From LSZ Reduction to Feynman Diagrams

Relating the *S*-matrix to Correlation Functions

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What LSZ (Lehmann–Symanzik–Zimmermann) Does

The LSZ reduction formula relates two central objects: S-matrix elements and time-ordered correlation functions.

A scattering amplitude has the form

$$\mathcal{A}_{fi} = \langle f, \mathrm{out} | i, \mathrm{in} \rangle \,.$$

A correlation function has the form

$$G_n(x_1, \ldots, x_n) = \langle \Omega | \mathrm{T}\{\phi(x_1) \cdots \phi(x_n)\} | \Omega \rangle \,.$$

LSZ says:

S-matrix element = on-shell amputated connected Green's function.

Motivation

LSZ reduction provides a relation between the S-matrix and the correlation functions.

- Let us start with an initial state.
- The operator $a^{\dagger}_{\vec{k}}$ creates a one-particle state when acting on the vacuum $|0\rangle$.
- But this will *not* be a number eigenstate of the interacting theory.
- Hence it cannot be used for building initial and final states in a scattering process.

Time-dependent operators

In a general Fourier expansion of the Heisenberg field ϕ (we will be working with scalar fields to start with), there is a time dependence:

$$\phi(\vec{x}, t) = \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega_k}} \left[a_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}(t)\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

The creation and annihilation operators $a^\dagger(t)$ and $a(t)$ acquire a **time dependence**.

But when we define asymptotic states, we know how the operators work.

Asymptotic States

Asymptotic one-particle states:

$$|k\rangle = \sqrt{2\omega_k}\, a^\dagger_{\vec{k}}(\pm\infty)\,|0\rangle$$

- $-\infty \;\rightarrow\;$ initial states
- $+\infty \;\rightarrow\;$ final states

With the property

$$a_{\vec{k}}(\pm\infty)\,|0\rangle = |0\rangle$$

and normalization

$$\left\langle k \middle| k' \right\rangle = (2\pi)^3\, 2\omega_k\, \delta^{(3)}(\vec{k} - \vec{k}')$$

Wave Packets

In order for the plane waves of different particles **not to interfere**, we build a **wave packet**:

$$\tilde{a}_j^\dagger(t) = \int \frac{d^3k}{(2\pi)^3} \, f(\vec{k}, \vec{k}_j) \, a_{\vec{k}}^\dagger(t)$$

with a Gaussian profile

$$f(\vec{k}, \vec{k}_j) = \frac{(2\pi)^{3/4}}{\sigma^{3/2}} \exp\left[-\frac{(\vec{k} - \vec{k}_j)^2}{4\sigma^2} \right]$$

Normalized such that

$$\langle 0 | \, \tilde{a}_j \, \tilde{a}_j^\dagger \, | 0 \rangle = 1$$

A $2 \to 2$ Scattering Process

Consider a $2 \to 2$ scattering process. The initial and final states are

$$|i\rangle = \tilde{a}_1^\dagger(-\infty)\,\tilde{a}_2^\dagger(-\infty)\,|0\rangle$$

$$\langle f| = \langle 0|\;\tilde{a}_3(+\infty)\,\tilde{a}_4(+\infty)$$

We now need to relate the scattering amplitude to a correlation function:

$$\langle f|i\rangle \;\longrightarrow\; \langle 0|\;\phi\,\phi\,\phi\,\phi\;|0\rangle$$

Inverting the Mode Expansion

From the Fourier expansion of ϕ, we can extract the creation operator:

$$a_{\vec{k}}^\dagger(t \to \pm\infty) = -\frac{1}{\sqrt{2\omega_k}} \int d^3x\, e^{-ik\cdot x}\, \overleftrightarrow{\partial_0}\, \phi(x)\,\Bigg|_{t\to\pm\infty}$$

where $A \overleftrightarrow{\partial_0} B \equiv A\,(\partial_0 B) - (\partial_0 A)\,B$.

Relating in- and out-Operators

To rewrite a scattering amplitude, it is convenient to express $\tilde{a}_j^\dagger(-\infty)$ in terms of $\tilde{a}_j^\dagger(+\infty)$.
Using the fundamental theorem of calculus:

$$\tilde{a}_j^\dagger(+\infty) - \tilde{a}_j^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt\, \partial_0\, \tilde{a}_j^\dagger(t)$$

Substituting the wave-packet definition:

$$a^\dagger(+\infty) - a^\dagger(-\infty) = \int \frac{d^3k}{(2\pi)^3}\, f(\vec{k}, \vec{k}_j) \int d^4x\, \frac{-i}{\sqrt{2\omega_k}}\, \partial_0 \left[e^{-ik\cdot x}\, \overleftrightarrow{\partial_0}\, \phi(x) \right]$$

Expanding the bracket:

$$\partial_0 \left[e^{-ik\cdot x}\, \overleftrightarrow{\partial_0}\, \phi(x) \right] = e^{-ik\cdot x}\, \partial_0^2 \phi(x) - \phi(x)\, \partial_0^2\, e^{-ik\cdot x}$$

From Time Derivatives to Klein–Gordon

Because $k^2 = m^2$ on-shell,

$$\partial_0^2 \, e^{-ik\cdot x} = \left(\nabla^2 - m^2\right) e^{-ik\cdot x}$$

Integrating by parts (the boundary terms drop for wave packets),

$$\cdots = \int \frac{d^3k}{(2\pi)^3} f(\vec{k}, \vec{k}_j) \int d^4x \, \frac{-i}{\sqrt{2\omega_k}} \, e^{-ik\cdot x} \left[\Box + m^2\right] \phi(x)$$

Key point: this operator annihilates free fields, but *not* interacting ones — that is exactly what produces non-trivial scattering.

Reducing One External Particle: Starting Point

Let us see in detail what we do with this relation between in and out asymptotic operators. Consider a $2 \to 2$ scattering amplitude:

$$\mathcal{A} = \langle p_3, p_4, \mathrm{out} | p_1, p_2, \mathrm{in} \rangle$$

Write the asymptotic states using creation/annihilation operators:

$$|p_1, p_2, \mathrm{in}\rangle = a^\dagger_{\mathrm{in}}(p_1) a^\dagger_{\mathrm{in}}(p_2) |0\rangle$$

$$\langle p_3, p_4, \mathrm{out}| = \langle 0 | a_{\mathrm{out}}(p_3) a_{\mathrm{out}}(p_4)$$

Therefore

$$\boxed{\mathcal{A} = \langle 0 | a_{\mathrm{out}}(p_3) a_{\mathrm{out}}(p_4) a^\dagger_{\mathrm{in}}(p_1) a^\dagger_{\mathrm{in}}(p_2) |0\rangle}$$

Goal of LSZ:

replace asymptotic operators by interacting fields

The LSZ Identity for One Operator

The LSZ relation is

$$a^\dagger_{\rm out}(p) - a^\dagger_{\rm in}(p) = iZ^{-1/2}\int d^4x\, e^{-ipx}(\Box + m^2)\phi(x)$$

The factor $Z^{-1/2}$ corrects this normalisation so that the creation/annihilation operators produce correctly normalised one-particle states. Rearrange:

$$\boxed{a^\dagger_{\rm in}(p) = a^\dagger_{\rm out}(p) - iZ^{-1/2}\int d^4x\, e^{-ipx}(\Box + m^2)\phi(x)}$$

Interpretation:

- $a^\dagger_{\rm in}$: incoming particle operator
- $a^\dagger_{\rm out}$: outgoing particle operator
- the integral term measures the effect of interactions

If there were no interactions:

$$(\Box + m^2)\phi = 0$$

Insert the Identity into the Amplitude

Start from

$$\mathcal{A} = \langle 0 | a_{\rm out}(p_3) a_{\rm out}(p_4) a_{\rm in}^\dagger(p_1) a_{\rm in}^\dagger(p_2) | 0 \rangle$$

Replace only the first incoming operator:

$$a_{\rm in}^\dagger(p_1) = a_{\rm out}^\dagger(p_1) - iZ^{-1/2} \int d^4x \, e^{-ip_1 x} (\Box + m^2) \phi(x)$$

Substitute directly:

$$\mathcal{A} = \langle 0 | a_{\rm out} a_{\rm out} \left[a_{\rm out}^\dagger - iZ^{-1/2} \int d^4x \, e^{-ipx} (\Box + m^2) \phi(x) \right] a_{\rm in}^\dagger | 0 \rangle$$

Now distribute the operators.

After Expanding the Bracket

We obtain two terms:

$$\mathcal{A} = \mathcal{A}_{\rm disc} + \mathcal{A}_{\rm int}$$

with

$$\mathcal{A}_{\rm disc} = \langle 0 | a_{\rm out}\, a_{\rm out}\, a^\dagger_{\rm out}\, a^\dagger_{\rm in} |0\rangle$$

and

$$\mathcal{A}_{\rm int} = -iZ^{-1/2} \int d^4x\, e^{-ipx} (\Box + m^2) \langle 0 | a_{\rm out}\, a_{\rm out}\, \phi(x) a^\dagger_{\rm in} |0\rangle$$

Interpretation:

- first term = disconnected/free propagation
- second term = genuine interaction contribution

The interacting physics is now encoded in the field insertion $\phi(x)$.

Analysing the Disconnected Term

Consider

$$\mathcal{A}_{\rm disc} = \langle 0 | a_{\rm out}(p_3) a_{\rm out}(p_4) a^\dagger_{\rm out}(p_1) a^\dagger_{\rm in}(p_2) | 0 \rangle$$

Use the commutation relation:

$$[a(p), a^\dagger(q)] = (2\omega_p)(2\pi)^3 \delta^3(\mathrm{p} - \mathrm{q})$$

Commute $a^\dagger_{\rm out}(p_1)$ leftward:

$$a_{\rm out}(p_3) a^\dagger_{\rm out}(p_1) = a^\dagger_{\rm out}(p_1) a_{\rm out}(p_3) + [a_{\rm out}(p_3), a^\dagger_{\rm out}(p_1)]$$

The commutator produces delta functions.
These correspond to particles propagating freely without interaction.

Vacuum Annihilation

Recall:

$$a_{\text{out}}|0\rangle = 0$$

and

$$\langle 0|a^{\dagger}_{\text{out}} = 0$$

Therefore many reordered terms vanish after commuting.
Only terms generated by commutators survive.
Thus the disconnected term reduces to products of delta functions:

$$\delta^3(\mathrm{p}_i - \mathrm{p}_j)$$

These are non-scattering contributions.
LSZ is mainly interested in the connected interacting part.

Focus on the Interacting Term

The important term is

$$\mathcal{A}_{\rm int} = -iZ^{-1/2} \int d^4x \, e^{-ipx} (\Box + m^2) \langle 0 | a_{\rm out} a_{\rm out} \phi(x) a_{\rm in}^\dagger | 0 \rangle$$

Notice what happened:

One external particle operator has been replaced by:

$$(\Box + m^2)\phi(x)$$

This is the essence of LSZ reduction.

We now repeat the procedure for the remaining asymptotic operators.

Reducing the Next Operator

Apply the same identity again:

$$a_{\rm out}(p) = a_{\rm in}(p) + iZ^{-1/2} \int d^4y\, e^{ipy} (\Box + m^2)\phi(y)$$

Each reduction introduces:

- one spacetime integral
- one Fourier factor $e^{\pm ipx}$
- one Klein–Gordon operator
- one field insertion $\phi(x)$

After reducing all external operators:

$$\mathcal{A} \sim \prod_i \int d^4x_i\, e^{\pm ip_i x_i} (\Box_{x_i} + m^2) \langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

Where Time Ordering Comes From

Repeated commutations generate step functions:

$$\theta(x^0 - y^0)$$

For two fields:

$$T\phi(x)\phi(y) = \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)$$

Thus the operator algebra reorganizes naturally into:

$$\boxed{\langle 0| T\phi(x_1)\cdots\phi(x_n)|0\rangle}$$

which is precisely the Green function computed using Feynman diagrams.

Conceptual Picture of One Reduction Step

The LSZ reduction step does:

$$\boxed{a^\dagger_{\mathrm{in/out}} \quad \longrightarrow \quad (\Box + m^2)\phi(x)}$$

Meaning:

- external asymptotic particles are converted into field insertions,
- operator algebra converts matrix elements into vacuum correlators,
- the dynamics becomes encoded entirely in Green functions.

After all reductions:

$$\boxed{\text{Scattering amplitude} = \text{Amputated time-ordered correlator}}$$

Narrow Wave-Packet Limit

If we assume the wave packets to be **narrow**,

$$f(\vec{k}, \vec{k}_j) \simeq (2\pi)^3 \sqrt{2\omega_{k_j}}\, \delta^{(3)}(\vec{k} - \vec{k}_j)$$

so that the $\tilde{a}$ are normalised as the a states.

LSZ Master Formula for Scalars

$$\langle f | i \rangle = \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\; e^{-ik_1\cdot x_1}\, e^{-ik_2\cdot x_2}\, e^{+ik_3\cdot x_3}\, e^{+ik_4\cdot x_4}$$

$$\times (\Box_{x_1} + m^2)(\Box_{x_2} + m^2)(\Box_{x_3} + m^2)(\Box_{x_4} + m^2)$$

$$\times \langle 0|\; T\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\; |0\rangle$$

Strategy: use the relation between a and ϕ, and the time ordering T to give the correct $\pm\infty$ limits.

LSZ for Fermions

For fermionic external states (incoming spinors u, outgoing $\bar{u}$; antifermions $v, \bar{v}$), the LSZ formula reads

$$\langle \text{out}; (p_1', s_1'), \ldots, (\bar{p}_n', \bar{s}_n') \,\big|\, (p_1, s_1), \ldots, (\bar{p}_n, \bar{s}_n); \text{ in} \rangle =$$

$$= \text{disc. terms} + \big(-iZ_2^{-1/2}\big)^n \big(-iZ_2^{-1/2}\big)^{n'} \int d^4x_1 \cdots d^4x_n \, d^4y_1 \cdots d^4y_{n'}$$

$$\times \; e^{-i\sum p_i \cdot x_i} \, e^{-i\sum \bar{p}_i \cdot y_i} \, e^{+i\sum p_i' \cdot x_i'} \, e^{+i\sum \bar{p}_i' \cdot y_i'}$$

$$\times \; \bar{u}(p_1', s_1')(i\partial\!\!\!/_{x_1'} - m) \cdots \bar{v}(\bar{p}_1, \bar{s}_1)(i\partial\!\!\!/_{y_1} - m)$$

$$\times \; \langle 0| \; T\left[\cdots \bar{\psi}(y_i') \cdots \psi(x_i') \, \bar{\psi}(x_i) \cdots \psi(y_i) \cdots \right] |0\rangle$$

$$\times \; (-i\overleftarrow{\partial\!\!\!/}_{x_1} - m)\, u(p_1, s_1) \cdots (+i\overleftarrow{\partial\!\!\!/}_{y_1'} - m)\, v(\bar{p}_1', \bar{s}_1')$$

LSZ for Fermions

- Each external fermion leg contributes a Dirac operator $(i\partial\!\!\!/ - m)$ acting on the corresponding spacetime point.
- Each leg comes with the appropriate spinor $u, \bar{u}, v, \bar{v}$.
- The wavefunction renormalization $Z_2^{-1/2}$ replaces the scalar $Z^{-1/2}$.
- The exponentials carry the standard $e^{-ip\cdot x}$ (incoming) and $e^{+ip\cdot x}$ (outgoing) phases.
- "Disconnected terms" correspond to non-scattering pieces and are dropped when computing the connected S-matrix.

Scalar Versus Fermionic LSZ

	Scalar field	Dirac field
Field	$\phi(x)$	$\psi(x), \bar{\psi}(x)$
Free equation	$(\Box + m^2)\phi = 0$	$(i\partial\!\!\!/ - m)\psi = 0$
Inverse propagator	$p^2 - m^2$	$p\!\!\!/ - m$
External wavefunction	1	$u, \bar{u}, v, \bar{v}$
LSZ action	Amputate scalar legs	Amputate spinor legs

In both cases,

S-matrix = on-shell amputated connected correlator.

From $G(x_1, \ldots, x_n)$ to Feynman Diagrams

Goal: evaluate the time-ordered correlator

$$G(x_1, \ldots, x_n) = \langle 0| \; T\,\phi(x_1)\cdots\phi(x_n)\; |0\rangle$$

perturbatively, by expanding in powers of the interaction.

Idea: pass from Heisenberg-picture fields $\phi(x)$ to interaction-picture fields $\phi_{\text{in}}(x)$, where the time evolution is governed only by the interaction Hamiltonian.

Setting the Stage: The Interaction Picture

Split the Hamiltonian:

$$H = H_0 + H_{\text{int}}$$

- H_0: free part, exactly solvable
- H_{int}: perturbation

Interaction-picture definitions:

$$|\psi_I(t)\rangle = e^{iH_0 t}|\psi_S(t)\rangle, \qquad \mathcal{O}_I(t) = e^{iH_0 t}\,\mathcal{O}_S\,e^{-iH_0 t}$$

Schrödinger-like equation for states:

$$i\frac{\partial}{\partial t}|\psi_I(t)\rangle = H_I(t)\,|\psi_I(t)\rangle$$

with

$$H_I(t) = e^{iH_0 t}\,H_{\text{int}}\,e^{-iH_0 t}$$

The Evolution Operator

Define $U(t, t_0)$ so that $|\psi_I(t)\rangle = U(t, t_0)|\psi_I(t_0)\rangle$:

$$i\frac{\partial}{\partial t}U(t, t_0) = H_I(t)\,U(t, t_0), \qquad U(t_0, t_0) = 1$$

Integral form:

$$U(t, t_0) = 1 - i\int_{t_0}^{t} dt_1\; H_I(t_1)\,U(t_1, t_0)$$

Catch: $H_I(t_1)$ and $H_I(t_2)$ do *not* commute in general.

- Cannot simply write $U = \exp\left[-i\int H_I\,dt'\right]$
- Need to iterate the integral equation

Iterative Solution

Substitute the equation into itself repeatedly:

$$U(t, t_0) = 1 + (-i)\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)H_I(t_2) + \cdots$$

General n-th order term:

$$U^{(n)} = (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1)\cdots H_I(t_n)$$

Nested limits: $t \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq t_0$. Latest time on the left.

Dyson's Time-Ordering Trick

Define the time-ordered product:

$$T\{H_I(t_1)\,H_I(t_2)\} = \begin{cases} H_I(t_1)H_I(t_2) & t_1 > t_2 \\ H_I(t_2)H_I(t_1) & t_2 > t_1 \end{cases}$$

The nested simplex is one of $n!$ equivalent regions filling the hypercube:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)H_I(t_2) = \frac{1}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\ T\{H_I(t_1)H_I(t_2)\}$$

General n-th order:

$$U^{(n)} = \frac{(-i)^n}{n!}\int_{t_0}^{t} dt_1 \cdots dt_n\ T\{H_I(t_1)\cdots H_I(t_n)\}$$

The Dyson Formula

Dyson series

$$U(t, t_0) = T \exp\left[-i \int_{t_0}^{t} dt' \; H_I(t')\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \cdots dt_n \; T\{H_I(t_1) \cdots H_I(t_n)\}$$

Covariant form for the S-matrix ($t_0 \to -\infty$, $t \to +\infty$):

$$S = T \exp\left[-i \int d^4x \, \mathcal{H}_I(x)\right] = T \exp\left[i \int d^4x \, \mathcal{L}_I(x)\right]$$

Lorentz invariance requires microcausality: $[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0$ for spacelike $(x - y)$.

Three Pictures, One Reference Time

At a reference time t_0, all pictures coincide:

$$\phi_S = \phi_H(t_0) = \phi_I(t_0)$$

Heisenberg field:

$$\phi_H(x) = e^{iH(t-t_0)}\,\phi_S(\vec{x})\,e^{-iH(t-t_0)}$$

Interaction-picture field:

$$\phi_I(x) = e^{iH_0(t-t_0)}\,\phi_S(\vec{x})\,e^{-iH_0(t-t_0)}$$

Solve the second for ϕ_S and substitute into the first.

The Bridge Between Pictures

After substitution:

$$\phi_H(x) = e^{iH(t-t_0)}\, e^{-iH_0(t-t_0)}\, \phi_I(x)\, e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$

Define

$$U(t,t_0) \equiv e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$

which satisfies $i\partial_t U = H_I(t)U$ with $U(t_0,t_0) = 1$ — the Dyson operator!

Key relation

$$\phi_H(x) = U^\dagger(t,t_0)\, \phi_I(x)\, U(t,t_0)$$

The Dyson operator is exactly what dresses interaction-picture fields into Heisenberg fields.

Time-Ordered Products of Heisenberg Fields

Consider (with $t_1 > t_2$):

$$\phi_H(x_1)\phi_H(x_2) = U^\dagger(t_1, t_0)\,\phi_I(x_1)\,U(t_1, t_2)\,\phi_I(x_2)\,U(t_2, t_0)$$

using $U(t_a, t_b) = U(t_a, t_0)U^\dagger(t_b, t_0)$.

Extend to large times $T \to -\infty$, $T' \to +\infty$:

$$\phi_H(x_1)\phi_H(x_2) = U^\dagger(T', t_0)\,U(T', t_1)\phi_I(x_1)U(t_1, t_2)\phi_I(x_2)U(t_2, T)\,U(T, t_0)$$

The middle factor is exactly:

$$T\{\phi_I(x_1)\phi_I(x_2)\,U(T', T)\}$$

because the time-ordered exponential $U(T', T)$ distributes its H_I insertions around the fields by their time arguments.

Adiabatic Switching: Selecting $|\Omega\rangle$

Expand the free vacuum in eigenstates of full H:

$$|0\rangle = c_\Omega|\Omega\rangle + \sum_n c_n|n\rangle, \qquad E_n > E_\Omega$$

Evolve from $T \to -\infty$:

$$e^{-iH(t_0-T)}|0\rangle = c_\Omega e^{-iE_\Omega(t_0-T)}|\Omega\rangle + \sum_n c_n e^{-iE_n(t_0-T)}|n\rangle$$

$i\epsilon$ **prescription:** take $T \to -\infty(1 - i\epsilon)$.

- Phases become $e^{-iE_n|T|}e^{-\epsilon E_n|T|}$
- Excited states damped more strongly (since $E_n > E_\Omega$)
- Only $|\Omega\rangle$ survives

Vacuum Limits

Result:

$$|\Omega\rangle \propto \lim_{T \to -\infty(1-i\epsilon)} U(t_0, T)\,|0\rangle$$

$$\langle\Omega| \propto \lim_{T' \to +\infty(1-i\epsilon)} \langle 0|\,U(T', t_0)$$

Proportionality constants $\sim 1/c_\Omega$ and phases will cancel between numerator and denominator.

Physical assumption: $\langle 0|\Omega\rangle \neq 0$.

- Fails for spontaneous symmetry breaking
- Fails for confinement / non-perturbative vacua
- Path-integral / Euclidean methods sidestep this

The Gell-Mann–Low Formula

Substituting and using $\langle\Omega|\Omega\rangle = 1$ to fix the normalization:

Gell-Mann–Low formula

$$\langle\Omega| T\{\phi_H(x_1)\cdots\phi_H(x_n)\}|\Omega\rangle = \frac{\langle 0| T\{\phi_I(x_1)\cdots\phi_I(x_n)\, e^{-i\int d^4x\, \mathcal{H}_I(x)}\}|0\rangle}{\langle 0| T\{e^{-i\int d^4x\, \mathcal{H}_I(x)}\}|0\rangle}$$

Everything on the right is computable:

- Free vacuum $|0\rangle$
- Interaction-picture (free) fields ϕ_I
- Dyson series in the exponent

Expanding the Numerator

Power-series expansion:

$$\text{Num} = \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \int d^4y_1 \cdots d^4y_N \, \langle 0| T\{\phi_I(x_1) \cdots \phi_I(x_n) \, \mathcal{H}_I(y_1) \cdots \mathcal{H}_I(y_N)\}$$

Wick's theorem: a vacuum expectation value of a T-product of free fields equals the sum of all complete pairings into Feynman propagators:

$$\langle 0| T\{\phi_I(x)\phi_I(y)\}|0\rangle = D_F(x-y)$$

Each term $\Rightarrow$ a Feynman diagram:

- External points: $x_1, \ldots, x_n$
- Internal vertices: $y_1, \ldots, y_N$ (from $\mathcal{H}_I$)
- Edges: propagators from Wick contractions

Vacuum Bubble Cancellation

Every diagram in the numerator factorizes:

$$\text{(piece with external legs)} \times \text{(vacuum bubbles)}$$

Summing all numerator diagrams:

$$\text{Num} = \left[\sum \text{(diagrams with external legs)}\right] \times \langle 0| T\{e^{-i\int \mathcal{H}_I}\}|0\rangle$$

The denominator is exactly the sum of all vacuum bubbles:

$$\langle 0| T\{e^{-i\int \mathcal{H}_I}\}|0\rangle = \exp\left[\sum \text{(connected vacuum bubbles)}\right]$$

Result

$$G^{(n)}(x_1, \ldots, x_n) = \sum \text{(diagrams with external legs, no vacuum bubbles)}$$

Worked Example: Three Scalar Fields

Goal: compute

$$G^{(3)}(x_1, x_2, x_3) = \langle\Omega| T\{\phi_H(x_1)\phi_H(x_2)\phi_H(x_3)\}|\Omega\rangle$$

$$t_1 > t_2 > t_3$$

so that

$$T\{\phi_H(x_1)\phi_H(x_2)\phi_H(x_3)\} = \phi_H(x_1)\phi_H(x_2)\phi_H(x_3)$$

Why three fields? Minimal non-trivial case showing the chain structure of U's.

Step 1: Dress Each Heisenberg Field

Apply $\phi_H(x_i) = U^\dagger(t_i, t_0)\, \phi_I(x_i)\, U(t_i, t_0)$ to each:

$$\phi_H(x_1)\phi_H(x_2)\phi_H(x_3) =$$

$$U^\dagger(t_1, t_0)\phi_I(x_1)\, \underbrace{U(t_1, t_0)U^\dagger(t_2, t_0)}_{\to U(t_1, t_2)}\, \phi_I(x_2)\, \underbrace{U(t_2, t_0)U^\dagger(t_3, t_0)}_{\to U(t_2, t_3)}\, \phi_I(x_3)U(t_3, t_0)$$

Using the composition rule $U(t_a, t_0)U^\dagger(t_b, t_0) = U(t_a, t_b)$:

$$\boxed{\phi_H(x_1)\phi_H(x_2)\phi_H(x_3) = U^\dagger(t_1, t_0)\, \phi_I(x_1)\, U(t_1, t_2)\, \phi_I(x_2)\, U(t_2, t_3)\, \phi_I(x_3)\, U(t_3}$$

Each ϕ_I is sandwiched by evolution operators bridging adjacent times.

Step 2: Extend to Asymptotic Times

Introduce $T \to -\infty$ and $T' \to +\infty$. Insert $1 = U^\dagger(T', t_0)U(T', t_0)$ on the left and $1 = U^\dagger(T, t_0)U(T, t_0)$ on the right:

- Left: $U^\dagger(t_1, t_0) = U^\dagger(T', t_0)\, U(T', t_1)$
- Right: $U(t_3, t_0) = U(t_3, T)\, U(T, t_0)$

Result:

$$\phi_H(x_1)\phi_H(x_2)\phi_H(x_3) = U^\dagger(T', t_0)\, \mathcal{C}\, U(T, t_0)$$

where the central chain is:

$$\mathcal{C} = U(T', t_1)\, \phi_I(x_1)\, U(t_1, t_2)\, \phi_I(x_2)\, U(t_2, t_3)\, \phi_I(x_3)\, U(t_3, T)$$

Step 3: Recognize the Time-Ordered Exponential

Read the chain $\mathcal{C}$ left-to-right:

$$\underbrace{U(T',t_1)}_{T'\to t_1}\,\underbrace{\phi_I(x_1)}_{t_1}\,\underbrace{U(t_1,t_2)}_{t_1\to t_2}\,\underbrace{\phi_I(x_2)}_{t_2}\,\underbrace{U(t_2,t_3)}_{t_2\to t_3}\,\underbrace{\phi_I(x_3)}_{t_3}\,\underbrace{U(t_3,T)}_{t_3\to T}$$

Operators ordered from latest (T') to earliest (T) — exactly the time-ordering pattern.

Therefore,

$$\mathcal{C} = T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\,U(T',T)\}$$

The T-symbol distributes the H_I insertions of $U(T',T)$ into the gaps: $T \to t_3,\ t_3 \to t_2,\ t_2 \to t_1,\ t_1 \to T'$.

Step 4: The Substitution Equation

Putting it together:

$$\phi_H(x_1)\phi_H(x_2)\phi_H(x_3) = U^\dagger(T', t_0)\, T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\, U(T', T)\}\, U(T, t_0)$$

Sandwich between vacua using

$$|\Omega\rangle = \lim_{T} \frac{1}{\mathcal{C}_-(T)}\, U(t_0, T)|0\rangle, \quad \langle\Omega| = \lim_{T'} \frac{1}{\mathcal{C}_+^*(T')} \langle 0| U(T', t_0)$$

$$\langle\Omega| T\{\phi_H(x_1)\phi_H(x_2)\phi_H(x_3)\}|\Omega\rangle = \mathcal{N}\, \langle 0|\, U(T', t_0)U^\dagger(T', t_0)$$
$$T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\, U(T', T)\}\, U(T, t_0)U(t_0, T)\, |0\rangle$$

with $\mathcal{N} = 1/[\mathcal{C}_+^*(T')\mathcal{C}_-(T)]$ collecting phases and overlaps.

Step 5: Telescoping

The U's collapse against the vacuum-projector pieces:

On the left:

$$\langle 0|\, U(T', t_0)\, U^\dagger(T', t_0) = \langle 0|$$

On the right:

$$U(T, t_0)\, U(t_0, T)\, |0\rangle = U(T, T)\, |0\rangle = |0\rangle$$

Reference time t_0 disappears — as it must, being arbitrary. Result:

$$\langle\Omega|\, T\{\phi_H(x_1)\phi_H(x_2)\phi_H(x_3)\}|\Omega\rangle = \mathcal{N}'\, \langle 0|\, T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\, U(T', T)\}\, |0\rangle$$

Step 6: Fix the Normalization

Apply the same identity with *no* field insertions:

$$1 = \langle\Omega|\Omega\rangle = \mathcal{N}' \langle 0| T\{U(T', T)\} |0\rangle = \mathcal{N}' \langle 0| U(T', T) |0\rangle$$

Therefore:

$$\mathcal{N}' = \frac{1}{\langle 0| U(T', T) |0\rangle}$$

This denominator is precisely what cancels the unknown overlap $|c_\Omega|^2$ and vacuum energy phases.

Diagrammatically: it will remove vacuum bubbles.

The Three-Field Gell-Mann–Low Formula

Taking $T' \to +\infty(1 - i\epsilon)$, $T \to -\infty(1 - i\epsilon)$, and using $U(T', T) = T \exp\left[-i \int d^4x\, \mathcal{H}_I\right]$:

Result for three fields

$$\langle\Omega| T\{\phi_H(x_1)\phi_H(x_2)\phi_H(x_3)\}|\Omega\rangle = \frac{\langle 0| T\{\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\, e^{-i\int d^4x\, \mathcal{H}_I(x)}\}|0\rangle}{\langle 0| T\{e^{-i\int d^4x\, \mathcal{H}_I(x)}\}|0\rangle}$$

Everything on the right involves only:

- Free vacuum $|0\rangle$
- Free fields ϕ_I
- Dyson exponential of $\mathcal{H}_I$

Lessons from the Three-Field Case

1. **Chain structure is explicit:** $n-1$ intermediate $U(t_i, t_{i+1})$ factors plus two end-cap U's reaching to $\pm\infty$.
2. **T-symbol as a filler:** each gap in the field chain is filled by the time-ordered exponential restricted to that interval. The total assembles into $U(T', T)$.
3. **Symmetry restored:** though we assumed $t_1 > t_2 > t_3$, the final formula is symmetric in the three fields — the T-symbol absorbs all $3! = 6$ orderings.
4. **Minimal non-trivial vertex:**

$$G^{(3)} = G_c^{(3)} + [G_c^{(1)} G_c^{(2)} + 2 \text{ perms}] + G_c^{(1)} G_c^{(1)} G_c^{(1)}$$

If $\langle\Omega|\phi|\Omega\rangle = 0$ then $G^{(3)} = G_c^{(3)}$ automatically.

The ϕ^4 Theory

Consider real scalar ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - \mu^2\phi^2\right) - \frac{\lambda}{4!}\phi^4.$$

The interaction Lagrangian is

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{4!}\phi^4.$$

We want to calculate the diagrams contributing to the $2 \to 2$ scattering amplitude

$$q_1 + q_2 \longrightarrow p_1 + p_2.$$

The lowest non-trivial contribution is order λ.

The 2 → 2 S-Matrix Element

The relevant S-matrix element is

$$S_{fi} = \langle p_1, p_2; \mathrm{out} | q_1, q_2; \mathrm{in} \rangle \, .$$

Using LSZ, this can be written in terms of the four-point Green's function:

$$G(y_1, y_2, x_1, x_2) = \langle \Omega | T\{\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\} | \Omega \rangle \, .$$

Here:

x_1, x_2 are associated with incoming particles,

y_1, y_2 are associated with outgoing particles.

The physical scattering amplitude comes from the connected part of G.

LSZ for 2 $\rightarrow$ 2 Scattering

With external momenta q_1, q_2 incoming and p_1, p_2 outgoing, LSZ gives

$$S_{fi}^{\mathrm{conn}} = \int d^4y_1\, d^4y_2\, d^4x_1\, d^4x_2\, e^{+ip_1\cdot y_1} e^{+ip_2\cdot y_2} e^{-iq_1\cdot x_1} e^{-iq_2\cdot x_2}$$

$$\times(\Box_{y_1}+\mu^2)(\Box_{y_2}+\mu^2)(\Box_{x_1}+\mu^2)(\Box_{x_2}+\mu^2)\, G_{\mathrm{conn}}(y_1, y_2, x_1, x_2).$$

There are also disconnected terms, schematically denoted by

$$S_{fi} = S_{fi}^{\mathrm{conn}} + S_{fi}^{\mathrm{disc}}.$$

For genuine scattering, we focus on S_{fi}^{conn}.

Four-Point Function in Perturbation Theory

The four-point Green's function is

$$G(y_1, y_2, x_1, x_2) = \frac{\langle 0| \, T \left\{ \phi_I(y_1)\phi_I(y_2)\phi_I(x_1)\phi_I(x_2) e^{i\int d^4z \, \mathcal{L}_{\rm int}(z)} \right\} |0\rangle}{\langle 0| \, T \left\{ e^{i\int d^4z \, \mathcal{L}_{\rm int}(z)} \right\} |0\rangle}.$$

For

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{4!}\phi_I^4,$$

we expand

$$e^{i\int d^4z \, \mathcal{L}_{\rm int}(z)} = 1 - i\frac{\lambda}{4!}\int d^4z \, \phi_I^4(z) + O(\lambda^2).$$

The first connected contribution to $2 \to 2$ scattering appears at order λ.

Zeroth Order: No Interaction

At order λ^0,

$$G^{(0)}(y_1, y_2, x_1, x_2) = \langle 0 | T\{\phi_I(y_1)\phi_I(y_2)\phi_I(x_1)\phi_I(x_2)\} | 0 \rangle \,.$$

By Wick's theorem,

$$G^{(0)} = \Delta_F(y_1 - y_2)\Delta_F(x_1 - x_2) + \Delta_F(y_1 - x_1)\Delta_F(y_2 - x_2) + \Delta_F(y_1 - x_2)\Delta_F(y_2 - x$$

These are disconnected pairings.

They describe free propagation, not genuine interaction.

Therefore, the connected scattering amplitude starts at order λ.

First Order in λ

At first order,

$$G^{(1)}(y_1, y_2, x_1, x_2) = -\frac{i\lambda}{4!}\int d^4z\ \langle 0|\,T\{\phi_I(y_1)\phi_I(y_2)\phi_I(x_1)\phi_I(x_2)\phi_I^4(z)\}|0\rangle\,.$$

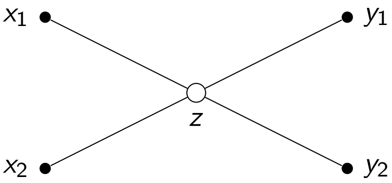
The connected part comes from contracting each external field with one of the four fields at z.

There are 4! such contractions. Therefore,

$$G^{(1)}_{\rm conn} = -i\lambda \int d^4z\, \Delta_F(y_1 - z)\Delta_F(y_2 - z)\Delta_F(x_1 - z)\Delta_F(x_2 - z).$$

Tree-Level Diagram

The first-order connected contribution is the tree-level contact diagram.



Algebraically,

$$G^{(1)}_{\text{conn}} = -i\lambda \int d^4z \, \Delta_F(y_1 - z)\Delta_F(y_2 - z)\Delta_F(x_1 - z)\Delta_F(x_2 - z).$$

The factor 4! from Wick contractions cancels the $1/4!$ in the interaction Lagrangian.

Applying LSZ to the Tree Diagram

Insert

$$G^{(1)}_{\text{conn}} = -i\lambda \int d^4z\, \Delta_F(y_1 - z)\Delta_F(y_2 - z)\Delta_F(x_1 - z)\Delta_F(x_2 - z)$$

into the LSZ formula.

Each external operator

$$\Box + \mu^2$$

acts on an external propagator. Using

$$(\Box_x + \mu^2)\Delta_F(x - z) = -i\delta^{(4)}(x - z),$$

up to convention-dependent factors of i, LSZ amputates the four external propagators.

After the x_1, x_2, y_1, y_2 integrations, only the vertex integral remains:

$$-i\lambda \int d^4z\, e^{i(p_1+p_2-q_1-q_2)\cdot z}.$$

Momentum Conservation

The remaining spacetime integral gives the momentum-conserving delta function:

$$\int d^4z\, e^{i(p_1+p_2-q_1-q_2)\cdot z} = (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2).$$

Therefore,

$$S_{fi}^{\text{conn}} = (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)\, i\mathcal{M}.$$

For ϕ^4 theory at tree level,

$$i\mathcal{M}_{\text{tree}} = -i\lambda.$$

Equivalently,

$$\mathcal{M}_{\text{tree}} = -\lambda.$$

The precise overall i-convention depends on how $S = 1 + iT$ is defined.

Next Order: One-Loop $2 \to 2$ Contribution

At order λ^2, the four-point function contains two interaction vertices:

$$G^{(2)}(y_1, y_2, x_1, x_2) = \frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^2 \int d^4 z_1\, d^4 z_2$$

$$\times \langle 0| T\{\phi_I(y_1)\phi_I(y_2)\phi_I(x_1)\phi_I(x_2)\phi_I^4(z_1)\phi_I^4(z_2)\}|0\rangle\,.$$

One important connected contribution is the bubble diagram:

$$x_1, x_2 \longrightarrow z_1 \longrightarrow z_2 \longrightarrow y_1, y_2.$$

This is the first loop correction to the $2 \to 2$ amplitude.

Bubble Contraction in Coordinate Space

One connected Wick contraction gives the structure

$$\Delta_F(x_1 - z_1)\Delta_F(x_2 - z_1)\,[\Delta_F(z_1 - z_2)]^2\,\Delta_F(y_1 - z_2)\Delta_F(y_2 - z_2).$$

There is also the contribution with the two vertices exchanged:

$$z_1 \leftrightarrow z_2.$$

Thus, schematically,

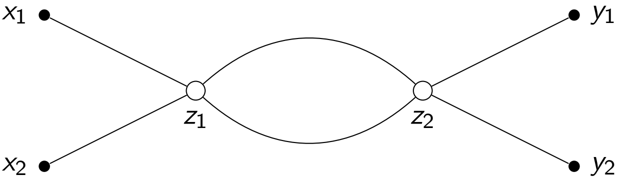
$$G^{(2)}_{\text{bubble}} \sim \frac{1}{2!}(-i\lambda)^2 \int d^4z_1\, d^4z_2\, \Delta_F(x_1 - z_1)\Delta_F(x_2 - z_1)$$

$$\times\, [\Delta_F(z_1 - z_2)]^2\, \Delta_F(y_1 - z_2)\Delta_F(y_2 - z_2) + (z_1 \leftrightarrow z_2).$$

The factor $1/2!$ comes from expanding the exponential to second order.

Bubble Diagram

The coordinate-space bubble contribution is represented by



The two internal lines between z_1 and z_2 produce the factor

$$[\Delta_F(z_1 - z_2)]^2 .$$

This is a connected one-loop correction to the four-point Green's function.

Applying LSZ to the Bubble Diagram

The LSZ operators amputate the external propagators:

$$(\Box_{x_1} + \mu^2)\Delta_F(x_1 - z_1) = -i\delta^{(4)}(x_1 - z_1),$$

$$(\Box_{x_2} + \mu^2)\Delta_F(x_2 - z_1) = -i\delta^{(4)}(x_2 - z_1),$$

$$(\Box_{y_1} + \mu^2)\Delta_F(y_1 - z_2) = -i\delta^{(4)}(y_1 - z_2),$$

$$(\Box_{y_2} + \mu^2)\Delta_F(y_2 - z_2) = -i\delta^{(4)}(y_2 - z_2).$$

After amputation, only the internal bubble remains:

$$[\Delta_F(z_1 - z_2)]^2 \, .$$

So the order-λ^2 contribution becomes an integral over the two vertex positions z_1, z_2.

Order λ^2 Contribution to S_{fi}

For the bubble channel shown above, after LSZ we obtain schematically

$$S_{fi}^{(2)} = \frac{1}{2!}(-i\lambda)^2 \int d^4z_1\, d^4z_2\, e^{-i(q_1+q_2)\cdot z_1} e^{+i(p_1+p_2)\cdot z_2}$$

$$\times \left[\Delta_F(z_1 - z_2)\right]^2 .$$

Equivalently,

$$S_{fi}^{(2)} = \frac{1}{2}(-i\lambda)^2 \int d^4z_1\, d^4z_2\, e^{i(p_1+p_2)\cdot z_2} e^{-i(q_1+q_2)\cdot z_1} \left[\Delta_F(z_1 - z_2)\right]^2 .$$

This is the coordinate-space form before extracting the momentum-conserving delta function.

Change of Variables

Define

$$P = q_1 + q_2 = p_1 + p_2.$$

Now make the change of variables

$$r = z_1 - z_2, \qquad Z = z_2.$$

Then

$$z_1 = Z + r, \qquad z_2 = Z,$$

and

$$d^4 z_1 \, d^4 z_2 = d^4 r \, d^4 Z.$$

The phase factor becomes

$$e^{-iP\cdot z_1} e^{+iP\cdot z_2} = e^{-iP\cdot(Z+r)} e^{+iP\cdot Z} = e^{-iP\cdot r}.$$

The Z-integral gives momentum conservation:

$$\int d^4 Z \, e^{i(p_1+p_2-q_1-q_2)\cdot Z} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2).$$

Bubble Integral in Coordinate Space

After the change of variables, the order-λ^2 bubble contribution becomes

$$S_{fi}^{(2)} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \frac{1}{2}(-i\lambda)^2 \int d^4r \, e^{-iP\cdot r} \, [\Delta_F(r)]^2 \, .$$

Thus the loop correction is the Fourier transform of

$$[\Delta_F(r)]^2 \, .$$

In terms of the invariant

$$s = P^2 = (q_1 + q_2)^2,$$

this is the s-channel one-loop correction.

Bubble Integral in Momentum Space

Using

$$\Delta_F(r) = \int \frac{d^4\ell}{(2\pi)^4} \frac{i\, e^{-i\ell\cdot r}}{\ell^2 - \mu^2 + i\epsilon},$$

the Fourier transform gives

$$\int d^4r\, e^{-iP\cdot r}\, [\Delta_F(r)]^2 = \int \frac{d^4\ell}{(2\pi)^4} \frac{i}{\ell^2 - \mu^2 + i\epsilon} \frac{i}{(P-\ell)^2 - \mu^2 + i\epsilon}.$$

Therefore,

$$i\mathcal{M}_s^{(2)} = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{i}{\ell^2 - \mu^2 + i\epsilon} \frac{i}{(P-\ell)^2 - \mu^2 + i\epsilon}.$$

The factor $1/2$ is the symmetry factor of the bubble diagram.

The Three One-Loop Channels

For identical scalar particles, there are three one-loop bubble channels $s = (q_1 + q_2)^2$, $t = (q_1 - p_1)^2$ and $u = (q_1 - p_2)^2$. Thus the full one-loop four-point amplitude has the schematic form

$$i\mathcal{M}^{(2)} = \frac{1}{2}(-i\lambda)^2 \left[I(s) + I(t) + I(u) \right],$$

where

$$I(P^2) = \int \frac{d^4\ell}{(2\pi)^4} \frac{i}{\ell^2 - \mu^2 + i\epsilon} \frac{i}{(P-\ell)^2 - \mu^2 + i\epsilon}.$$

These are the one-loop corrections to the tree-level result

$$i\mathcal{M}_{\text{tree}} = -i\lambda.$$

Amplitude Through Order λ^2

Combining tree level and the one-loop bubble corrections,

$$i\mathcal{M} = -i\lambda + \frac{1}{2}(-i\lambda)^2 \left[I(s) + I(t) + I(u)\right] + O(\lambda^3).$$

The corresponding connected S-matrix element is

$$S_{fi}^{\text{conn}} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)\, i\mathcal{M}.$$

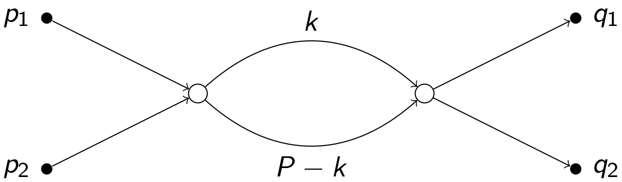
So perturbation theory gives

$$\boxed{S = 1 + \text{tree diagrams} + \text{one-loop diagrams} + \cdots}$$

Each order in λ corresponds to adding more interaction vertices.

Diagram in Momentum Space

The s-channel one-loop diagram is



$$P = p_1 + p_2$$

The loop momentum k is integrated over:

$$\int \frac{d^4 k}{(2\pi)^4}.$$

Each internal line contributes a propagator.

Degree of Divergence

At large loop momentum,

$$k^2 \gg \mu^2, \qquad (P-k)^2 \sim k^2.$$

Therefore,

$$I(P^2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\epsilon} \frac{i}{(P-k)^2-\mu^2+i\epsilon}$$

behaves as

$$I(P^2) \sim \int d^4k \, \frac{1}{k^2} \frac{1}{k^2} = \int d^4k \, \frac{1}{k^4}.$$

Using four-dimensional spherical coordinates in momentum space,

$$d^4k \sim k^3 dk.$$

Thus,

$$I(P^2) \sim \int^{\Lambda} k^3 dk \, \frac{1}{k^4} = \int^{\Lambda} \frac{dk}{k}.$$

Logarithmic Divergence

The integral

$$\int^{\Lambda} \frac{dk}{k}$$

diverges logarithmically:

$$\int^{\Lambda} \frac{dk}{k} \sim \log \Lambda.$$

Therefore the one-loop bubble in four-dimensional ϕ^4 theory is logarithmically divergent:

$$I(P^2) \sim \log \Lambda.$$

This divergence must be handled by renormalisation.

In ϕ^4 theory, the one-loop four-point divergence is absorbed into a redefinition of the coupling:

$$\lambda \longrightarrow \lambda_{\text{ren}}.$$

Result Through One Loop

Including tree level and the one-loop s, t, u channels, the amplitude is

$$i\mathcal{M} = -i\lambda + \frac{1}{2}(-i\lambda)^2 \left[I(s) + I(t) + I(u)\right] + O(\lambda^3),$$

where

$$I(P^2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \frac{i}{(P-k)^2 - \mu^2 + i\epsilon}.$$

The integrals $I(s)$, $I(t)$, and $I(u)$ are logarithmically divergent in four spacetime dimensions.

Renormalization makes the final physical amplitude finite.

Example: QED

As another example, consider QED.
The interaction Lagrangian is

$$\mathcal{L}_{\rm int} = -e\,\bar{\psi}\gamma^\mu\psi A_\mu .$$

Here:

$$\psi = \text{electron field}, \qquad A_\mu = \text{photon field},$$

and

$$e = \text{electric charge}.$$

We consider electron-electron scattering:

$$e^- e^- \to e^- e^- .$$

This process is also called Møller scattering.

QED Scattering Process

We label the process as

$$e^-(q_1, r_1) + e^-(q_2, r_2) \longrightarrow e^-(p_1, s_1) + e^-(p_2, s_2).$$

The relevant S-matrix element is

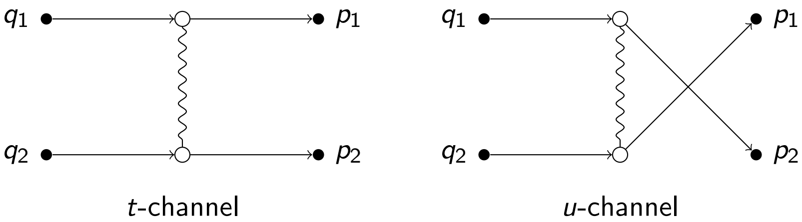
$$S_{fi} = \langle p_1, s_1;\ p_2, s_2; \mathrm{out} | q_1, r_1;\ q_2, r_2; \mathrm{in} \rangle \,.$$

At tree level, there are two diagrams because the outgoing electrons are identical fermions:

t-channel and u-channel exchange.

Tree-Level QED Diagrams

The two tree-level diagrams are:



The minus sign between these two contributions comes from exchanging identical fermions.

Four-Point Fermion Green's Function

The relevant Green's function contains four fermion fields:

$$G(y_1, y_2, x_1, x_2) = \langle 0 | T\{\psi(y_1)\psi(y_2)\bar{\psi}(x_1)\bar{\psi}(x_2)\} | 0 \rangle \,.$$

For scattering, LSZ attaches external spinors:

$$u(q_i, r_i) \quad \text{for incoming electrons,}$$

and

$$\bar{u}(p_j, s_j) \quad \text{for outgoing electrons.}$$

The amplitude is obtained from the connected amputated four-point function:

$$\mathcal{A} = \bar{u}(p_1, s_1)\bar{u}(p_2, s_2) G_{\rm amp} u(q_1, r_1) u(q_2, r_2).$$

Perturbative Expansion in QED

The interacting Green's function is generated by

$$\exp\left[i\int d^4z\,\mathcal{L}_{\rm int}(z)\right].$$

For QED,

$$\mathcal{L}_{\rm int}(z) = -e\bar{\psi}(z)\gamma^\mu\psi(z)A_\mu(z).$$

Thus,

$$i\int d^4z\,\mathcal{L}_{\rm int}(z) = -ie\int d^4z\,\bar{\psi}(z)\gamma^\mu\psi(z)A_\mu(z).$$

The first non-zero contribution to $e^-e^- \to e^-e^-$ is second order:

$$\frac{1}{2!}(-ie)^2\int d^4z_1\,d^4z_2.$$

One photon must be emitted at one vertex and absorbed at the other.

Order e^2 Green's Function

At order e^2,

$$G^{(2)} = \frac{1}{2!}(-ie)^2 \int d^4z_1\, d^4z_2$$

$$\times \langle 0| T\{\psi(y_1)\psi(y_2)\bar{\psi}(x_1)\bar{\psi}(x_2)\bar{\psi}(z_1)\gamma^\mu\psi(z_1)A_\mu(z_1)\bar{\psi}(z_2)\gamma^\nu\psi(z_2)A_\nu(z_2)\}|0\rangle \,.$$

The photon fields contract as

$$\langle 0| T\{A_\mu(z_1)A_\nu(z_2)\}|0\rangle = D_{\mu\nu}(z_1 - z_2).$$

The fermion fields contract into electron propagators connecting external points to the vertices.

Photon Propagator

In Feynman gauge, the photon propagator is

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2+i\epsilon} e^{-ik\cdot(x-y)}.$$

In momentum space,

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2+i\epsilon}.$$

Each QED vertex contributes

$$-ie\gamma^\mu.$$

Therefore, the tree-level QED amplitude comes from:

fermion current × photon propagator × fermion current.

Fermionic Wick Contractions

For fermion fields, Wick contractions acquire signs because the fields anticommute. For example,

$$T\{\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4\}$$

has contractions such as

$$\contraction{}{\psi_1}{}{\bar{\psi}_3}\psi_1\bar{\psi}_3\contraction{}{\psi_2}{}{\bar{\psi}_4}\psi_2\bar{\psi}_4$$

and

$$-\contraction{}{\psi_1}{}{\bar{\psi}_4}\psi_1\bar{\psi}_4\contraction{}{\psi_2}{}{\bar{\psi}_3}\psi_2\bar{\psi}_3.$$

The relative minus sign is the origin of the minus sign between the two diagrams for identical outgoing electrons.

This is a key difference between scalar and fermionic scattering.

Direct Channel Amplitude

For the direct channel,

$$q_1 \to p_1, \qquad q_2 \to p_2,$$

the momentum transferred through the photon is

$$k = q_1 - p_1 = p_2 - q_2.$$

The amplitude is

$$i\mathcal{M}_t = [\bar{u}(p_1, s_1)(-ie\gamma^\mu)u(q_1, r_1)] \, \frac{-ig_{\mu\nu}}{(q_1 - p_1)^2 + i\epsilon} \, [\bar{u}(p_2, s_2)(-ie\gamma^\nu)u(q_2, r_2)]$$

This is the t-channel contribution.

Exchange Channel Amplitude

Because the final electrons are identical, there is another contribution:

$$q_1 \to p_2, \qquad q_2 \to p_1.$$

The momentum transferred is

$$k = q_1 - p_2 = p_1 - q_2.$$

The corresponding amplitude is

$$i\mathcal{M}_u = [\bar{u}(p_2, s_2)(-ie\gamma^\mu)u(q_1, r_1)] \frac{-ig_{\mu\nu}}{(q_1 - p_2)^2 + i\epsilon} [\bar{u}(p_1, s_1)(-ie\gamma^\nu)u(q_2, r_2)]$$

This term enters with a relative minus sign:

$$i\mathcal{M} = i\mathcal{M}_t - i\mathcal{M}_u.$$

Tree-Level Møller Scattering

Putting both channels together,

$$i\mathcal{M} = [\bar{u}(p_1, s_1)(-ie\gamma^\mu)u(q_1, r_1)]\,\frac{-ig_{\mu\nu}}{(q_1 - p_1)^2 + i\epsilon}\,[\bar{u}(p_2, s_2)(-ie\gamma^\nu)u(q_2, r_2)]$$

$$-\,[\bar{u}(p_2, s_2)(-ie\gamma^\mu)u(q_1, r_1)]\,\frac{-ig_{\mu\nu}}{(q_1 - p_2)^2 + i\epsilon}\,[\bar{u}(p_1, s_1)(-ie\gamma^\nu)u(q_2, r_2)]\,.$$

The connected S-matrix element is

$$S_{fi}^{\mathrm{conn}} = (2\pi)^4\delta^{(4)}(p_1 + p_2 - q_1 - q_2)\, i\mathcal{M}.$$

QED Four-Point Function at Order e^2

For electron-electron scattering, the relevant connected Green's function appears at order e^2:

$$G(y_1, y_2, x_1, x_2) = \frac{(-ie)^2}{2!} \int d^4z_1\, d^4z_2\ \langle 0| T\{\psi(y_1)\psi(y_2)\bar{\psi}(x_1)\bar{\psi}(x_2)\}|0\rangle_{e^2} .$$

More explicitly,

$$G^{(2)} = \frac{(-ie)^2}{2!} \int d^4z_1\, d^4z_2\ \langle 0| T\{\psi(y_1)\psi(y_2)\bar{\psi}(x_1)\bar{\psi}(x_2)\}|0\rangle$$

$$\times \bar{\psi}(z_1)\gamma^\mu \psi(z_1) A_\mu(z_1) \bar{\psi}(z_2)\gamma^\nu \psi(z_2) A_\nu(z_2).$$

The photon fields must contract with each other:

$$A_\mu(z_1) \quad \text{with} \quad A_\nu(z_2).$$

Direct and Exchange Fermion Contractions

Using Wick's theorem, the photon contraction gives

$$D_{\mu\nu}(z_1 - z_2).$$

The fermion contractions give two terms.

The direct term is

$$S(y_1 - z_1)\gamma^\mu S(z_1 - x_1)S(y_2 - z_2)\gamma^\nu S(z_2 - x_2).$$

The exchange term is

$$S(y_2 - z_1)\gamma^\mu S(z_1 - x_1)S(y_1 - z_2)\gamma^\nu S(z_2 - x_2).$$

Because the external particles are identical fermions, the exchange term enters with a minus sign.

Thus,

$$G^{(2)}_{\rm conn} = \frac{(-ie)^2}{2!} \int d^4 z_1\, d^4 z_2\, D_{\mu\nu}(z_1 - z_2)\, [\text{direct} - \text{exchange}]\,.$$

Connected QED Green's Function

The connected order-e^2 four-point function is

$$G^{(2)}_{\rm conn}(y_1, y_2, x_1, x_2) = \frac{(-ie)^2}{2!} \int d^4z_1\, d^4z_2\, D_{\mu\nu}(z_1 - z_2)$$

$$\times \Big[S(y_1 - z_1)\gamma^\mu S(z_1 - x_1) S(y_2 - z_2)\gamma^\nu S(z_2 - x_2)$$

$$- S(y_2 - z_1)\gamma^\mu S(z_1 - x_1) S(y_1 - z_2)\gamma^\nu S(z_2 - x_2) \Big].$$

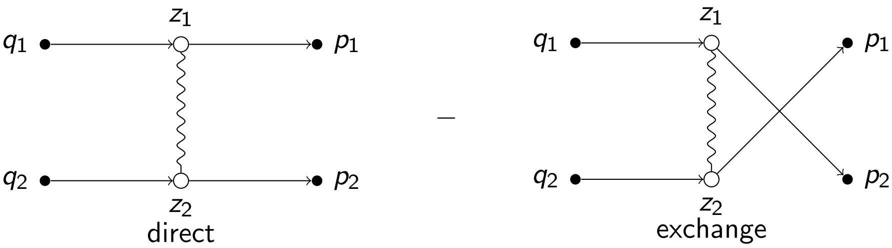
The two terms correspond to the two tree-level diagrams:

direct and exchange.

The relative minus sign is a fermionic sign from exchanging identical electrons.

Diagrammatic Form

The two terms are represented by



The wavy line is the photon propagator

$$D_{\mu\nu}(z_1 - z_2).$$

The solid lines are fermion propagators.

Applying Fermionic LSZ

For external fermions, LSZ amputates the external fermion propagators. The relevant identities are

$$(i\partial\!\!\!/_y - m)S(y-z) = i\delta^{(4)}(y-z),$$

and

$$S(z-x)(-i\overleftarrow{\partial\!\!\!/}_x - m) = i\delta^{(4)}(z-x).$$

After LSZ amputation, the external propagators are removed and replaced by spinors:

$$S(y-z) \longrightarrow \bar{u}(p,s)e^{+ip\cdot z},$$

$$S(z-x) \longrightarrow u(q,r)e^{-iq\cdot z}.$$

Thus the Green's function becomes a product of spinor currents connected by a photon propagator.

Direct Term After LSZ

For the direct term,

$$q_1 \to p_1, \qquad q_2 \to p_2,$$

LSZ gives

$$S_{fi}^{\text{direct}} = \frac{(-ie)^2}{2!} \cdot 2 \int d^4 z_1 \, d^4 z_2 \, D_{\mu\nu}(z_1 - z_2)$$

$$\times e^{i(p_1 - q_1)\cdot z_1} e^{i(p_2 - q_2)\cdot z_2}$$

$$\times \left[\bar{u}(p_1, s_1)\gamma^{\mu} u(q_1, r_1)\right] \left[\bar{u}(p_2, s_2)\gamma^{\nu} u(q_2, r_2)\right].$$

The factor of 2 cancels the 1/2! from the expansion:

$$\frac{1}{2!} \cdot 2 = 1.$$

Change of Variables

For the direct term, define

$$x = z_1 - z_2, \qquad y = \frac{z_1 + z_2}{2}.$$

Then

$$z_1 = y + \frac{x}{2}, \qquad z_2 = y - \frac{x}{2},$$

and the Jacobian is

$$d^4 z_1 \, d^4 z_2 = d^4 x \, d^4 y.$$

The phase becomes

$$e^{i(p_1 - q_1)\cdot z_1} e^{i(p_2 - q_2)\cdot z_2}$$

$$= e^{i(p_1 + p_2 - q_1 - q_2)\cdot y} e^{\frac{i}{2}(p_1 - q_1 - p_2 + q_2)\cdot x}.$$

The integral over y gives total momentum conservation.

Momentum Conservation

The y-integral gives

$$\int d^4y\, e^{i(p_1+p_2-q_1-q_2)\cdot y} = (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2).$$

Thus,

$$S_{fi}^{\rm direct} = (2\pi)^4\delta^{(4)}(p_f-p_i)(-ie)^2$$

$$\times\left[\bar{u}(p_1,s_1)\gamma^\mu u(q_1,r_1)\right]\left[\bar{u}(p_2,s_2)\gamma^\nu u(q_2,r_2)\right]$$

$$\times\int d^4x\, e^{i(p_1-q_1)\cdot x} D_{\mu\nu}(x),$$

where momentum conservation has been used to simplify the phase.

Takeaway

The practical recipe is:

1. Write the interaction Lagrangian.
2. Expand the time-ordered exponential.
3. Apply Wick's theorem.
4. Draw the corresponding Feynman diagrams.
5. Keep connected diagrams for scattering.
6. Amputate external legs using LSZ.
7. Put external momenta on shell.
8. Read off $\mathcal{M}$.

In short:

Feynman rules are the diagrammatic form of perturbative Green's functions

Relative Minus Signs in QED

In QED, relative signs between diagrams can appear because fermion fields anticommute.

When two diagrams differ by an exchange of fermionic operators, Wick's theorem gives a relative minus sign.

This is especially important for processes such as:

$$e^- e^- \to e^- e^-,$$

and

$$e^- e^+ \to e^- e^+.$$

The first is Møller scattering. The second is Bhabha scattering. For Bhabha scattering, two tree-level diagrams contribute:

t-channel photon exchange

and

s-channel annihilation and creation.

Bhabha Scattering

Consider

$$e^-(h_1) + e^+(h_2) \longrightarrow e^-(h_1') + e^+(h_2').$$

Here:

h_1 = incoming electron momentum, h_2 = incoming positron momentum,

h_1' = outgoing electron momentum, h_2' = outgoing positron momentum.

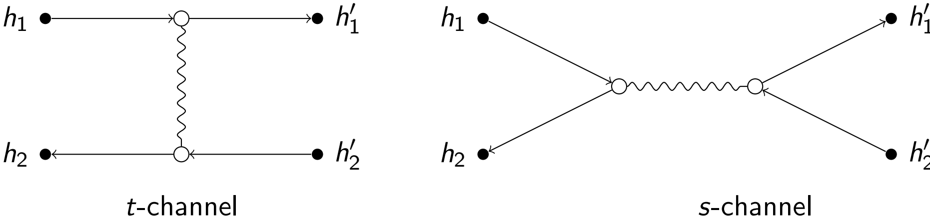
The connected S-matrix element has the form

$$S_{fi}^{\text{conn}} = i(2\pi)^4 \delta^{(4)}(h_1 + h_2 - h_1' - h_2')\mathcal{M}.$$

We now compute the tree-level invariant amplitude.

Tree-Level Diagrams for Bhabha Scattering

There are two tree-level diagrams.



The t-channel is photon exchange.
The s-channel is annihilation into a virtual photon followed by pair creation.

QED Ingredients

The QED vertex is

$$-ie\gamma^{\mu}.$$

The photon propagator in Feynman gauge is

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\epsilon}.$$

External spinors are:

$$u(h) \quad \text{for electrons,}$$

$$v(h) \quad \text{for positrons.}$$

For outgoing particles we use adjoint spinors:

$$\bar{u}(h'), \qquad \bar{v}(h').$$

Thus each diagram becomes a product of two fermion currents connected by a photon propagator.

The t-Channel Contribution

In the t-channel, the electron line carries

$$h_1 \to h_1',$$

and the positron line carries

$$h_2 \to h_2'.$$

The momentum transferred by the photon is

$$k_t = h_1' - h_1 = h_2 - h_2'.$$

Thus

$$t = k_t^2 = (h_1' - h_1)^2.$$

The corresponding amplitude is

$$i\mathcal{M}_t = \left[\bar{u}(h_1')(-ie\gamma^\mu)u(h_1)\right] \frac{-ig_{\mu\nu}}{(h_1' - h_1)^2 + i\epsilon} \left[\bar{v}(h_2)(-ie\gamma^\nu)v(h_2')\right].$$

This is the photon-exchange diagram.

The s-Channel Contribution

In the s-channel, the incoming electron and positron annihilate into a virtual photon:

$$e^- e^+ \to \gamma^* \to e^- e^+.$$

The photon momentum is

$$k_s = h_1 + h_2 = h_1' + h_2'.$$

Thus

$$s = k_s^2 = (h_1 + h_2)^2.$$

The corresponding amplitude is

$$i\mathcal{M}_s = [\bar{v}(h_2)(-ie\gamma^\mu)u(h_1)] \frac{-ig_{\mu\nu}}{(h_1 + h_2)^2 + i\epsilon} \left[\bar{u}(h_1')(-ie\gamma^\nu)v(h_2')\right].$$

This is the annihilation channel.

Relative Sign Between the Diagrams

The two contributions are not simply added with arbitrary signs. The relative sign is fixed by the ordering of fermionic operators in Wick's theorem. For the convention used here, the tree-level Bhabha amplitude is

$$i\mathcal{M} = i\mathcal{M}_t + i\mathcal{M}_s.$$

Equivalently,

$$i\mathcal{M} = \left[\bar{u}(h_1')(-ie\gamma^\mu)u(h_1)\right] \frac{-ig_{\mu\nu}}{(h_1' - h_1)^2 + i\epsilon} \left[\bar{v}(h_2)(-ie\gamma^\nu)v(h_2')\right]$$

$$+ \left[\bar{v}(h_2)(-ie\gamma^\mu)u(h_1)\right] \frac{-ig_{\mu\nu}}{(h_1 + h_2)^2 + i\epsilon} \left[\bar{u}(h_1')(-ie\gamma^\nu)v(h_2')\right].$$

Different choices for ordering external fermion operators can shift signs between intermediate expressions, but the physical amplitude is fixed once a convention is chosen.

Bhabha Scattering Amplitude

Putting everything together,

$$S_{fi}^{\mathrm{conn}} = i(2\pi)^4 \delta^{(4)}(h_1 + h_2 - h_1' - h_2')\mathcal{M}.$$

The tree-level invariant amplitude is

$$i\mathcal{M} = \left[\bar{u}(h_1')(-ie\gamma^\mu)u(h_1)\right] \frac{-ig_{\mu\nu}}{t + i\epsilon} \left[\bar{v}(h_2)(-ie\gamma^\nu)v(h_2')\right]$$

$$+ \left[\bar{v}(h_2)(-ie\gamma^\mu)u(h_1)\right] \frac{-ig_{\mu\nu}}{s + i\epsilon} \left[\bar{u}(h_1')(-ie\gamma^\nu)v(h_2')\right],$$

where

$$t = (h_1' - h_1)^2, \qquad s = (h_1 + h_2)^2.$$

Comparison with Møller Scattering

For Møller scattering,

$$e^- e^- \to e^- e^-,$$

the two diagrams are t- and u-channel photon exchange.
The amplitude has the structure

$$i\mathcal{M}_{\text{Moller}} = i\mathcal{M}_t - i\mathcal{M}_u.$$

For Bhabha scattering,

$$e^- e^+ \to e^- e^+,$$

the two diagrams are t- and s-channel:

$$i\mathcal{M}_{\text{Bhabha}} = i\mathcal{M}_t + i\mathcal{M}_s.$$

Comparison with ϕ^4 Theory

	ϕ^4 theory	QED
Interaction	$-\frac{\lambda}{4!}\phi^4$	$-e\bar{\psi}\gamma^\mu\psi A_\mu$
Tree-level vertex	$-i\lambda$	$-ie\gamma^\mu$
Propagator exchanged	None at tree level	Photon
Lowest $2 \to 2$ order	λ	e^2
External wavefunctions	Scalar LSZ factors	Spinors $u, \bar{u}$
Identical-particle sign	No fermion sign	Minus from exchange