From Einstein's Equations to Primordial Gravitational Waves

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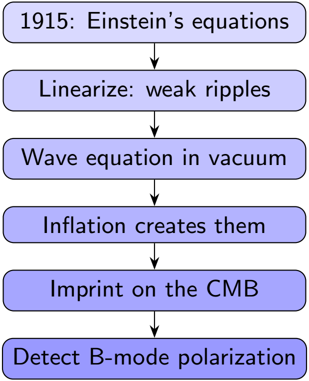
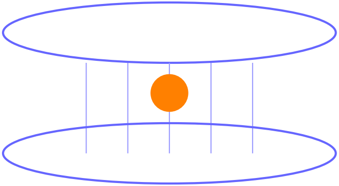
The Journey



From 1915 geometry to 21st-century telescopes.

Einstein's Equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



mass curves space

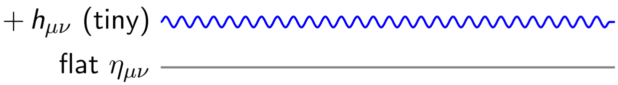
Wheeler: matter tells spacetime how to curve; spacetime tells matter how to move.

10 coupled, nonlinear equations. To find waves, we look at small ripples.

Small Ripples on Flat Space

Write the metric as flat plus a tiny perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1.$$

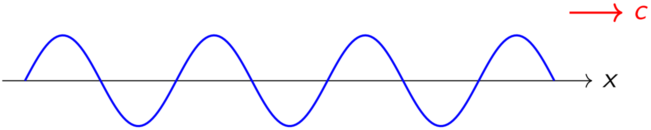


Keep only terms linear in h. With a clever gauge choice (Lorenz gauge), the equations simplify dramatically.

A Wave Equation Appears

In vacuum:

$$\Box \bar{h}_{\mu\nu} = 0$$



Solutions travel at the speed of light:

$$\bar{h}_{\mu\nu}(x) = A_{\mu\nu}\, e^{ik\cdot x}, \qquad k_\mu k^\mu = 0.$$

Gravitational waves exist – predicted by Einstein in 1916.

Linearized Gravity and the Wave Equation

From Einstein's equations to GW propagation

Perturb a flat FLRW background:

$$g_{\mu\nu} = a^2(\eta)\left[\eta_{\mu\nu} + h_{\mu\nu}\right], \qquad |h_{\mu\nu}| \ll 1$$

In the **transverse-traceless (TT) gauge**, $\partial^i h_{ij} = 0$ and $h^i{}_i = 0$, leaving two physical polarizations $h_+, h_\times$.

The linearized Einstein equation gives:

$$\Box\, h_{ij}^{\mathrm{TT}} \;=\; -\,\frac{16\pi G}{c^4}\,\Pi_{ij}^{\mathrm{TT}}$$

where Π_{ij}^{TT} is the **TT-projected anisotropic stress** of the source.

In an expanding background (conformal time η):

$$h_{ij}'' + 2\mathcal{H}\, h_{ij}' - \nabla^2 h_{ij} \;=\; 16\pi G\, a^2\, \Pi_{ij}^{\mathrm{TT}}$$

Two messages:

- Only the **TT part of the stress tensor** radiates — isotropic pressure does *not* source GWs.

The Big Picture

Three ingredients, one story:

1. **The Higgs field** fills all of space and gives particles their mass.
2. In the very early universe (about 10^{-11} seconds after the Big Bang), the Higgs field **changed state** — like water freezing into ice.
3. This change may have shaken the universe so violently that we can **still hear the echo today** — as gravitational waves.

- Light cannot reach us from that early epoch — the universe was opaque.
- Gravitational waves *can* reach us — they pass through everything.
- A future space mission, **LISA**, may detect them in the 2030s.

What Are Gravitational Waves?

Einstein's idea (1916):

Gravity is not a force — it is the **curvature of spacetime**. Mass and energy bend spacetime, the way a heavy ball bends a stretched rubber sheet.

When something massive accelerates, the curvature changes, and these changes travel outward at the **speed of light**. We call them gravitational waves.

Key properties:

- They **stretch and squeeze space** as they pass — distances between objects oscillate.
- They carry **energy** away from their source.
- They are **extremely weak**: even a colliding pair of black holes makes spacetime wobble by less than the size of an atomic nucleus over kilometers.
- They **barely interact** with matter, so they travel across the universe almost unchanged.

What Makes Gravitational Waves?

Not every motion produces gravitational waves.

A simple rule of thumb:

- A **perfectly symmetric** motion (like a pulsating sphere) does *not* make GWs.
- Motion needs to be **asymmetric** — different in different directions.

Examples:

- Two stars orbiting each other ⇒ yes (the mass distribution is lopsided and rotating).
- A spinning, perfectly round neutron star ⇒ no.
- A supernova collapsing perfectly spherically ⇒ no. But real supernovae are messy ⇒ yes.

For the early universe: A uniform, expanding plasma does not produce gravitational waves. But **turbulent, swirling, colliding motions** of the cosmic fluid *do*.

This is the key: *we need something violent and disordered to happen in the early universe.*

The Higgs Field and Phase Transitions

Analogy: water and ice

- At high temperature, water is liquid — molecules move freely, no preferred direction.
- Cool it below 0°C, and it **freezes** into ice — a new state with structure.

The same idea applies to the Higgs field:

- In the very hot early universe, the Higgs field had **zero average value** — all particles were massless.
- As the universe cooled below $T \sim 100$ GeV, the Higgs field **"switched on"** and developed a nonzero value everywhere.
- Particles like the W, Z, and the electron acquired their masses.

This event is called the **electroweak phase transition**.

The big question: *How* did this transition happen?

- **Smooth** (like a gradual change) $\Rightarrow$ no gravitational waves.
- **Violent** (like boiling, with bubbles) $\Rightarrow$ gravitational waves!

Bubbles in the Early Universe

What does a "violent" phase transition look like?

Imagine superheating water in a clean microwave: it can stay liquid above 100°C until, suddenly, **bubbles of steam** explode through it.

The same can happen with the Higgs field:

1. The universe gets "stuck" in the old state (Higgs $= 0$) even as it cools.
2. Tiny **bubbles of the new state** (Higgs $\neq 0$) appear randomly.
3. These bubbles **grow rapidly**, pushing against the surrounding hot plasma.
4. Bubbles **collide and merge**, until the whole universe is in the new state.

This is dramatic! The expanding bubble walls slam through the cosmic fluid, pushing matter around like a snowplow.
The result: the plasma is left full of **sound waves** — compressions and rarefactions traveling at the speed of sound (about $c/\sqrt{3}$ in a relativistic plasma).

Sound Waves Make Gravitational Waves

Why does sound become gravity here?

- Sound waves are organized motions of matter — bits of plasma swooshing back and forth.
- Moving matter carries **energy and momentum**.
- Energy and momentum, by Einstein's equations, **curve spacetime**.
- Rapidly changing, directional motion ⇒ **ripples in spacetime** ⇒ gravitational waves.

Why are sound waves so effective?

- The bubble collisions themselves are brief.
- But the sound waves they leave behind **persist for a long time** — many bubble-crossing times.
- Long-lasting source ⇒ more total gravitational wave energy produced.

In short: bubbles do the initial stirring; sound waves do the long, steady singing that fills the universe with gravitational waves.

What the Signal Looks Like

The gravitational wave spectrum: a graph of intensity vs. frequency

For an electroweak phase transition, the predicted signal has:

- A **peak frequency** of about **1 to 10 millihertz** (a thousandth of a hertz).
- A characteristic **shape**: rising on the low-frequency side, falling on the high-frequency side.
- An **amplitude** that depends on:
  - How violent the transition was (how much energy was released).
  - How big the bubbles grew before colliding.
  - How fast the bubble walls moved.

Why millihertz?

The wavelengths of these gravitational waves were originally tiny (size of bubbles), but the **expansion of the universe** has stretched them enormously — by a factor of about 10^{15} — to wavelengths of millions of kilometers today.

This is exactly the range that **LISA** is designed to detect.

LISA: Listening to the Early Universe

The Laser Interferometer Space Antenna

- A **space-based gravitational wave detector** by ESA and NASA.
- Three spacecraft flying in formation, separated by **2.5 million kilometers**.
- Lasers measure the tiny changes in their separation as gravitational waves pass.
- Planned launch: **mid-2030s**.

Why space?

Ground-based detectors (LIGO, Virgo) work at high frequencies ($\sim$ 100 Hz) — great for black hole mergers but blind to the millihertz signals from the early universe. Going to space allows much longer arms, and therefore sensitivity to much lower frequencies.

What LISA can do:

- Detect mergers of supermassive black holes.
- Observe thousands of compact binaries in our galaxy.
- **Search for the stochastic background** from an electroweak phase

Why This Matters

What would a detection mean?

The Standard Model of particle physics predicts the electroweak phase transition was **smooth** — no bubbles, no gravitational waves.

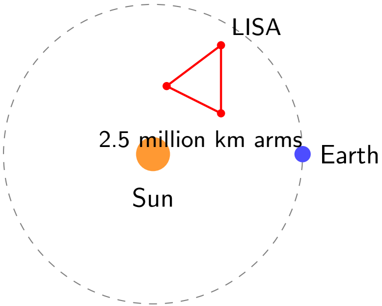
So if LISA sees the signal, it means:

- There is **new physics beyond the Standard Model** — new particles or new interactions modifying the Higgs.
- The transition was violent enough to perhaps explain the **matter–antimatter asymmetry** (why the universe is made of matter and not equal parts antimatter).
- We get a **direct window** into the universe at 10^{-11} seconds after the Big Bang — far earlier than any light-based observation.

A phase transition in the Higgs field, billions of years ago, could have left a faint humming background of gravitational waves filling all of space. We are about to build the instrument to hear it. If we do, it would be one of the most direct connections ever made between **particle physics, cosmology, and Einstein's theory of gravity**.

LISA – A Space-Based GW Observatory

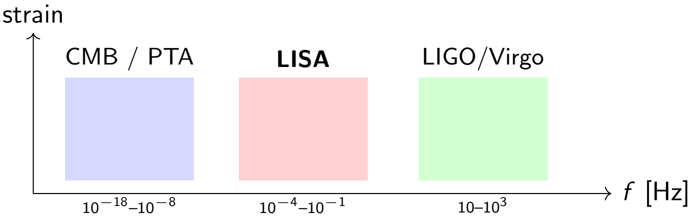
LISA (Laser Interferometer Space Antenna): ESA-led mission, launch $\sim$2035.



- Three spacecraft in a triangular formation, trailing Earth around the Sun.
- Laser interferometry between free-falling test masses.
- Sensitive to GW frequencies $\sim 10^{-4}$ to 10^{-1} Hz – well below LIGO's band.

LISA and the Primordial Background

LISA opens a **new frequency window** for gravitational waves.



Cosmological sources LISA could detect:

- **First-order phase transitions** in the early universe (e.g. electroweak), which produce bubble collisions and turbulence.
- **Cosmic strings** – topological defects from symmetry breaking.
- **Inflationary GWs** at much higher frequencies than the CMB probes – a complementary handle on the inflaton potential.

CMB B-modes + LISA together span $\sim$20 decades in frequency.

Motivation and Overview

Why connect these three topics?

The electroweak phase transition is one of the few moments in cosmic history where **particle physics, thermodynamics, and gravity** intersect in an observable way.

- The **Higgs potential** determines the structure of the electroweak vacuum.
- Its finite-temperature behavior dictates the **nature of the phase transition** in the early universe.
- A sufficiently violent transition imprints a **stochastic gravitational wave background** detectable today.

Three open questions this connects:

1. Is the Standard Model Higgs sector complete, or are there hidden scalars?
2. What is the origin of the baryon asymmetry ($\eta_B \sim 6 \times 10^{-10}$)?
3. Can we probe physics at $T \sim 100$ GeV through cosmology, complementary to colliders?

The Higgs Potential at Finite Temperature

Tree-level and thermal corrections

Zero-temperature potential:

$$V_0(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

At finite temperature, the **effective potential** receives loop corrections:

$$V_{\text{eff}}(\phi, T) = V_0(\phi) + V_{\text{CW}}(\phi) + V_T(\phi, T) + V_{\text{daisy}}(\phi, T)$$

The leading high-temperature expansion gives:

$$V_{\text{eff}}(\phi, T) \approx D(T^2 - T_0^2)\phi^2 - E\,T\phi^3 + \frac{\lambda(T)}{4}\phi^4$$

Key terms:

- $D(T^2 - T_0^2)\phi^2$ — thermal mass; restores symmetry at high T.
- $-E\,T\phi^3$ — **cubic term from bosonic loops**; this is what creates a **barrier** between vacua.
- $\lambda(T)$ — running quartic coupling.

In the Standard Model: E comes only from gauge bosons (W, Z) and is too small for $m_h = 125$ GeV $\Rightarrow$ predicts a **crossover**, not a true phase transition (lattice: Kajantie et al., 1996).

Characterizing a First-Order Phase Transition

The four key thermodynamic parameters

The universe tunnels from false to true vacuum via **bubble nucleation** (Coleman, Linde). The dynamics are captured by:

Parameter	**Definition**	**Physical meaning**
T_*	Transition temperature	Roughly the nucleation temperature T_n
α	$\rho_{\text{vac}}/\rho_{\text{rad}}$ at T_*	Strength (latent heat vs. radiation)
β/H_*	$\beta = -\,dS_E/dt\,\|_{t_*}$	Inverse duration in Hubble units
v_w	Bubble wall velocity	Sets efficiency of energy transfer

Bounce action governs nucleation rate:

$$\Gamma(T) \sim T^4 \left(\frac{S_3}{2\pi T}\right)^{3/2} e^{-S_3/T}$$

Nucleation occurs when $\Gamma \sim H^4$, equivalent to $S_3/T \approx 140$ at the electroweak scale.

Strong transition criterion (for baryogenesis): $\phi(T_c)/T_c \gtrsim 1$ — suppresses sphaleron washout in the broken phase.

Beyond the Standard Model Scenarios

Models that yield a strong first-order EWPT

Since the SM gives only a crossover, new physics must enhance the barrier:

1. **Singlet scalar extension** (xSM): adds S coupled to H; tree-level barrier from mixing.
2. **Two-Higgs-Doublet Model (2HDM)**: extra scalars contribute to V_T; favored regions have heavy A^0, $H^\pm$.
3. **Effective field theory**: add ϕ^6/Λ^2 term — strong PT requires $\Lambda \lesssim 850$ GeV (Grojean, Servant, Wells 2005).
4. **MSSM with light stop** — now largely excluded by LHC.
5. **Composite Higgs / dilaton-like potentials** — nearly conformal sectors give *very* strong, supercooled transitions.

Complementarity with colliders:

- Modified **trilinear Higgs coupling** λ_{hhh} (deviations of 20–100% expected).
- HL-LHC and FCC-hh can constrain *hh* production.
- Direct searches for new scalars.

$\Rightarrow$ Gravitational waves + colliders + EDM searches form a **triangulation** of the EW vacuum structure.

Gravitational Wave Production

Three sources, one spectrum

$$\Omega_{\text{GW}}(f) = \Omega_{\text{col}}(f) + \Omega_{\text{sw}}(f) + \Omega_{\text{turb}}(f)$$

1. Bubble collisions (envelope approximation):

- Dominant only for runaway walls (rare in thermal plasmas).
- $\Omega_{\text{col}} \propto (H_*/\beta)^2[\alpha/(1+\alpha)]^2$

2. Sound waves (Hindmarsh et al. 2014–2017) — **usually dominant**:

$$\Omega_{\text{sw}} h^2 \sim 2.65 \times 10^{-6} \left(\frac{H_*}{\beta}\right) \left(\frac{\kappa_v \alpha}{1+\alpha}\right)^2 \left(\frac{100}{g_*}\right)^{1/3} v_w \, S_{\text{sw}}(f)$$

3. MHD turbulence: subdominant but contributes at high frequencies.

Redshifted peak frequency today:

$$f_{\text{peak}} \approx 26\,\mu\text{Hz} \left(\frac{\beta/H_*}{1}\right) \left(\frac{T_*}{100\ \text{GeV}}\right) \left(\frac{g_*}{100}\right)^{1/6}$$

$\Rightarrow$ Lands squarely in the **LISA band** for electroweak-scale transitions.

Detection Prospects and Outlook

The observational landscape

LISA (ESA/NASA, launch ∼2035):

- Peak sensitivity at $\sim$ 1–10 mHz.
- Can detect $\Omega_{\mathrm{GW}} h^2 \sim 10^{-12}$ — sufficient for strong EWPT scenarios.
- Signal-to-noise depends on $(\alpha, \beta/H_*, T_*, v_w)$.

Complementary detectors:

- **DECIGO, BBO** (future, deci-Hz) — probe transitions at $T \sim 10^4$–10^6 GeV.
- **Pulsar timing arrays** (NANOGrav, EPTA) — already see a signal at nHz; some interpretations invoke a QCD-scale or dark sector phase transition.
- **Einstein Telescope, Cosmic Explorer** — higher frequencies, useful for very high-scale transitions.

Theoretical frontiers:

- Improved **lattice calculations** of $V_{\rm eff}$ in BSM models.
- Reducing the notorious **gauge and renormalization-scheme dependence** of perturbative computations (Croon et al. 2020).
- Better modeling of bubble wall dynamics and friction.
- Connection to **baryogenesis** via CP-violating sources at the wall.

Bottom line: A LISA detection consistent with a first-order EWPT would be a *direct cosmological signature of physics beyond the Standard Model* — and possibly the key to baryogenesis.