Cosmologia Física

Homework 4 due 24 April 2026 (iatereno@fc.ul.pt)

Exercise 1: Power spectrum

Assume a flat CDM universe (no dark energy) where the matter power spectrum today is given by the following expression, where k_p is a characteristic scale chosen to be $k_p = 0.02$ h/Mpc (corresponding to a size $R_p = 2\pi/k_p$):

$$P_\delta(k, z=0) = \frac{A_{eq}\, k/k_p}{1+\beta(k/k_p)^4}\left(\frac{D_+^2(z=0)}{D_+^2(z=z_{eq})}\right).$$

Remember also that the corresponding dimensionless power spectrum is defined as $\Delta^2(k) = P_\delta(k)\, k^3$. Assume that at the end of the radiation-dominated epoch ($z = z_{eq} = 3499$) the value of the amplitude is $A_{eq} = 0.00340136484(\mathrm{Mpc/h})^3$ (it is important to use these exact values in the computations).

a) What is the value of the spectral index parameter n_s in this universe?

b) Write the expression of the transfer function in this universe.

c) If we choose the characteristic scale k_p to be the scale of the peak, then this choice fixes the parameter β. Show that $\beta = 1/3$.

d) What is the amplitude of the power spectrum today at the peak?

e) Find out which scale is collapsing today (find both its k value and its R size).

f) Is the scale of the peak already collapsed today? If not, when does it collapse?

g) Find out at which redshift does the scale of R=0.1 Mpc/h collapse (which is a typical scale of a galaxy). Is this result consistent with the observations of the real universe?

h) What is the feature in the transfer function used in this exercise that led to the unrealistic result found in g)? How could you mathematically improve the transfer function to obtain a more realistic description of our universe?

Hint: Plot the dimensional power spectrum of this exercise $\Delta^2(k)$ to better visualize its strange behaviour.

Exercise 2: Newtonian perturbed fluid equations

(In the following, "boldface"denotes a vector).

2.1) Consider the continuity equation for the dark matter fluid in an expanding background in comoving coordinates:

$$\frac{\partial\rho}{\partial t} - \frac{\dot a}{a}\mathbf{x}.\nabla\rho + \frac{1}{a}\nabla.(\rho\mathbf{u}) = 0\,.$$

Inserting the density and velocity perturbations through, $\rho = \bar{\rho}(1+\delta)$ and $\mathbf{u} = \dot{a}\mathbf{x} + \mathbf{v}$, derive the perturbed linearized comoving continuity equation (i.e., the first equation given in 1.3 below).

2.2) Consider the Euler equation for the dark matter fluid in an expanding background in comoving coordinates:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{\dot{a}}{a}(\mathbf{x}.\nabla)\mathbf{u} + (\mathbf{u}.\frac{1}{a}\nabla)\mathbf{u} = -\frac{1}{a}\nabla\Phi\,.$$

Inserting the density and velocity perturbations through, $\rho = \bar{\rho}(1+\delta)$ and $\mathbf{u} = \dot{a}\mathbf{x} + \mathbf{v}$, derive the perturbed linearized comoving Euler equation (i.e., the second equation given in 1.3 below).

2.3) Consider the system of perturbed linearized comoving Newtonian equations of fluid mechanics (continuity, Euler and Poisson):

$$\dot{\delta} + \frac{1}{a}\nabla.\mathbf{v} = 0\,, \qquad \frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} = -\frac{1}{a}\nabla\Phi\,, \qquad \nabla^2\Phi = 4\pi G a^2 \bar{\rho}\delta\,.$$

Combine the linearized equations to derive the equation of motion of δ,

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi\, G\, \bar{\rho}\, \delta = 0\,.$$