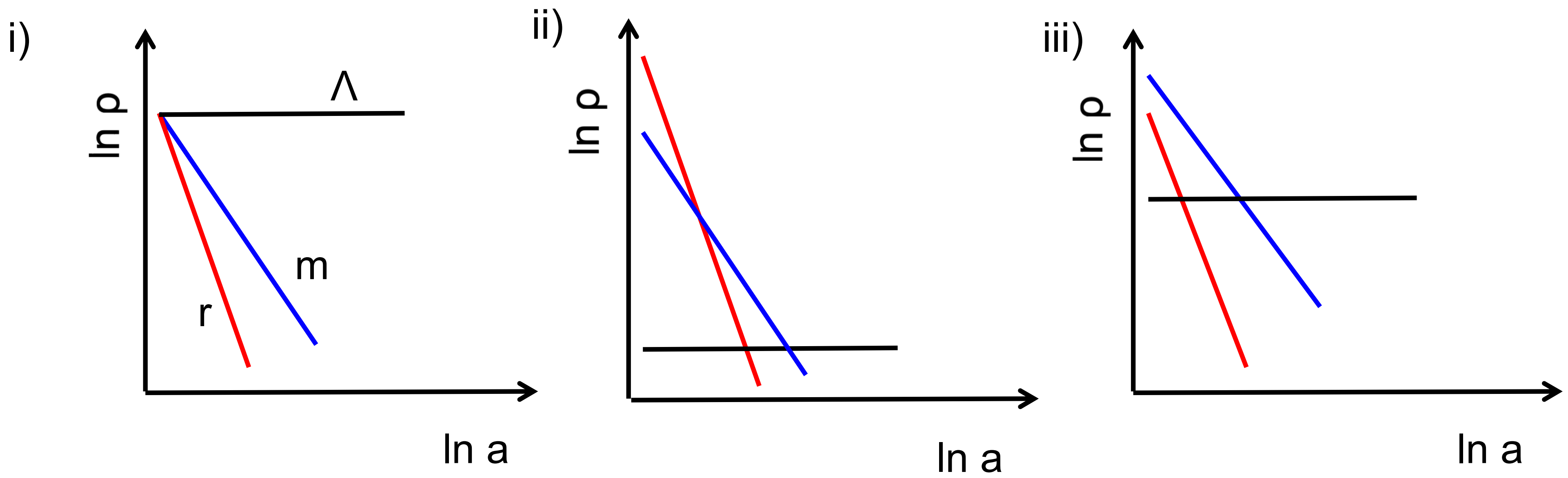
Inserting $p = -\rho$ in the continuity equation, we find the solution,

ρ = constant

i.e. the dark energy cosmological constant does not dilute as the Universe expands.

The evolution rate of the densities is not all the information needed to determine which is the dominating component, since the amplitudes of the densities depend on the initial conditions.

For example, consider the three following ΛCDM models:



In model i) all components start with the same amplitude → this model is always dominated by dark energy

In model iii) radiation never dominates

The concordance model looks like model ii)
we saw that from the measurement of the CMB temperature → $\Omega_r \sim 0.00008$,
and also that the various cosmological probes point to $\Omega_m \sim 0.3$ and $\Omega_\Lambda \sim 0.7$

Concordance model: given these amplitudes at a=1, we can search for valid times in the history of the Universe (with a>0 and a<1) where the densities were identical:

$\rho_r (a_{eq}) = \rho_m (a_{eq}) \rightarrow a_{eq} = \Omega_r / \Omega_m \sim 0.00027 \rightarrow z_{eq} \sim 3500$

$\rho_\Lambda (a_t) = \rho_r (a_t) \rightarrow a_t = (\Omega_r / \Omega_\Lambda)^{1/4} \sim 0.1 \rightarrow z_t \sim 8$

$\rho_\Lambda (a_\Lambda) = \rho_m (a_\Lambda) \rightarrow a_\Lambda = (\Omega_m / \Omega_\Lambda)^{1/3} \sim 0.75 \rightarrow z_\Lambda \sim 0.3$

We see that the concordance Universe starts to be radiation dominated, until the equality redshift z ~ 3500, where it starts to be matter dominated. At z ~8 the decreasing radiation density reaches the value of the cosmological constant, but this does not define another regime since matter continues to dominate (at z ~8, $\Omega_r = \Omega_\Lambda \sim 0.7$ and $\Omega_m \sim 270$). Finally, from z ~0.3 until today and into the future, the universe is dark energy dominated → **this defines the three epochs of the concordance Universe.**

(Note that the dark energy transition redshift does not mean that the Universe expansion started to accelerate at z = 0.3. It is just the redshift where dark energy and matter have the same mean densities).