Interactions

Rui Santos

FCUL & CFTC

2026

Interacting fields

Free fields are interesting, but they have no interactions. Let us start with a scalar field and introduce small perturbations.

Consider a general scalar-field Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2}\mu^2 \phi^2 - \sum_m \lambda_m \phi^m,$$

where λ_m are called coupling constants.

We work in natural units, $\hbar = c = 1$. Since the action is dimensionless, $[S] = 0$, and

$$S = \int d^4x \, \mathcal{L},$$

we have

$$[d^4x] = -4, \qquad [\mathcal{L}] = 4.$$

From the kinetic and mass terms,

$$[\partial_\mu] = 1, \qquad [\mu] = 1, \qquad [\phi] = 1.$$

Therefore, for an interaction term

$$\lambda_m \phi^m,$$

we get

$$[\lambda_m] + m[\phi] = 4,$$

so that

$$\boxed{[\lambda_m] = 4 - m.}$$

Thus, saying that a coupling is "small" only makes sense when it is dimensionless.

- For ϕ^3:
$$[\lambda_3] = 1.$$
The coupling has dimensions of energy. It is small if $\lambda_3 \ll E$, where E is the energy scale of interest.
- For ϕ^4:
$$[\lambda_4] = 0.$$
The coupling is dimensionless, so perturbation theory is valid when
$$\lambda_4 \ll 1.$$
- For ϕ^m, $m > 4$:
$$[\lambda_m] < 0.$$
These are non-renormalisable interactions: they are small at low energies and become large at high energies.

Example: weakly coupled ϕ^4 theory

An important example of a weakly coupled theory with a dimensionless coupling is ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The coupling has mass dimension

$$[\lambda] = 0.$$

Hence perturbation theory is controlled by the condition

$$\boxed{\lambda \ll 1}$$

The first two terms describe the free scalar field, while

$$-\frac{\lambda}{4!}\phi^4$$

introduces a self-interaction of the scalar field.

Interactions, conserved quantities, and pictures

By expanding ϕ^4 in terms of creation and annihilation operators, $a^\dagger_{\mathrm{k}}$ and a_{k}, we obtain interaction terms of the form

$$a^\dagger a^\dagger a^\dagger a^\dagger, \qquad a^\dagger a^\dagger a^\dagger a, \qquad a^\dagger a^\dagger a a, \qquad \ldots$$

These terms suggest a theory in which the total number of particles is not conserved. Indeed,

$$[H, N] \neq 0.$$

Scalar Yukawa theory
Consider a complex scalar field ψ coupled to a real scalar field ϕ:

$$\mathcal{L} = \partial_\mu \psi^\dagger \partial^\mu \psi + \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - M^2 \psi^\dagger \psi - \frac{1}{2}\mu^2 \phi^2 - g\, \psi^\dagger \psi\, \phi,$$

with

$$g \ll M, \mu.$$

This interaction couples the complex scalar ψ to the real scalar ϕ. The individual particle numbers of ψ and ϕ are no longer conserved. However, electric charge remains a good quantum number:

$$[Q, H] = 0.$$

This means that the number of ψ-particles minus the number of anti-ψ-particles is conserved.

Example of a strongly coupled theory - QCD

- QCD describes the interactions of quarks and gluons through the gauge group $SU(3)_c$.
- At high energies, the strong coupling constant α_s becomes small, perturbation theory works for deep inelastic scattering or high-energy collider physics.
- At low energies, coupling grows, perturbation theory breaks down.
- QCD is a paradigmatic example of how a quantum field theory can transition from a weakly coupled perturbative regime at high energies to a strongly coupled nonperturbative regime at low energies, where phenomena such as confinement and hadronization emerge.

Interaction picture and Dyson formula

In quantum mechanics we can describe a system with small perturbations and a well-defined Hamiltonian H.
In the Schrödinger picture, states evolve as

$$i\frac{d}{dt}|\psi\rangle_S = H|\psi\rangle_S,$$

while operators are independent of time:

$$\frac{dO_S}{dt} = 0.$$

In the Heisenberg picture, states are fixed and operators evolve in time:

$$O_H(t) = e^{iHt} O_S e^{-iHt}.$$

Interaction picture and Dyson formula

The interaction picture is a hybrid of the Schrödinger and Heisenberg pictures. We split the Hamiltonian as

$$H = H_0 + H_{\rm int}.$$

The time dependence of operators is governed by H_0, while the time dependence of states is governed by $H_{\rm int}$.

$$|\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S,$$
$$O_I(t) = e^{iH_0 t} O_S e^{-iH_0 t}.$$

Although it is still possible to have ambiguity in how one splits $H = H_0 + H_{\rm int}$, this is a useful choice in free-field theory, where H_0 is exactly solvable.

The interaction Hamiltonian in the interaction picture is

$$H_I(t) = \left(H_{\rm int}\right)_I = e^{iH_0 t}\left(H_{\rm int}\right)_S e^{-iH_0 t}.$$

The Schrödinger equation for states in the interaction picture is obtained from the Schrödinger picture:

$$i\frac{d}{dt}|\psi(t)\rangle_S = H_S|\psi(t)\rangle_S.$$

Using

$$|\psi(t)\rangle_S = e^{-iH_0 t}|\psi(t)\rangle_I,$$

we find

$$i\frac{d}{dt}\left(e^{-iH_0 t}|\psi(t)\rangle_I\right) = \left(H_0 + H_{\rm int}\right)e^{-iH_0 t}|\psi(t)\rangle_I.$$

Therefore,

$$i\frac{d}{dt}|\psi(t)\rangle_I = H_I(t)|\psi(t)\rangle_I.$$

Dyson formula

Let us write the solution as

$$|\psi(t)\rangle_I = U(t, t_0)|\psi(t_0)\rangle_I,$$

where $U(t, t_0)$ is a unitary time-evolution operator satisfying

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3), \qquad U(t, t) = 1.$$

The time-evolution operator in the interaction picture obeys

$$i\frac{dU(t, t_0)}{dt} = H_I(t)\, U(t, t_0).$$

The formal solution is

$$U(t, t_0) = T \exp\left[-i \int_{t_0}^{t} dt'\, H_I(t')\right],$$

where T denotes time ordering.

Why Non-Commutativity Matters

Expand the naive exponential:

$$U = 1 - i\int dt_1\, H_I(t_1) + \frac{(-i)^2}{2!}\int dt_1 dt_2\, H_I(t_1)H_I(t_2) + \cdots$$

The second-order term treats t_1 and t_2 symmetrically.

But physical time evolution is sequential:

- interactions occurring earlier act first
- later interactions act afterwards

Therefore the operator ordering matters whenever

$$H_I(t_1)H_I(t_2) \neq H_I(t_2)H_I(t_1)$$

Physical Interpretation

The Dyson series describes quantum evolution as a sequence of interactions occurring in time.

- The earliest interaction acts first
- Later interactions act afterwards
- Non-commuting operators must therefore be ordered

The time-ordering operator ensures:

- causality
- consistent chronological evolution
- correct perturbation theory

Without time ordering, the evolution operator would not satisfy the Schrödinger equation for non-commuting Hamiltonians.

Dyson expansion

Expanding perturbatively,

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)H_I(t_2) + \cdots .$$

Equivalently,

$$U(t,t_0) = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1\cdots dt_n\, T\left\{H_I(t_1)\cdots H_I(t_n)\right\}.$$

Interlude: scattering in Yukawa theory

In scalar Yukawa theory, the interaction term is

$$H_{\text{int}} = g \int d^3x \, \psi^\dagger \psi \, \phi.$$

The fields have the schematic expansions

$$\phi \sim a + a^\dagger,$$

so ϕ can create or destroy a meson.

For the complex scalar field,

$$\psi \sim b + c^\dagger, \qquad \psi^\dagger \sim b^\dagger + c.$$

Thus:

$$\psi: \quad \text{destroys a nucleon } b \text{ or creates an anti-nucleon } c^\dagger,$$

$$\psi^\dagger: \quad \text{creates a nucleon } b^\dagger \text{ or destroys an anti-nucleon } c.$$

The conserved charge is

$$Q = N_b - N_c,$$

so the number of nucleons minus anti-nucleons remains conserved.
At first order in perturbation theory, terms in

$$H_{\mathrm{int}} \sim \psi^\dagger \psi \phi \quad (c^\dagger b^\dagger a)$$

can describe processes such as

$$a \longrightarrow b + c,$$

that is, a meson is destroyed and a nucleon–anti-nucleon pair is produced:

$$\phi \longrightarrow \psi + \psi^\dagger.$$

At second order, more complicated processes appear, for example

$$\psi + \psi^\dagger \longrightarrow \phi \longrightarrow \psi + \psi^\dagger.$$

Adiabatic approximation and the S-matrix

Adiabatic approximation

To calculate amplitudes, we assume the initial state $|i\rangle$ at $t \to -\infty$ and the final state $|f\rangle$ at $t \to +\infty$ are eigenstates of the free Hamiltonian H_0. As the particles approach each other, they interact briefly, and then move apart again. The transition amplitude is

$$\lim_{t\to\pm\infty} \langle f|U(t_f, t_i)|i\rangle \equiv \langle f|S|i\rangle,$$

where S is the scattering matrix.

Example: meson decay

Consider the decay

$$\phi \longrightarrow \psi + \psi^\dagger.$$

The initial state contains one meson with momentum k:

$$|i\rangle = \sqrt{2\omega_\mathrm{k}}\, a^\dagger_\mathrm{k}|0\rangle.$$

The final state contains a particle–antiparticle pair with momenta q_1 and q_2:

$$|f\rangle = \sqrt{4\omega_{\mathbf{q}_1}\omega_{\mathbf{q}_2}}\, b^\dagger_{\mathbf{q}_1} c^\dagger_{\mathbf{q}_2}|0\rangle.$$

At first order in perturbation theory,

$$S = 1 - i\int d^4x\, H_{\rm int}(x) + \cdots\,,$$

with

$$H_{\rm int}(x) = g\,\psi^\dagger(x)\psi(x)\phi(x).$$

Therefore the decay amplitude is

$$\langle f|S|i\rangle = -ig\int d^4x\,\langle f|\psi^\dagger(x)\psi(x)\phi(x)|i\rangle.$$

Using the field expansions, only the term

$$c_{\mathrm{q}_2}\, b_{\mathrm{q}_1}\, a_{\mathrm{k}}^{\dagger}$$

contributes, giving

$$\langle f|S|i\rangle = -ig \int d^4x\, e^{i(q_1+q_2-k)\cdot x}.$$

Hence

$$\boxed{\langle f|S|i\rangle = -ig\,(2\pi)^4 \delta^{(4)}(q_1+q_2-k)}$$

This is the scattering amplitude together with four-momentum conservation.

The delta function constrains the possible decays. In particular, the decay is kinematically allowed only if

$$\boxed{m_\phi > 2M}$$

where m_ϕ is the meson mass and M is the mass of the ψ-particle.

QED and the S-matrix

Now we want the full QED Lagrangian, with an interaction term:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi - eQ\,\bar{\psi}\gamma^\mu \psi A_\mu .$$

Here $e > 0$ is the absolute value of the electron charge, and Q is the electric charge in units of e. For example,

$$Q = -1 \quad \text{electron}, \qquad Q = \frac{2}{3} \quad \text{up quark}, \qquad Q = -\frac{1}{3} \quad \text{down quark}.$$

We assume that the electromagnetic field becomes free in the far past and far future:

$$\lim_{t\to-\infty} A^\mu(t,\mathrm{x}) = A^\mu_{\mathrm{in}}(t,\mathrm{x}), \qquad \lim_{t\to+\infty} A^\mu(t,\mathrm{x}) = A^\mu_{\mathrm{out}}(t,\mathrm{x}),$$

where A^μ_{in} and A^μ_{out} are free fields.

The S-matrix is defined by a unitary operator satisfying

$$S^\dagger = S^{-1},$$

such that

$$|\mathrm{in}\rangle = S|\mathrm{out}\rangle, \qquad |\mathrm{out}\rangle = S^{-1}|\mathrm{in}\rangle.$$

The fields are related by

$$A^\mu_{\mathrm{out}} = S^{-1} A^\mu_{\mathrm{in}} S.$$

Equivalently, if A^μ and A^μ_{in} obey the same canonical commutation relations, there exists a unitary operator $U(t)$ such that

$$A^\mu(t, \mathrm{x}) = U^{-1}(t) A^\mu_{\mathrm{in}}(t, \mathrm{x}) U(t).$$

In the asymptotic future,

$$\lim_{t \to +\infty} A^\mu(t, \mathrm{x}) = A^\mu_{\mathrm{out}}(t, \mathrm{x}), \qquad \lim_{t \to +\infty} U(t) = 1.$$

If we define the full QED Lagrangian as

$$\mathcal{L}_{\rm QED} = \mathcal{L}_0 + \mathcal{L}_{\rm int},$$

then

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi,$$

and

$$\boxed{\mathcal{L}_{\rm int} = -eQ\, \bar{\psi} \gamma^\mu \psi A_\mu.}$$

Because the interaction Lagrangian does not depend on the field derivatives, in this case

$$H_{\rm int}(x) = -\mathcal{L}_{\rm int}.$$

Taking the limits $t_0 \to -\infty$ and $t \to +\infty$, the S-matrix is

$$S = \lim_{\substack{t \to +\infty \\ t_0 \to -\infty}} U(t, t_0) = T \exp\left[-i \int d^4x \, \mathcal{H}_I(x)\right].$$

We now want to compute quantities of the form

$$\langle f | T \{H_I(x_1) \cdots H_I(x_n)\} | i \rangle,$$

where $|i\rangle$ and $|f\rangle$ are free-particle states.
The ordering of operators is fixed by T, but the interaction Hamiltonian contains creation and annihilation operators. We therefore need a systematic way to move operators around so that they can act on the initial and final states.
Wick's theorem tells us how to rewrite time-ordered products as normal-ordered products plus contractions.

Why do we need Wick's theorem?

In interacting QFT, scattering amplitudes are computed from the Dyson expansion of the S-matrix:

$$S = T\exp\left(-i\int d^4x\,\mathcal{H}_I(x)\right) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n\,T\{\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)\}.$$

At each order we must evaluate vacuum expectation values of *time-ordered* products of field operators, e.g.

$$\langle 0|\,T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}\,|0\rangle.$$

Problem: time-ordered products are awkward to handle directly, while *normal-ordered* products are easy — they annihilate the vacuum.

Wick's theorem provides the bridge between the two.

The free scalar field

The free real Klein–Gordon field is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathrm{k}}}} \left[a_{\mathrm{k}}\, e^{-ik\cdot x} + a^\dagger_{\mathrm{k}}\, e^{+ik\cdot x} \right],$$

with $k^0 = \omega_{\mathrm{k}} = \sqrt{\mathrm{k}^2 + m^2}$ and

$$[a_{\mathrm{k}}, a^\dagger_{\mathrm{k}'}] = (2\pi)^3\, \delta^{(3)}(\mathrm{k} - \mathrm{k}').$$

We split the field into positive- and negative-frequency parts:

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

$$\phi^+(x) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{a_{\mathrm{k}}}{\sqrt{2\omega_{\mathrm{k}}}}\, e^{-ik\cdot x}, \quad \phi^-(x) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{a^\dagger_{\mathrm{k}}}{\sqrt{2\omega_{\mathrm{k}}}}\, e^{+ik\cdot x}.$$

By construction, $\phi^+|0\rangle = 0$ and $\langle 0|\phi^- = 0$.

Normal ordering

The **normal-ordered product** $:\phi(x_1)\cdots\phi(x_n):$ is defined by reordering all ϕ^- (creation parts) to the *left* of all ϕ^+ (annihilation parts), as if they commuted.

Example, for two fields:

$$:\phi(x)\phi(y): = \phi^-(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^+(x)\phi^+(y).$$

Crucial property: the vacuum expectation value of any non-trivial normal-ordered product vanishes,

$$\langle 0| :\phi(x_1)\cdots\phi(x_n): |0\rangle = 0 \quad (n \geq 1).$$

Each term has at least one annihilation operator on the right or one creation operator on the left.

Time ordering

The **time-ordered product** of two scalar fields is

$$T\{\phi(x)\phi(y)\} = \begin{cases} \phi(x)\phi(y), & x^0 > y^0, \\ \phi(y)\phi(x), & x^0 < y^0. \end{cases}$$

Equivalently,

$$T\{\phi(x)\phi(y)\} = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x).$$

For n fields, T places them in order of *decreasing* time from left to right. For scalar (bosonic) fields, no extra signs appear; this differs from the fermionic case, where every transposition picks up a minus sign.

The contraction

The **contraction** of two scalar fields is defined as

$$\overbrace{\phi(x)\phi(y)} \equiv T\{\phi(x)\phi(y)\} - :\phi(x)\phi(y): .$$

The contraction is a c-number (proportional to the identity operator), not an operator. We can therefore evaluate it by taking the vacuum expectation value:

$$\overbrace{\phi(x)\phi(y)} = \langle 0| T\{\phi(x)\phi(y)\} |0\rangle = D_F(x-y),$$

where $D_F(x-y)$ is the Feynman propagator,

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}.$$

Wick's theorem for two fields

We want to express $T\{\phi(x)\phi(y)\}$ in terms of $:\phi\phi:$ plus contractions. Assume $x^0 > y^0$, so $T\{\phi(x)\phi(y)\} = \phi(x)\phi(y)$.

Split each field:

$$\phi(x)\phi(y) = \left[\phi^+(x) + \phi^-(x)\right]\left[\phi^+(y) + \phi^-(y)\right].$$

Of the four resulting terms, three are already normal-ordered. Only $\phi^+(x)\phi^-(y)$ is not. Reorder it using the commutator:

$$\phi^+(x)\phi^-(y) = \phi^-(y)\phi^+(x) + [\phi^+(x), \phi^-(y)].$$

The commutator is a c-number.

Wick's theorem for two fields

Collecting terms,

$$\phi(x)\phi(y) \; = :\phi(x)\phi(y): \; + [\phi^+(x), \phi^-(y)].$$

Taking the vacuum expectation value of both sides, the normal-ordered piece vanishes, leaving

$$[\phi^+(x), \phi^-(y)] = \langle 0|\phi(x)\phi(y)|0\rangle = D_F(x-y) \qquad (x^0 > y^0).$$

A symmetric argument handles $x^0 < y^0$, giving the propagator in either order.

Result (Wick's theorem for two fields):

$$\boxed{T\{\phi(x)\phi(y)\} \; = :\phi(x)\phi(y): \; + \overbrace{\phi(x)\phi(y)}}$$

with $\overbrace{\phi(x)\phi(y)} = D_F(x-y)$.

Wick's theorem

For any collection of scalar fields

$$\phi_1 = \phi(x_1), \qquad \phi_2 = \phi(x_2), \qquad \dots, \qquad \phi_n = \phi(x_n),$$

Wick's theorem states that

$$\boxed{T(\phi_1\phi_2\cdots\phi_n) =: \phi_1\phi_2\cdots\phi_n : + \text{all possible contractions}}$$

More explicitly,

$$T(\phi_1\phi_2\cdots\phi_n) =: \phi_1\phi_2\cdots\phi_n : + \sum_{\text{single contractions}} : \Delta_F(x_i - x_j)\, \phi_1\cdots\widehat{\phi_i}\cdots\widehat{\phi_j}\cdots\phi_n : + \cdots\,,$$

where the hats mean that the contracted fields are omitted.

For $n = 4$, we have

$$\begin{aligned}
T(\phi_1\phi_2\phi_3\phi_4) = \; & :\phi_1\phi_2\phi_3\phi_4: \\
& + \Delta_F(x_1 - x_2) :\phi_3\phi_4: + \Delta_F(x_1 - x_3) :\phi_2\phi_4: \\
& + \Delta_F(x_1 - x_4) :\phi_2\phi_3: + \Delta_F(x_2 - x_3) :\phi_1\phi_4: \\
& + \Delta_F(x_2 - x_4) :\phi_1\phi_3: + \Delta_F(x_3 - x_4) :\phi_1\phi_2: \\
& + \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) \\
& + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \\
& + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3).
\end{aligned}$$

The last three terms are the fully contracted terms.

Example: nucleon scattering

Let us go back to the theory with a nucleon field ψ and a meson field ϕ. We consider elastic nucleon–nucleon scattering:

$$\psi + \psi \longrightarrow \psi + \psi.$$

The initial and final states are

$$|i\rangle = \sqrt{2\omega_{\mathrm{p}_1}}\sqrt{2\omega_{\mathrm{p}_2}}\, b^\dagger_{\mathrm{p}_1} b^\dagger_{\mathrm{p}_2}|0\rangle \equiv |\mathrm{p}_1, \mathrm{p}_2\rangle,$$

$$|f\rangle = \sqrt{2\omega_{\mathrm{p}'_1}}\sqrt{2\omega_{\mathrm{p}'_2}}\, b^\dagger_{\mathrm{p}'_1} b^\dagger_{\mathrm{p}'_2}|0\rangle \equiv |\mathrm{p}'_1, \mathrm{p}'_2\rangle.$$

We want to compute

$$\langle f|(S-1)|i\rangle.$$

The first non-trivial contribution is second order in g:

$$\frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2\; T\Big\{\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\,\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\Big\}\,.$$

Using Wick's theorem, one relevant term is

$$:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):\;\; \contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2).$$

The contraction of the two meson fields gives the Feynman propagator:

$$\contraction{}{\phi}{(x_1)}{\phi}\phi(x_1)\phi(x_2) = \Delta_F(x_1 - x_2).$$

This term contributes to scattering because the two ψ fields annihilate the incoming nucleons, while the two $\psi^\dagger$ fields create the outgoing nucleons. Other orderings of the ψ and $\psi^\dagger$ operators give zero contributions for this matrix element.

Second-order amplitude: setup

From the Dyson expansion, the S-matrix at second order in the Yukawa coupling $\mathcal{L}_{\text{int}} = -g\,\psi^\dagger\psi\,\phi$ is

$$S^{(2)} = \frac{(-ig)^2}{2!}\int d^4x_1\,d^4x_2\;T\Big\{:\psi^\dagger\psi\phi:_{x_1}\,:\psi^\dagger\psi\phi:_{x_2}\Big\}\,.$$

For nucleon–nucleon scattering $N(p_1)N(p_2) \to N(p_1')N(p_2')$, the relevant Wick contraction connects the two meson fields, leaving the nucleon fields to act on the external states:

$$T^{(2)} = \frac{(-ig)^2}{2}\int d^4x_1\,d^4x_2\;\phi(x_1)\phi(x_2)\;\;:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):$$

Meson contraction: the propagator

The contraction of the two scalar fields is the Feynman propagator:

$$\phi(x_1)\phi(x_2) \;=\; \langle 0|\,T\{\phi(x_1)\phi(x_2)\}|0\rangle \;=\; D_F(x_1 - x_2),$$

with momentum-space representation

$$D_F(x_1 - x_2) \;=\; \int \frac{d^4 u}{(2\pi)^4}\,\frac{i\,e^{-iu\cdot(x_1 - x_2)}}{u^2 - \mu^2 + i\epsilon},$$

where μ is the meson mass and u is the exchanged four-momentum.

The amplitude becomes

$$T^{(2)} = \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2 \int \frac{d^4 u}{(2\pi)^4}\,\frac{i\,e^{-iu\cdot(x_1 - x_2)}}{u^2 - \mu^2 + i\epsilon}\,\mathcal{M}(x_1, x_2),$$

with $\mathcal{M}(x_1, x_2) = \langle p_1' p_2'|\,:\psi^\dagger\psi\psi^\dagger\psi:\,|p_1 p_2\rangle$.

Nucleon matrix element

Since $:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):=\psi^\dagger(x_1)\psi^\dagger(x_2)\psi(x_1)\psi(x_2)$, the two ψ's annihilate the in-state into the vacuum, so

$$\mathcal{M}(x_1, x_2) = \langle p_1' p_2' | \psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1 p_2\rangle .$$

Using $\langle 0|\psi(x)|p\rangle = e^{-ip\cdot x}$ and $\langle p|\psi^\dagger(x)|0\rangle = e^{ip\cdot x}$, with the two ways of pairing identical bosons,

$$\langle 0|\psi(x_1)\psi(x_2)|p_1 p_2\rangle = e^{-ip_1\cdot x_1 - ip_2\cdot x_2} + e^{-ip_1\cdot x_2 - ip_2\cdot x_1},$$

$$\langle p_1' p_2'|\psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle = e^{ip_1'\cdot x_1 + ip_2'\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1}.$$

Combining the four terms

Multiplying out, $\mathcal{M}(x_1, x_2)$ contains four exponentials:

$$\mathcal{M} = e^{i(p_1'-p_1)\cdot x_1}\, e^{i(p_2'-p_2)\cdot x_2} + e^{i(p_1'-p_2)\cdot x_1}\, e^{i(p_2'-p_1)\cdot x_2} \\ + e^{i(p_2'-p_1)\cdot x_1}\, e^{i(p_1'-p_2)\cdot x_2} + e^{i(p_2'-p_2)\cdot x_1}\, e^{i(p_1'-p_1)\cdot x_2}.$$

The integrand is symmetric under $x_1 \leftrightarrow x_2$ (the propagator depends only on $x_1 - x_2$ and $u \to -u$ is a relabelling). Therefore the 1st and 4th terms give equal contributions, as do the 2nd and 3rd. This doubling absorbs the $1/2!$ from the Dyson expansion:

$$T^{(2)} = (-ig)^2 \int d^4x_1\, d^4x_2 \int \frac{d^4u}{(2\pi)^4}\, \frac{i\, e^{-iu\cdot(x_1-x_2)}}{u^2-\mu^2+i\epsilon} \left[e^{i(p_1'-p_1)\cdot x_1} e^{i(p_2'-p_2)\cdot x_2} + (x_1 \leftrightarrow x_2) \right].$$

Final result: t- and u-channel exchange

Performing the u integral with one delta in each term, and factoring out overall momentum conservation $(2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)$:

$$T^{(2)} = (-ig)^2\,(2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)\left[\frac{i}{t - \mu^2 + i\epsilon} + \frac{i}{u - \mu^2 + i\epsilon}\right],$$

with the Mandelstam variables

$$t = (p_1' - p_1)^2, \qquad u = (p_1' - p_2)^2.$$

Interpretation. The two terms are the t- and u-channel diagrams for identical-nucleon scattering, with a single meson of mass μ exchanged. Their sum implements Bose symmetry of the two-nucleon final state.

Performing the u-integral gives

$$\langle f|S-1|i\rangle = i(-ig)^2(2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2)\left[\frac{1}{(p_1'-p_1)^2-\mu^2+i\epsilon}+\frac{1}{(p_1'-p_2)^2-\mu^2+i\epsilon}\right]$$

For physical external particles the $i\epsilon$ prescription can often be dropped, since the denominators do not vanish. It is, however, essential for internal propagators in general.

Some processes vanish because there are no allowed contractions:

$$\psi\psi \to \overline{\psi}\,\overline{\psi} \qquad \text{is forbidden by charge conservation.}$$

For processes such as

$$\psi\phi \to \phi\psi,$$

Wick's theorem gives non-zero contractions, for example

$$\psi^\dagger(x_1)\phi(x_1)\psi(x_2)\phi(x_2) \quad \rightsquigarrow \quad \psi(x_2)\psi^\dagger(x_1)\,\Delta_F(x_1-x_2).$$

Feynman rules for scalar Yukawa theory

Position-space rules

- Draw an external line for each particle in the initial state $|i\rangle$ and each particle in the final state $|f\rangle$.
- Use different line styles for mesons and nucleons:

$$\text{dotted line: } \phi, \qquad \text{solid line: } \psi.$$

- Assign a momentum to each external line.
- Add an arrow to solid lines to indicate charge flow:

$$\psi : \ \longrightarrow, \qquad \overline{\psi} : \ \longleftarrow .$$

- Join the external lines using allowed vertices.

For scalar Yukawa theory,

$$\mathcal{L}_{\text{int}} = -g\,\psi^\dagger\psi\phi,$$

the allowed vertex contains one ϕ, one ψ, and one $\psi^\dagger$.

Yukawa vertex:

$$\psi^\dagger\psi\phi$$

Momentum-space rules

- Assign a momentum k to each internal line.
- At each vertex, write a momentum-conservation factor

$$(-ig)(2\pi)^4\delta^{(4)}\left(\sum_i k_i\right),$$

where the sum is over all momenta flowing into the vertex.

- For each internal dotted line corresponding to a ϕ particle with momentum k, write

$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}.$$

- For each internal solid line corresponding to a ψ particle, write the same factor with the meson mass μ replaced by the nucleon mass M:

$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i\epsilon}.$$

Feynman rules and nucleon scattering revisited

Nucleon–nucleon scattering

At tree level, nucleon scattering receives two contributions, corresponding to the two possible ways of exchanging a meson between the external nucleon lines. If the exchanged meson has mass μ, then

$$\langle f|S-1|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_1'+p_2'-p_1-p_2),$$

where p_i are incoming momenta and p_i' are outgoing momenta.
For identical nucleons,

$$i\mathcal{A}_{fi} = (-ig)^2\left[\frac{i}{(p_1'-p_1)^2-\mu^2+i\epsilon}+\frac{i}{(p_1'-p_2)^2-\mu^2+i\epsilon}\right].$$

The internal exchanged particle is off-shell:

$$k^2 \neq \mu^2,$$

whereas external particles are on-shell:

$$p_i^2 = M^2, \qquad p_i'^2 = M^2.$$

Definition of the scattering amplitude

In general we define

$$\boxed{\langle f|S-1|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_f - p_i)}$$

where p_i and p_f are the total incoming and outgoing four-momenta.

Feynman rules for $i\mathcal{A}_{fi}$

- Draw all possible diagrams with the appropriate external legs.
- Impose four-momentum conservation at each vertex.
- Write a factor
$$(-ig)$$
for each Yukawa vertex.
- For each internal line, write down the corresponding propagator. For an internal meson line:
$$\frac{i}{k^2 - \mu^2 + i\epsilon}.$$
For an internal nucleon line:
$$\frac{i}{k^2 - M^2 + i\epsilon}.$$
- Integrate over each undetermined loop momentum:
$$\int \frac{d^4 k}{(2\pi)^4}.$$

Example: nucleon–meson scattering

For

$$\psi + \phi \longrightarrow \psi + \phi,$$

with incoming momenta p, k and outgoing momenta p', k', the tree-level amplitude is

$$i\mathcal{A} = (-ig)^2 \left[\frac{i}{(p+k)^2 - M^2 + i\epsilon} + \frac{i}{(p-k')^2 - M^2 + i\epsilon} \right].$$

More examples

Example 2: nucleon–anti-nucleon scattering
For

$$\psi + \overline{\psi} \longrightarrow \psi + \overline{\psi},$$

there are two tree-level contributions. The amplitude has the schematic form

$$i\mathcal{A} = (-ig)^2 \left[\frac{i}{(p_1' - p_1)^2 - \mu^2 + i\epsilon} + \frac{i}{(p_1 + p_2)^2 - \mu^2 + i\epsilon} \right].$$

The second term corresponds to annihilation into an intermediate meson and then recreation of the pair. If the exchanged momentum satisfies

$$(p_1 + p_2)^2 \simeq \mu^2,$$

the propagator becomes resonant.

Example 3: meson–meson scattering
For

$$\phi + \phi \longrightarrow \phi + \phi,$$

the first contribution arises at fourth order in the scalar Yukawa coupling:

$$(-ig)^4 \int \frac{d^4u}{(2\pi)^4} \frac{i}{u^2 - M^2 + i\epsilon} \frac{i}{(u + p_1 - p_1')^2 - M^2 + i\epsilon}$$

$$\times \frac{i}{(u + p_1 + p_2)^2 - M^2 + i\epsilon} \frac{i}{(u + p_1 - p_2')^2 - M^2 + i\epsilon}.$$

This is a one-loop process with virtual nucleons circulating in the loop.

QED Feynman Rules

The interaction Lagrangian is

$$\mathcal{L}_{\rm int} = -e\,\overline{\psi}\gamma^\mu\psi A_\mu .$$

The basic QED vertex gives

$$\boxed{-ie\gamma^\mu}$$

or, for a particle with charge Qe,

$$\boxed{-ieQ\gamma^\mu .}$$

Internal propagators:

$$\text{Photon:} \qquad \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

$$\text{Fermion:} \qquad \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

External lines are represented by wave functions:

$$\begin{aligned} \text{incoming fermion} &\quad : \quad u(p), \\ \text{outgoing fermion} &\quad : \quad \overline{u}(p), \\ \text{incoming anti-fermion} &\quad : \quad \overline{v}(p), \\ \text{outgoing anti-fermion} &\quad : \quad v(p), \end{aligned}$$

and for photons:

$$\epsilon_\mu(k) \qquad \text{or} \qquad \epsilon^*_\mu(k),$$

depending on whether the photon is incoming or outgoing.

QED Feynman rules: signs and examples

Continuing the QED Feynman rules:

1. For every loop momentum, include an integration
$$\int \frac{d^4\ell}{(2\pi)^4}.$$
2. Include a minus sign for every closed fermion loop:
$$\text{closed fermion loop} \quad \longrightarrow \quad (-1).$$
3. Include the appropriate relative minus signs coming from the interchange of identical fermions.

For example, for electron–electron scattering,
$$e^- e^- \to e^- e^-,$$
there are two diagrams. Since the final-state electrons are identical, the two contributions enter with a relative minus sign:
$$i\mathcal{M} = i\mathcal{M}_t - i\mathcal{M}_u.$$