Quantum Field Theory **2025/26**

Exame Modelo (does not mean that this is the exam...and it will be shorter)

1. Consider a real scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2.$$

a) Show that under an infinitesimal translation

$$x^\mu \to x^\mu + a^\mu,$$

the scalar field transforms as

$$\delta\phi(x) = -a^\mu\partial_\mu\phi(x).$$

b) Using Noether's theorem, derive the conserved energy-momentum tensor

$$T^{\mu\nu}.$$

c) Define

$$P^\nu = \int d^3x\, T^{0\nu}.$$

Show that

$$[P^\nu, \phi(x)] = -i\partial^\nu\phi(x).$$

d) Suppose the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \lambda x^\mu x_\mu\phi^2.$$

Discuss whether translation invariance and momentum conservation still hold.

2. Consider that $\phi(x)$ is a spin zero free field, solution of the Klein-Gordon equation. Show that $[\phi(x), \phi(y)] = 0$ for spacelike separated x, y. Why is this essential?

3. Consider the Dirac operator, solution of the Dirac equation

$$\psi(x) = \sum_s \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left[c_s(\vec{k})u_s(\vec{k})e^{-ik.x} + d_s^\dagger(\vec{k})v_s(\vec{k})e^{ik.x}\right],$$

written in terms of creation and annihilation operators, and of the spinors u and v.
a) Write the Hamiltonian, H, as a function of creation and annihilation operators.
b) Show that

$$i[H, \psi(x)] = \frac{\partial\psi(x)}{\partial t}.$$

4. Consider a real scalar field $\phi(x)$ described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

The exact interacting vacuum is denoted by $|\Omega\rangle$.
a) Explain the meaning of the asymptotic fields $\phi_{\text{in}}(x)$ and $\phi_{\text{out}}(x)$.
b) Starting from the scattering matrix element $\langle p_3, p_4\,\text{out}\,|\,p_1, p_2\,\text{in}\rangle$, derive the LSZ reduction formula for the connected $2 \to 2$ scattering amplitude:

$$i\mathcal{M} = \prod_{i=1}^{4}\int d^4x_i\, e^{ip_i\cdot x_i}(\partial\cdot\partial_{x_i} + m^2)G^{(4)}(x_1, x_2, x_3, x_4),$$

where

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle.$$

c) At order λ in perturbation theory, the connected four-point Green function is

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = (-i\lambda)\int d^4z\prod_{i=1}^{4}\Delta_F(x_i - z).$$

Using the LSZ formula and the identity

$$(\partial\cdot\partial + m^2)\Delta_F(x - z) = -i\,\delta^{(4)}(x - z),$$

compute the tree-level invariant amplitude $\mathcal{M}$.
d) Consider the process

$$\phi\phi \to \phi\phi$$

in the canter-of-mass frame. Assuming identical particles and neglecting masses whenever appropriate, show that the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{128\pi^2 s}.$$

5. Consider QED with gauge-fixing term

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - j^\mu A_\mu.$$

a) Derive the photon propagator in a general covariant gauge.
b) Show that the tree-level amplitude for the scattering of two conserved currents

$$i\mathcal{M} = -\frac{i}{k^2}\left[j_1\cdot j_2 - (1-\xi)\frac{(j_1\cdot k)(j_2\cdot k)}{k^2}\right].$$

is independent of the gauge parameter ξ (use current conservation $\partial\cdot j = 0$).
c) Explain why gauge fixing changes the propagator but not physical observables.

6. Consider the Lagrangian density

$$\mathcal{L} = (D_\mu \phi_1)^* D^\mu \phi_1 - m_1^2 \phi_1^* \phi_1 + (\partial_\mu \phi_2)^* \partial^\mu \phi_2 - m_2^2 \phi_2^* \phi_2 - \frac{1}{4} F^2$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, ϕ_1 and ϕ_2 are complex scalar fields and D_μ is the covariant derivative.

a) Show the Lagrangian is invariant under a global $U(1)$ transformation $\phi_1 \to e^{-i\epsilon}\phi_1$ and $\phi_2 \to e^{-i\epsilon}\phi_2$. Find the corresponding conserved current.

b) Show the Lagrangian is not invariant under a local $U(1)$ transformation (A_μ transforms as $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \epsilon$, where e is the electric charge).

c) How do we have to modify the Lagrangian to make it invariant under a local $U(1)$ transformation?

7. The Parity operator for a scalar field can be written as

$$P = e^X \quad \text{with} \quad X = \left[-\frac{i\pi}{2} \int d^3k \, \left\{ a^\dagger(k) a(k) - \eta_P a^\dagger(k) a(-k) \right\} \right]$$

where $\eta_P = \pm 1$. Show that the Parity operator does not commute with the Momentum $\vec{P}$,

$$\vec{P} = \int d^3k \, \vec{k} \, a^\dagger(k) a(k) \, .$$

Explain why.

8. Consider the following Lagrangian, where ϕ_1 is a real scalar field, ϕ_2 is a complex scalar field, ψ is a fermion field and A_μ is a spin 1 field,

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}F^2 + i\bar{\psi}\partial_\mu \gamma^\mu \psi - m_l \bar{\psi}\psi + \frac{1}{2}\partial_\mu \phi_1 \, \partial^\mu \phi_1 + \partial_\mu \phi_2^* \, \partial^\mu \phi_2 - m^2 \phi_2^* \phi_2 \\ & + \alpha \bar{\psi}\gamma^5 \psi \phi_1 + \beta \bar{\psi}\gamma_\mu \psi A^\mu + \kappa_1 \phi_1^2 (\phi_2^* \phi_2) + \kappa_2 A_\mu A^\mu \phi_2^* \phi_2 + \kappa_3 \partial_\mu A^\mu \phi_1^2 \, . \end{aligned}$$

a) Identify the particles in the Lagrangian by stating their mass and spin. Identify the interaction terms and the dimension of all coupling constants.

b) Assuming the Lagrangian is Parity invariant, determine the Parity quantum numbers of the integer spin particles.

c) What is the effect of the charge conjugation operator on ϕ_2? What is the C number of ϕ_1 considering the Lagrangian is C-invariant?

d) Write one dimension 3 and one dimension 4 terms in the Lagrangian that break P invariance.

e) Write one extra non-renormalisable term, with at least one fermion. Write a non-renormalisable P-breaking term with no fermions.

9. Consider the process $\mu^- e^- \to \mu^- e^-$ in the framework of QED .

a) Use the Feynman rules to find the amplitude for the process.
b) Show that the differential cross section $d\sigma/d\Omega$, can be written as a function of the Mandelstam variables, s, t and u as (before starting the calculation set the muon and electron masses to zero)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2\,s}\,\frac{s^2+u^2}{t^2}\,, \tag{1}$$

where $\alpha = e^2/(4\pi)$ is the fine-structure constant.
c) When is the above approximation valid?

10. Consider a real scalar field described by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

Let

$$G^{(4)}(x_1,x_2,x_3,x_4) = \langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle.$$

a) Using Wick's theorem, show that for the free theory ($\lambda = 0$)

$$\begin{aligned} G^{(4)}(x_1,x_2,x_3,x_4) &= \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) \\ &\quad + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) \\ &\quad + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3), \end{aligned}$$

where $\Delta_F(x-y)$ is the Feynman propagator. How many distinct contractions contribute?
b) The connected four-point Green's function is defined as

$$G_c^{(4)}(x_1,x_2,x_3,x_4) = G^{(4)}(x_1,x_2,x_3,x_4) - \sum_{\text{pairings}} G^{(2)}G^{(2)}.$$

Calculate $G_c^{(4)}$ in the free theory and explain the physical meaning of the result.
c) To first order in λ, write the expression for the connected four-point Green's function

$$G_c^{(4)}(x_1,x_2,x_3,x_4)$$

using perturbation theory. Draw the corresponding Feynman diagram and show that

$$G_c^{(4)} = -i\lambda \int d^4z\,\Delta_F(x_1-z)\Delta_F(x_2-z)\Delta_F(x_3-z)\Delta_F(x_4-z) + \mathcal{O}(\lambda^2).$$

Explain why this contribution is connected.