The Homogeneous Universe

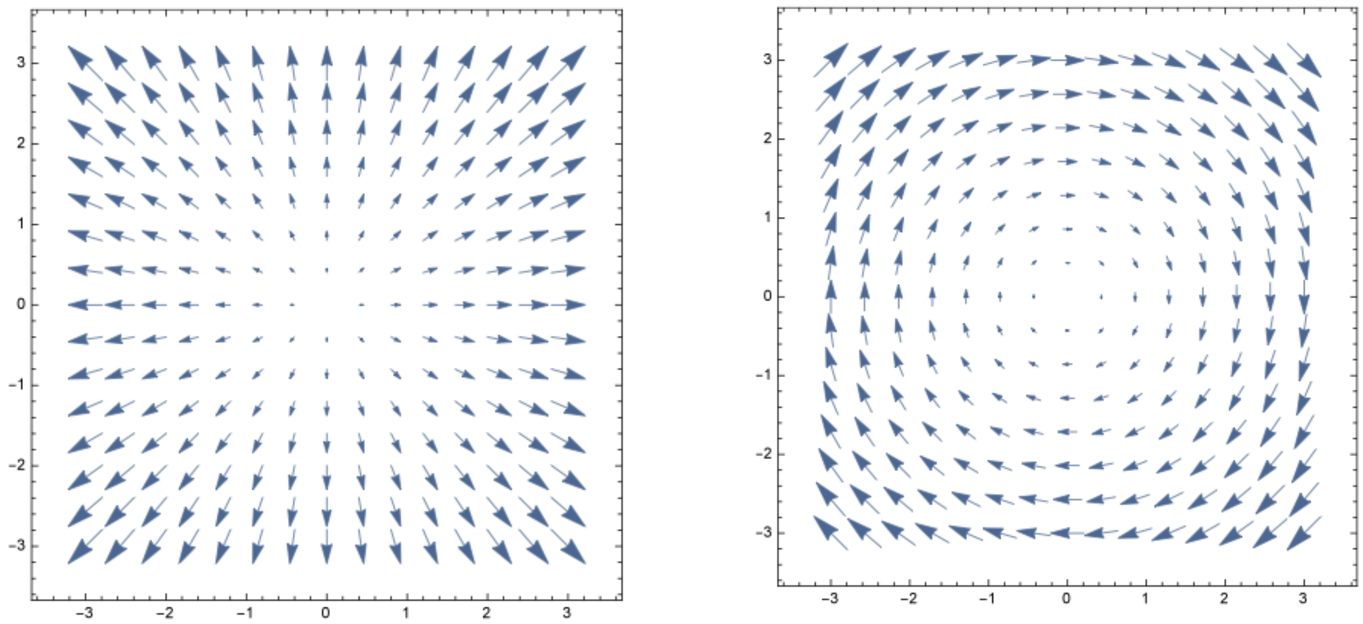
The zeroth order Universe

The zeroth-order Universe: Cosmological Principle

Isotropy

"The Universe observed in any direction (from an observing point) looks the same"

The observed properties are independent of direction (rotational invariance)



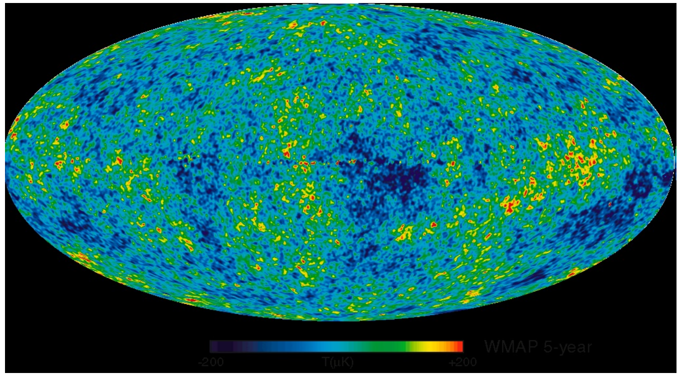
isotropic (but not homogeneous)

Isotropy observed

Except for the “nearby” structures, the observed spatial distribution of the Universe looks isotropic.

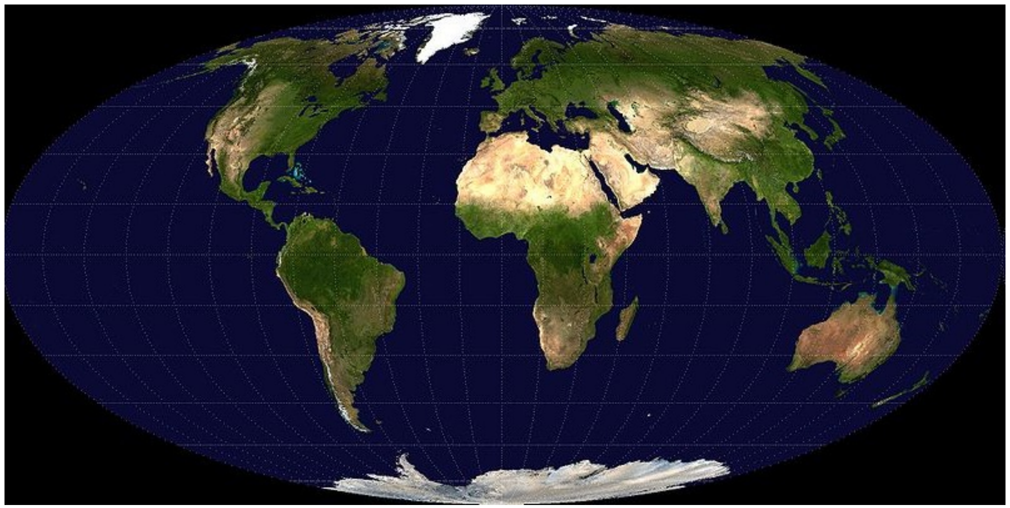
CMB (Cosmic Microwave Background) is isotropic

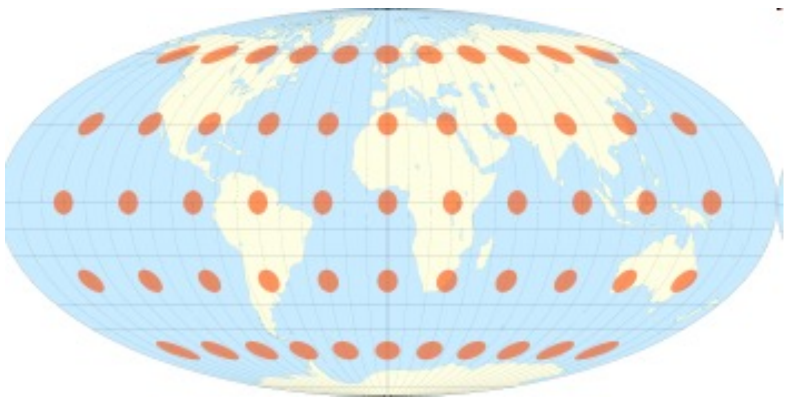
$\Delta T/T \sim 0.00001$



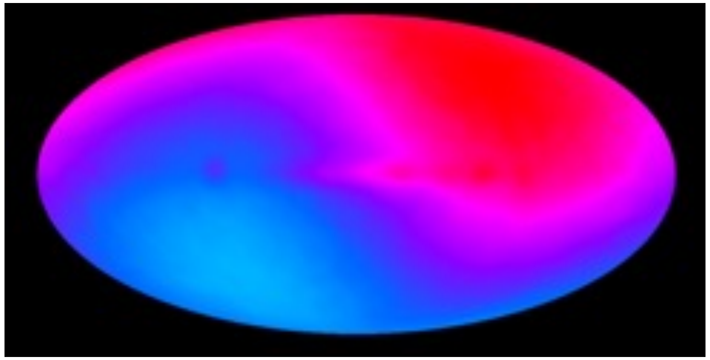
The sky shown in Mollweide projection
in galactic coordinates

(preserves areas e distorts shapes)





Anisotropy is for example



dipole in CMB $\Delta T/T \sim 0.001$

→ $\Delta\lambda/\lambda \sim 0.001 \sim v/c$

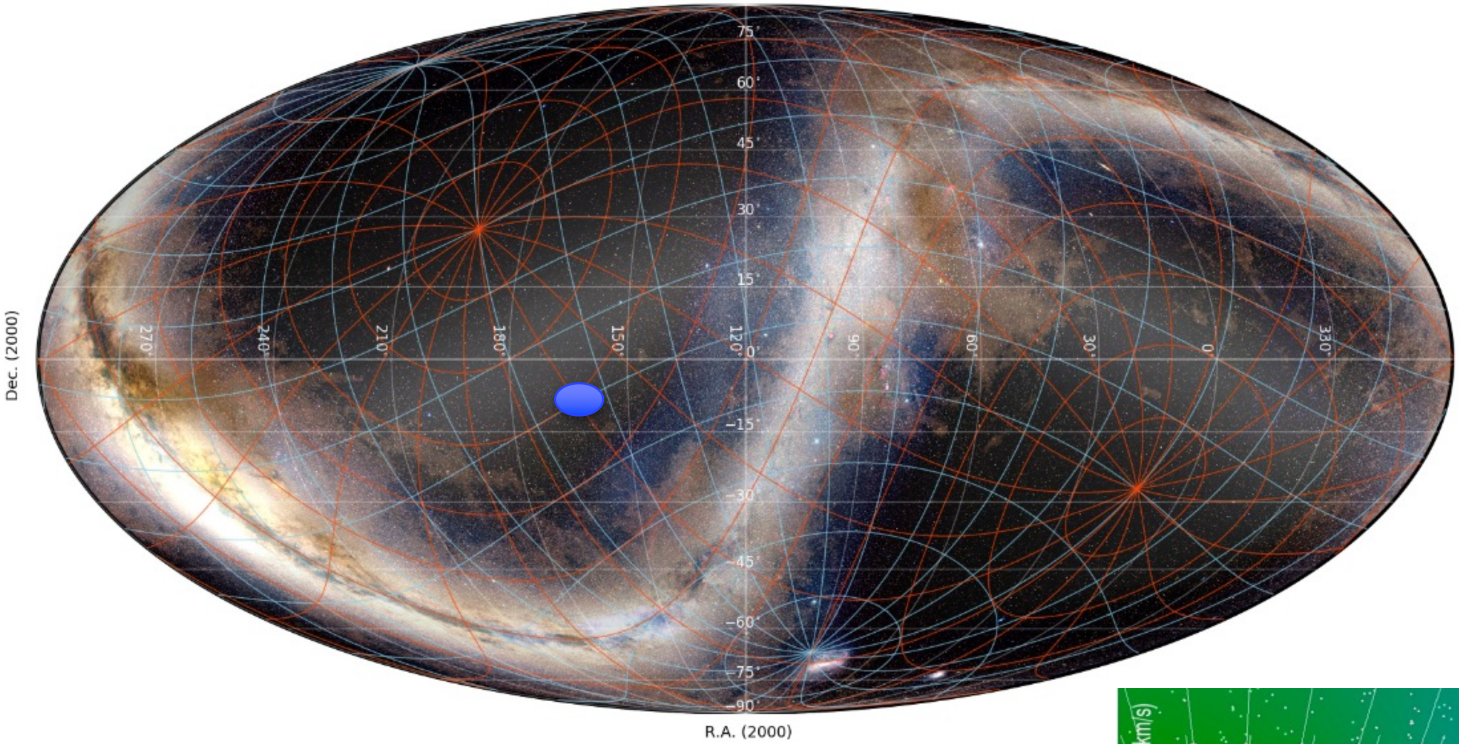
→ $v \sim 300$ Km/s

This is the total velocity of the Earth with respect to the CMB frame:

includes Earth's orbital movement + solar system movement in the galaxy + local galaxy movement → peculiar velocity of the galaxy
(it is a perturbation to Hubble's flow)

So, there is a "local" anisotropy that can be measured.

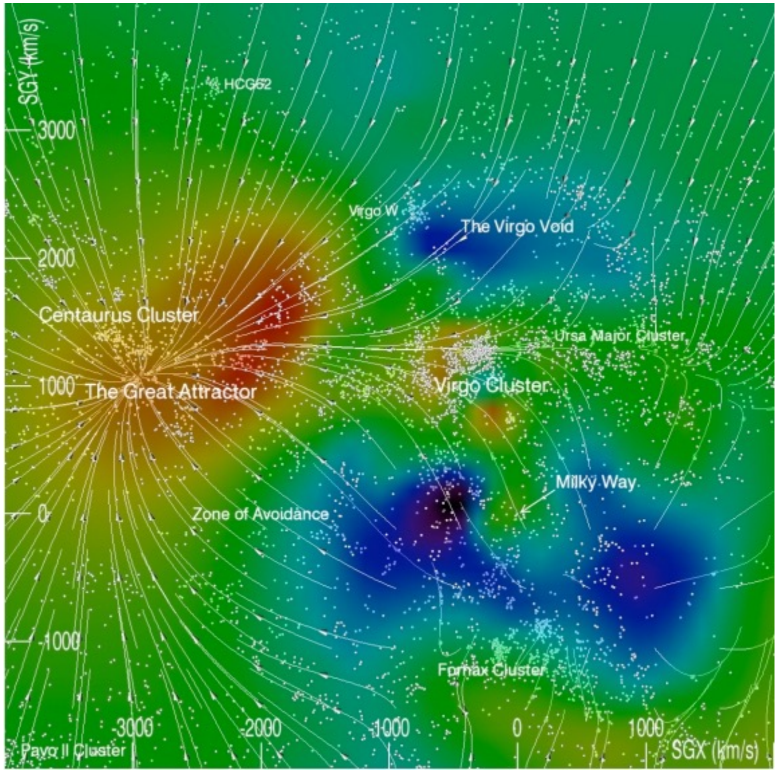
The movement is in the direction of the blue pole (ra, dec = 11h11min57s , -7.22º)
(Leo constellation) towards the Great Attractor.

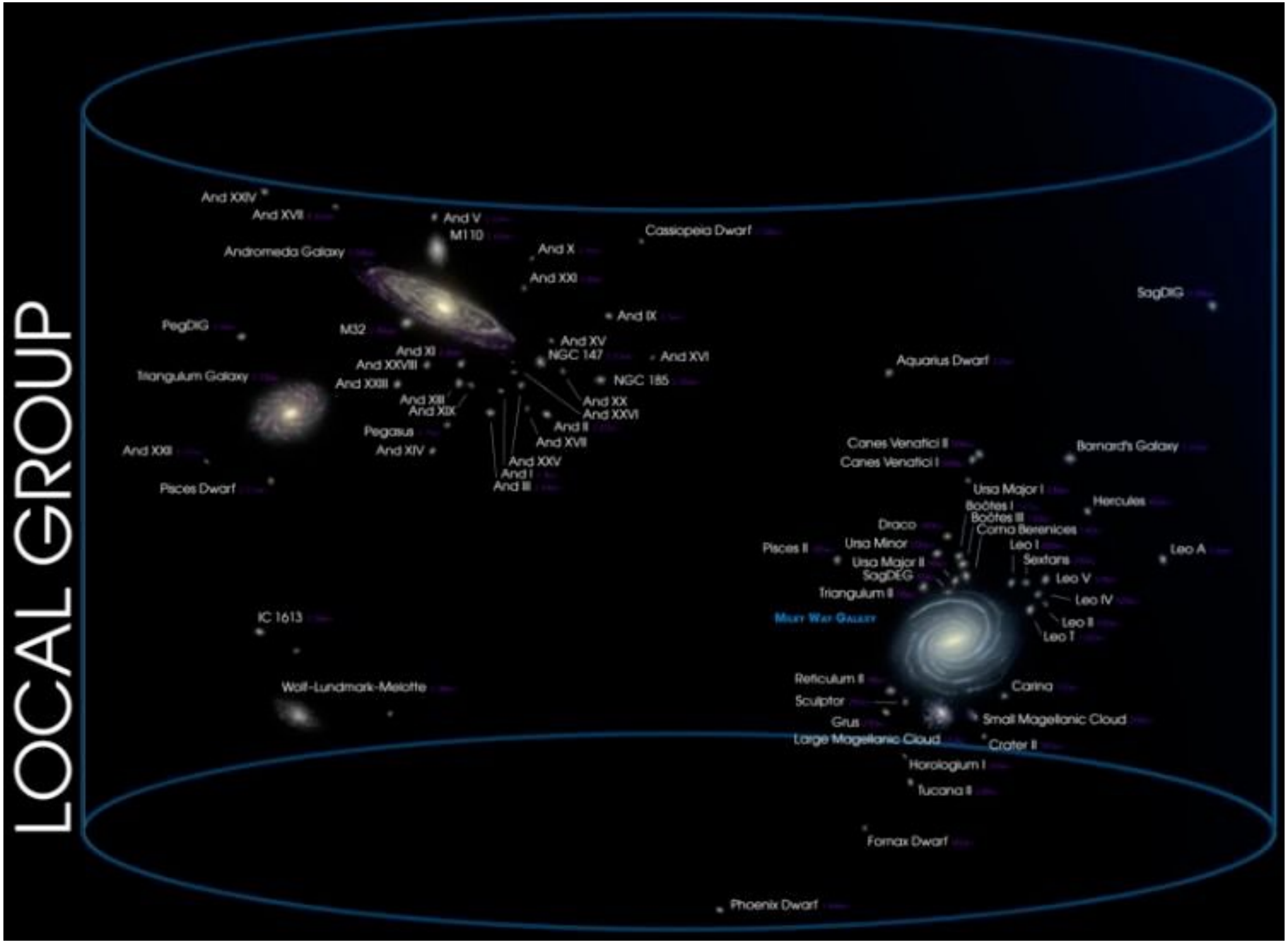


Looking from Earth, the Great Attractor lies on the zodiacal plane and close to the galactic plane → difficult to observe the extra-galactic sky → results are from radio-astronomy (2016)

It is at ~50 Mpc from us

Parsec is a historical unit of distance. It is the distance to a star that changes its apparent position due to the Earth's orbital movement (paralax) by 1 arcsec. It corresponds to 3.26 lyr.

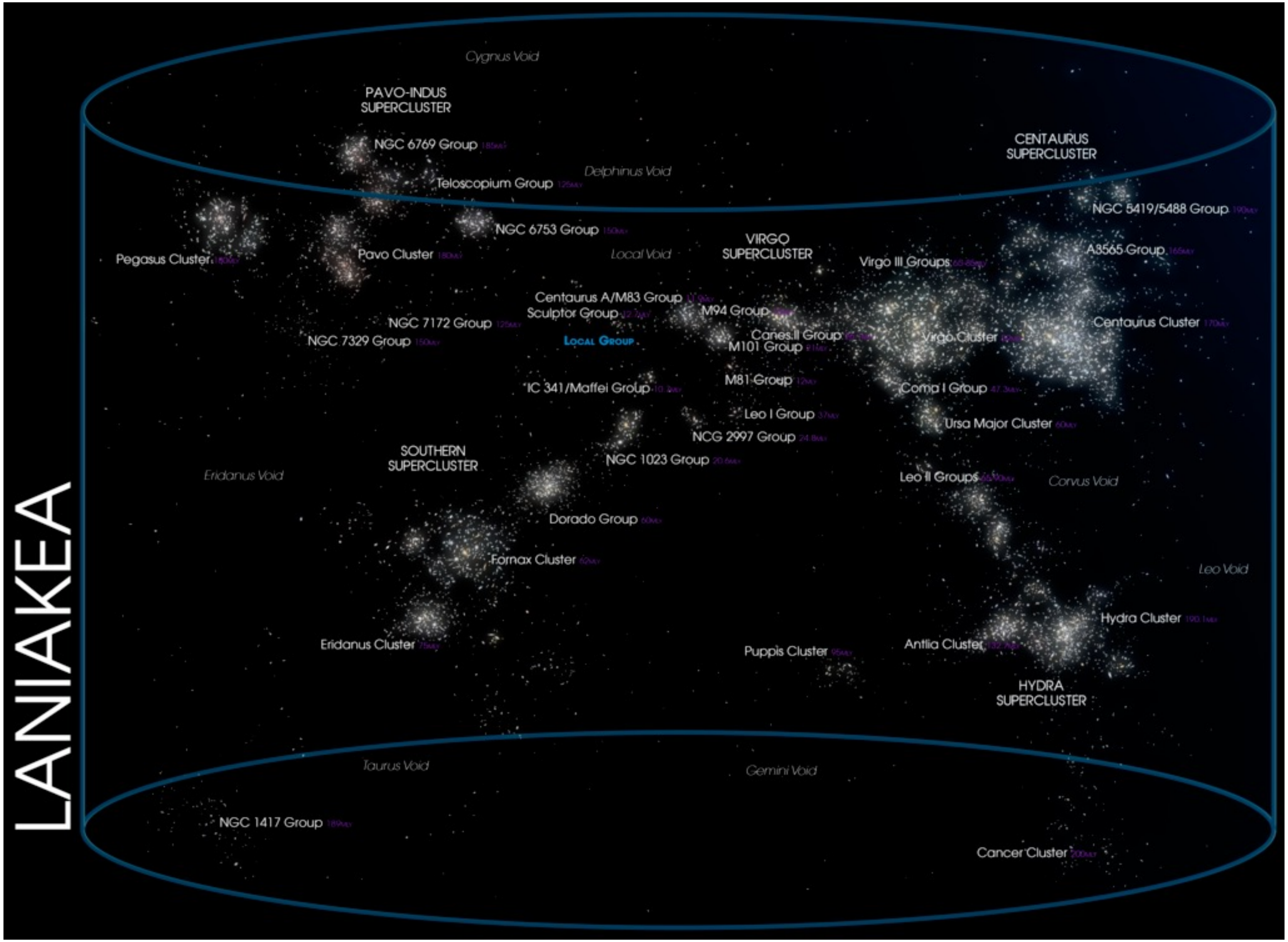




Diameter ~3 Mpc

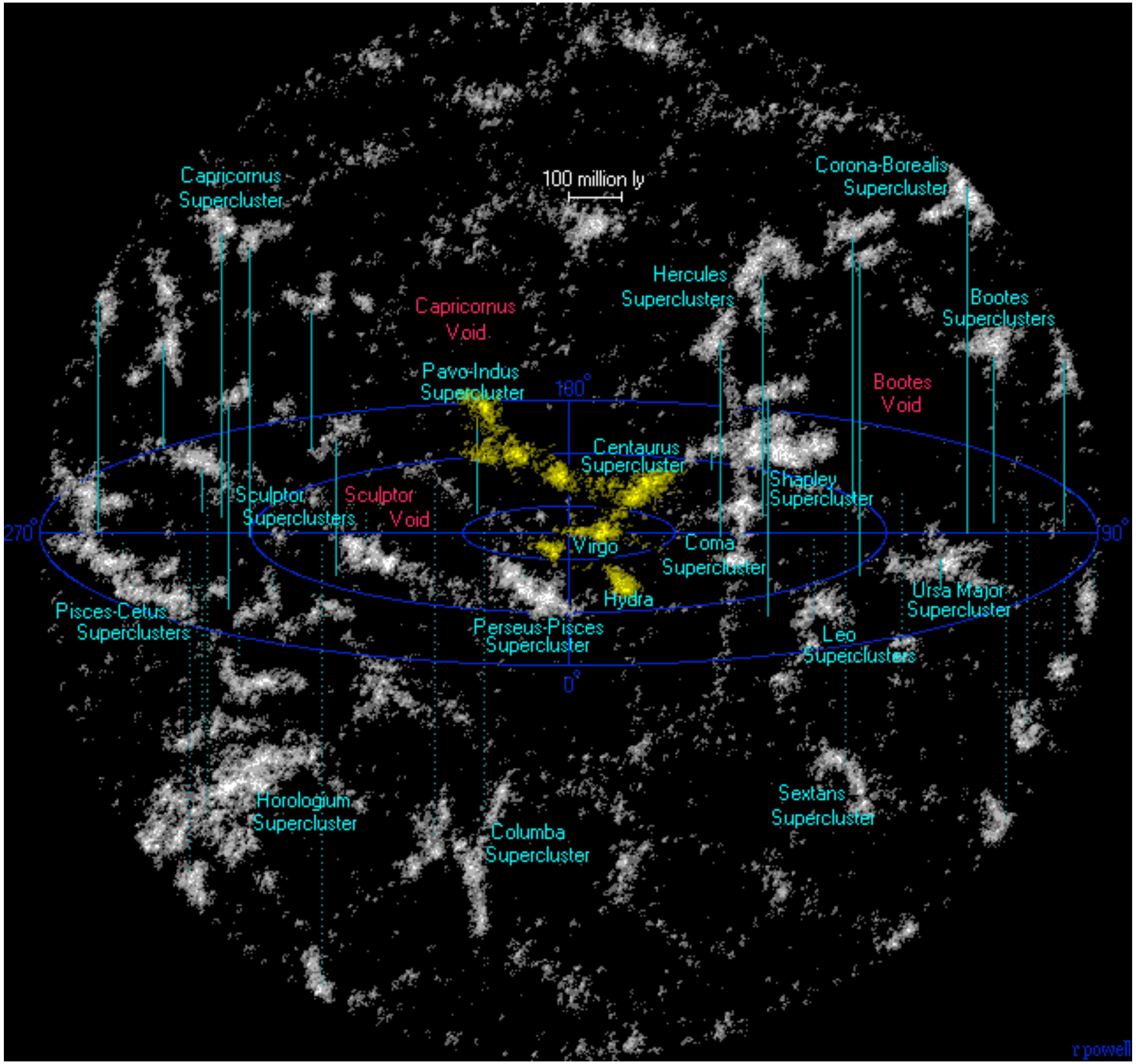
Gravitationally bound. Non-linear structure that contains many non-linear structures

Laniakea: the local super-cluster. Its central gravitational point is the Great Attractor.



Diameter ~170 Mpc

Loosely gravitationally bound. Linear structure that contains many non-linear structures.



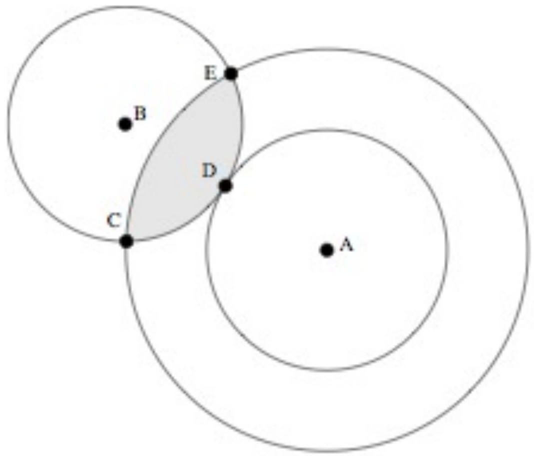
Beyond Laniakea (shown in yellow), the movements with respect to us start to be dominated by the Hubble flow and no longer by peculiar velocities → isotropy

This is roughly redshift $z \sim 0.1$

Cosmology starts beyond z ~0.1

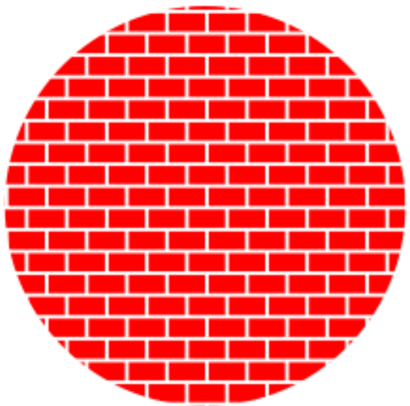
Extrapolation of the Copernican principle → we should not be in a special position. All points should observe isotropy.

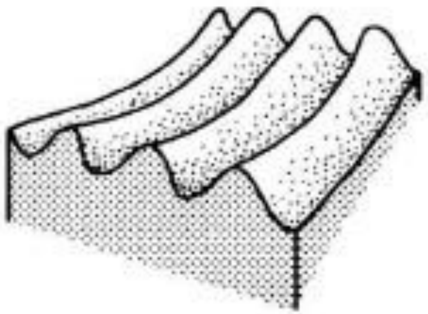
Isotropy in all points implies homogeneity.



isotropy around A and around B
implies that the grey zone is homogeneous.

On the contrary, homogeneity does not imply isotropy

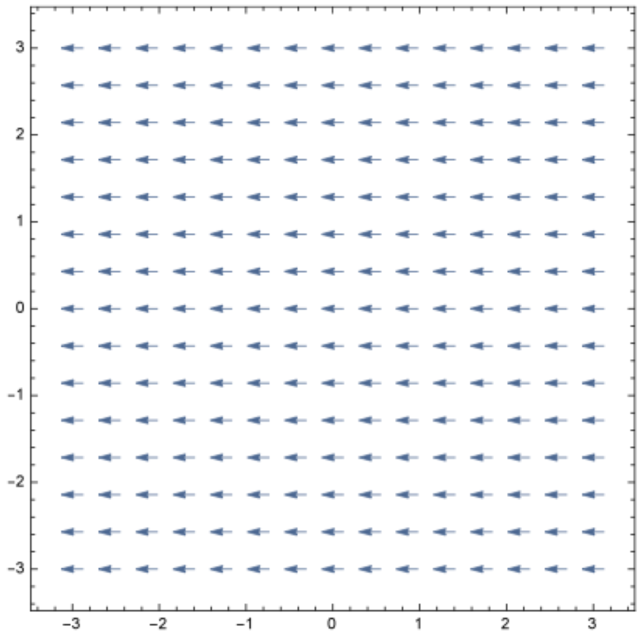




Homogeneity

"The Universe is identical in all points, at each instant"

The observed properties are independent of location (translational invariance)

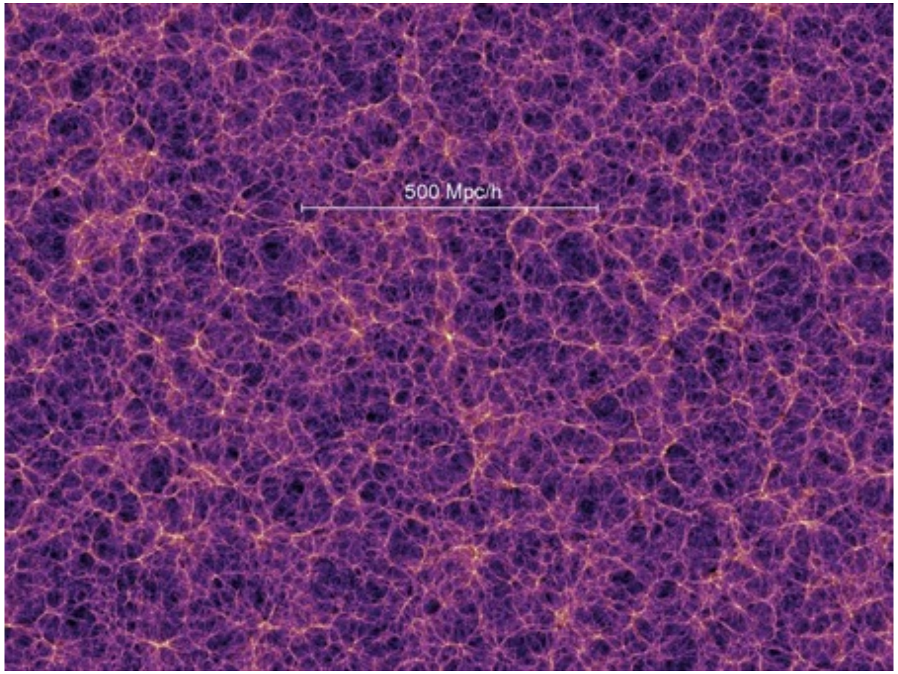


homogeneous (but not isotropic)

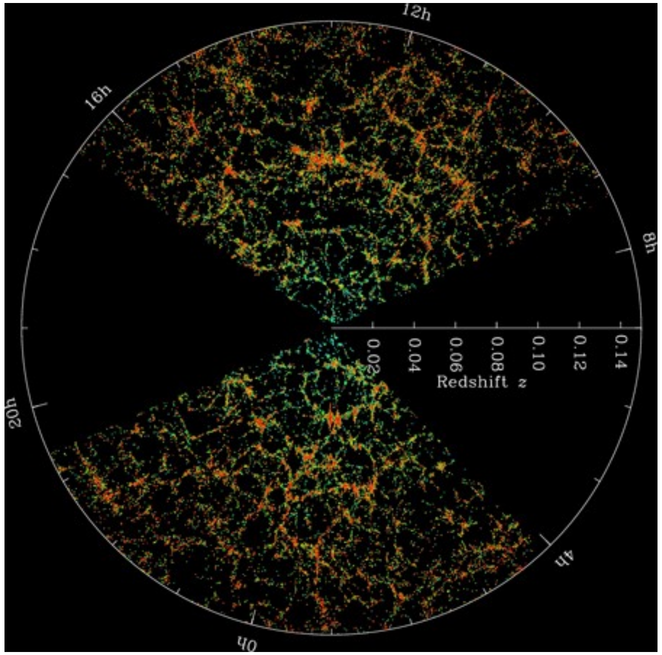
Homogeneity observed

- Galaxy counts as function of volume

- The absence of structures on “very large scales“ – the average matter density constrast on very large scales is very low.



(dark matter N-body simulation)



(observations of galaxies)

Homogeneity scale > 100 Mpc

Cosmological Principle

The Universe is homogeneous and isotropic (on “large-enough scales”)

This implies that there is a set of observers that have the same history of the Universe and to which all observables are independent of direction. This defines a fundamental reference frame where the physical properties are the same on all points. This is the comoving frame - that follows Hubble’s flow

Physical fields (matter density or CMB temperature) have the same values for all comoving observers.

The time rate is also identical, which allows to define an universal time and separate space and time coordinates.

In practice: $\rho(t,x) \rightarrow \rho(t)$ where **t** is universal

The first (and higher)-order Universe: Inhomogeneities

The cosmological principle is a first approximation to study the Universe.

It is not verified on smaller scales where “local” structures differ from point to point, defining local gravitational potentials.

Is it a good approximation? In other words, ***what is the amplitude of the gravitational potentials associated to the astrophysical structures?***

To address this question let us consider the Theorem of the Virial (for the dynamics of the gravitational collapse of a local system of N particles of masses m at positions x)

Sistema de N partículas com massas m_α em posição $\vec{x}_\alpha$

Tensor of Inertia is the matrix of the second-order moments of the mass distribution in the system

Momento de inércia: $$I_{jk} = \sum_{\alpha=1}^{N} m_\alpha \, x_j^\alpha x_k^\alpha$$

3D
j,k=1,2,3

Calculemos a 2ª derivada em relação ao tempo:

second-order derivative of the inertia tensor

$$\frac{\partial^2 I_{jk}}{\partial t^2} = \sum_{\alpha=1}^{N} m_\alpha \left(\ddot{x}_j^\alpha x_k^\alpha + 2\,\dot{x}_j^\alpha \dot{x}_k^\alpha + x_j^\alpha \ddot{x}_k^\alpha \right)$$

Consideremos a aceleração (Newtoniana) de uma partícula:

acceleration
of each particle:

$$\ddot{x}_j^{\alpha} = \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{N} \frac{G\, m_\beta (x_j^\beta - x_j^\alpha)}{|\vec{x}^\beta - \vec{x}^\alpha|^3}$$

$$\Rightarrow \frac{\partial^2 I_{jk}}{\partial t^2} = \underbrace{2\sum_{\alpha=1}^{N} m_\alpha \dot{x}_j^\alpha \dot{x}_k^\alpha}_{4K_{jk}} + \underbrace{\sum_{\substack{\alpha=1 \\ (\alpha\neq\beta)}}^{N}\sum_{\beta=1}^{N} \frac{G\, m_\alpha m_\beta}{|\vec{x}^\alpha - \vec{x}^\beta|^3}\left[(x_j^\beta - x_j^\alpha)\, x_k^\alpha + (x_k^\beta - x_k^\alpha)\, x_j^\alpha\right]}_{\text{Tensor de Energia Potencial}}$$

Potential energy tensor

$4K_{jk}$

Tensor de Energia Cinética

Kinetic energy tensor

Tensor de Energia Potencial

$$V_{jk} = \sum_{\substack{\alpha=1 \\ (\alpha\neq\beta)}}^{N}\sum_{\beta=1}^{N} G\, m_\alpha m_\beta \frac{x_j^\alpha (x_k^\beta - x_k^\alpha)}{|\vec{x}^\alpha - \vec{x}^\beta|^3}$$

Ou seja,

$$\frac{1}{2}\frac{\partial^2 I_{jk}}{\partial t^2} = 2K_{jk} + V_{jk}$$

This means that the evolution of the tensor of inertia of the set of gravitationally interacting particles is subject to this constraint (by definition).

Costuma-se calcular o traço para trabalhar com quantidades escalares

$$I = \sum_{\substack{j=1 \\ (j=k)}}^{3} I_{jk}$$

Introduce the trace, just to work with scalar quantities

$$K = T_r(K_{jk}) = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha v_\alpha^2$$

$$V = T_r(V_{jk}) = -\frac{1}{2} \sum \sum \frac{G m_\alpha m_\beta}{|\vec{x}^\alpha - \vec{x}^\beta|}$$

T. Virial: $\boxed{\frac{1}{2} \frac{d^2 I}{dt^2} = 2K + V}$

Theorem of the Virial

Sistema Virializado (ex: estruturas em equilíbrio após colapso)

When the system is virialized - the collapse has ended, the inertial tensor does not change anymore and the system remains with this energy condition E=V/2

$$V = -2K$$

$$(E = K + V) \rightarrow \quad E = -K \quad ; \quad E = \frac{V}{2}$$

Potential of an astrophysical structure $$\Phi = \frac{V}{m} = \frac{Gm}{r}$$

(note it has dimensions of velocity square: $[G M / r] = v^2$)

Theorem of the virial → $$\Phi = -\frac{2K}{m} = -v^2$$

i.e. the amplitude of the gravitational potential of a virialized structure is given by the velocity dispersion

Exemple of structures in the Universe : Clusters : v = 1000 km/s

Galaxies : v = 200 km/s

We need to compare these values with the amplitude of the "gravitational potential" of the homogeneous Universe.

But what is the potential of the Universe?

For the potential of the Universe we consider the following argument:

special relativity: accelerated frame → change of the time rate (the g_00 term of the metric) For example in Minkowski the accelerated frame has g_00 = $1-v^2/c^2$

From the Equivalence principle → the gravitational potential also changes the time rate

→ a potential (just like a velocity) affects the g_00 term of the metric → gravitational redshift

So the potential of the homogeneous Universe is just g_00, which is g_00 = c^2 → the kinetic velocity of the Universe (which is equivalent to a potential) is $v^2 = c^2$

$ds^2 = -c^2\, dt^2$ + spatial part

(potential + spatial curvature)

The existence of a local potential changes the term g_00 to: $\left(1-\frac{\Phi}{c^2}\right)^2$

We saw that galaxies and clusters have "small" dispersion velocities v << c → their gravitational potential is much smaller than the global potential of the homogeneous Universe $\Phi << c^2$ → **The astrophysical structures in the Universe only cause a perturbation in the homogeneous (Robertson-Walker) metric.**

Note: in the metric it is usual to approximate $\left(1-\frac{\Phi}{c^2}\right)^2 \approx 1-\frac{2\Phi}{c^2}$

Note:
Astrophysical structures are a scalar perturbation to the homogeneous metric of the Universe. There may be other types of perturbations to the metric. For example, gravitational waves are tensor perturbations to the spatial part of the homogeneous metric.

We conclude that the structures in the Universe can be considered perturbations to the cosmological principle

Homogeneous Universe - is in expansion - its gravitational dynamics are described by the homogeneous metric (Robertson-Walker).

Inhomogeneous Universe - consisting on global expansion + local linear clustering - its gravitational dynamics is described by the homogeneous metric with perturbation terms.

Collapsing structures - regions of space that are locally not expanding, they follow a non-linear collapse or are already collapsed - its gravitational dynamics is not described by the homogeneous metric with perturbation terms (they are weak gravitational fields, GR is not the best way to describe them. An exception are the black holes, which are strong gravitational fields and are described by a GR metric - but not the RW metric).

Fundamental properties of the zeroth-order Universe

Olbers' paradox - Luminosity distance – Universal redshift

Hypothesis: the fact that the night sky is dark may indicate that the Universe is not static.

Let us see why this is so.

Some definitions: Luminosity, Flux (L that reaches the observer), Surface Brightness (Flux concentration)

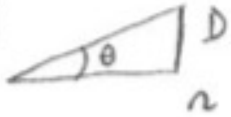
Luminosity $L = E/t$

Flux   Luminosity that "reaches us" $F = \frac{L}{4\pi r^2}$ (magnitudes)

Surface brightness S Flux per emission area $S = \frac{F}{\Omega}$
(brilho)

Solid angle (2d aperture) $\Omega = \theta^2$

Angular diameter (1d aperture)



$\theta \sim \mathrm{tg}\,\theta = \frac{D}{r}$

$$\Rightarrow S = \frac{L/r^2}{(D/r)^2} \sim \frac{L}{D^2}$$

An object of a given Luminosity and Size has a fixed S independent of its distance r.

Surface brightness is the ratio between 2 “apparent” quantities (flux - the apparent magnitude - and angular size - apparent size -) → **the brightness of an object is independent of its distance.**

Two objects of the same intrinsic size and with the same luminosity have the same surface brightness, regardless of its distance from the observer.

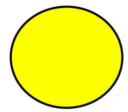
This fact has a very important consequence:

Let us consider two regions of the sky with a given angular size Ω that are completely filled with stars of equal luminosity and intrinsic sizes.

The fluxes of the regions are:

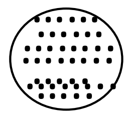
- region A with 1 large object that fills all the region (e.g. the Sun)

$F_A = S \times \Omega$



- region B filled with n stars identical to the Sun

$F_B = S \times \Omega_1 + S \times \Omega_2 + \ldots + S \times \Omega_n = S \times \Omega$



$$\sum_{i=1}^{n} \Omega\_i = \Omega$$

→ **the fluxes from the two regions are equal.**

Naturally, in region B, distant objects have a small angular size, but looking up to a faint magnitude limit (large distances) we can get an angular density of sources large enough to cover the full aperture.

Conclusion (for stars): If a sky aperture of the same size of the solar disk is filled with stars of luminosities similar to that of the Sun, the flux from that aperture is identical to the one coming from the Sun → **the sky should be always bright (day and night).**

The fact that this does not happen is known as the Olbers' paradox (1823)

Note that in fact this indeed happens for the observed Milky Way stars. In dense regions where stars "fill the regions", the "stellar sky" is bright.
There is no paradox here.

The eyes do not integrate for enough time and cannot detect the flux from faint stars, so most regions are not completely filled and the detected flux from them is lower to the naked eye.
But telescopes can saturate → the sky seen by a telescope is really bright!

However there is still a Olbers' paradox, but it applies only to **cosmologically distant objects** (like distant galaxies), so it applies to the "cosmological sky".
In that case, it is observed that even with an "infinite" integration time, the cosmological sky does not saturate, and this has implications for our modeling of the Universe. Let us see this in more detail.

The brightness of the sky can be computed in a more rigorous way:

Considerer the flux function: dN/dF, the number of objects per flux interval.
(Note that this type of functions - number counts per interval of a certain astronomical quantity - are very used in astrophysics: mass function, luminosity function, etc.)

Consider n galaxies per unit volume

How many galaxies have fluxes in the range F to F+dF

Radial shell

$$V_{shell} = 4\pi r^2 dr$$

Number of stars in the shell $\quad dN = n\, 4\pi r^2 dr$

Now, $F_{1\,gal} = \frac{L}{4\pi r^2} \Rightarrow \frac{dF}{dr} = -\frac{L}{4\pi} r^{-3}$

(assume Luminosity is constant, i.e., equal in all objects of the sample, and non-evolving, no r dependence)

Number of galaxies with flux between F and F+dF is $dN = \frac{dN}{dF} dF$ (definition)

$$dN = \frac{dN}{dF} dF = \frac{dN}{dr} \frac{dr}{dF} dF = \frac{n}{L} 4\pi r^2 \, 4\pi r^3 dF \propto r^5 dF$$

(there is a one-to-one relation between flux and distance)

Note that

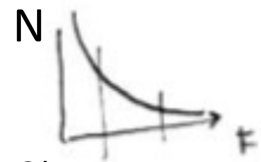
$$r = \sqrt{\frac{L}{4\pi F}} \propto F^{-1/2} \quad \Rightarrow \quad \boxed{dN \propto F^{-5/2} dF}$$ flux function

There are more objects with small flux - the distant ones - than with large flux - the closer ones - (at equal luminosity)

Integrate up to the detection flux-limit (magnitude limit):

Now we can compute the total flux of stars brighter than a certain minimum F_0 :

$$F_{\rm tot} = -\int_{F_{\rm min}}^{F_{\rm max}} F \frac{dN}{dF} dF \sim -\int_{F_{\rm min}}^{F_{\rm max}} F^{-3/2} dF \sim F_{\rm min}^{-1/2} - F_{\rm max}^{-1/2} \rightarrow \infty$$



(when F_min → 0)

(this is the standard way to compute a total or a weighted mean - the flux function is a weight function)

If we get the flux of objects up to $F \sim 0$ (i.e., including objects up to $r \rightarrow \infty$), then the total flux would be infinite → the bright night sky

Why is the (cosmological) night sky not bright ?

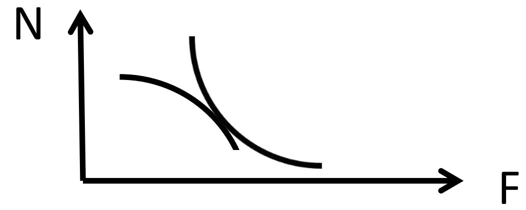
In reality we cannot integrate up to infinite distance (flux zero) if the object is not eternal (has an initial time). There is a cut-off $F_{min} > 0$ and the integral is finite. However it could still be very large → **Assuming an initial time does not solve the paradox.**

Perhaps there is absorption and part of the flux is lost?
True, but there would be re-emission of the absorbed flux that would still contribute to the total flux, even if in another form (such as with a different wavelength) →
Absorption is also not the solution.

Possible solution of the paradox:

To solve the problem in an absolute way, the best would be to obtain a total flux that would not go to infinity even in an infinite universe (i.e., even in the case F_min → 0). In that case it would be understandable that the night sky is not bright.

This can be achieved if the function dN/dF would be different, in particular if it would have a shallower slope → if the number of objects with small flux was smaller than predicted.



But their number on each spherical shell must increase with r^2 in a scenario of uniform distribution (homogeneity).

Moreover, even if some objects would disappear (end of life), others would appear to replace them (and why would this affect more the distant than the closer objects?) → **Finite life-time is also not a solution.**

However, what if the distant galaxies would contribute less to the flux? → meaning, they would have a smaller brightness → i.e., it would be like a smaller effective number of galaxies (even though the number would not change). But we saw that brightness does not depend on distance … or does it?

Could S become distance-dependent?

i) A possibility would be if Luminosity L or size D were distance-dependent → **all objects would evolve in time** (since the more distant ones are in the past) **in a universal way**, such that luminosity would always increase (smaller in the past) - or the intrinsic size would decrease (larger in the past) → **universal intrinsic evolution of luminosities or sizes.**

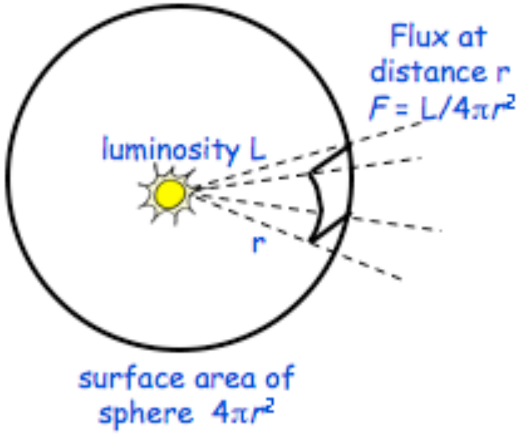
It seems unlikely to happen! and in fact this is not observed

ii) Another possibility would be that the flux (the numerator in the expression for S) does not change with r^2, but with a different f(r). This could happen if there exists a mechanism that would make the luminosity emitted by the distant objects to be somehow diluted *during propagation* → **universal loss of luminosity.**

Note that this is different than the first possibility, where the intrinsic luminosities of all objects would decrease (an astrophysical evolution).

This loss of luminosity during propagation would need to **increase with distance**, for the effect to go in the right direction.

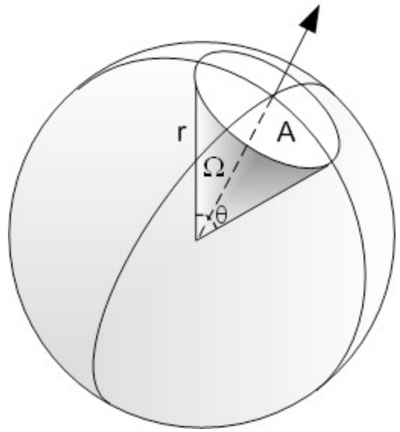
This also seems unlikely to happen! Needs to be tested with observations!



In other words, the hypothesis is that

flux(r) at a distance r from the source is less than L/r^2

while the angular size of a source of intrinsic area D^2 is the usual D^2/r^2



$$S = \frac{L/r^2}{(D/r)^2}$$

d_L

d_A

The angular size / intrinsic size relation would be the true geometrical distance 'r' → the "angular diameter distance" d_A

The flux / luminosity relation would depend not only on the geometrical distance but also on an extra factor of "luminosity loss" → by convention, this factor is absorbed in an effective 'r' in the numerator, defining an effective distance different from 'r' → the "luminosity distance" d_L

For this mechanism to solve the paradox the two distances must be related as

$$d_L = f(r)\, d_A \qquad \text{(i.e., the extra factor must be function of 'r').}$$

What mechanism could produce this effect?

Hypothesis: a universal change in all photons wavelength as they propagate from source (e) to observer (o) can produce this effect.

In particular, we need a **redshift** (not a blueshift), because the goal is to decrease the contribution of distant sources (not nearby ones).

Redshift is defined as $z = \dfrac{\lambda_o - \lambda_e}{\lambda_e} = \dfrac{\Delta t_o - \Delta t_e}{\Delta t_e}$ (if $z<0 \Rightarrow$ blueshift)

The existence of a redshift alters the luminosity propagation in two ways:

- increase of the photons wavelength → decrease of frequency → universal loss of energy $E_0 = E_e /(1+z)$

- decrease of frequency → universal increase of the time interval between two pulses → $\Delta t_0 = \Delta t_e (1+z)$ → less photons arriving per unit of time

Remember that

$$L_0 = \frac{E_0}{\Delta t_0}$$

and so the combination of the two effects creates a **luminosity loss of $(1+z)^2$**

By convention, this factor is absorbed in the definition of a new distance → the "luminosity distance" d_L

$$\mathbf{d_L = (1+z(r))^2 \, d_A}$$

This relation is known as **Etherington's distance-duality relation**

Measurements of d_A and d_L are used to test this relation at various redshifts.
If a deviation from $(1+z)^2$ is found, it means that the luminosity loss is not caused by redshift (or *only* by redshift), but there are other effects contributing to it:

non-conservation of photon number? → it would be a hint for **new physics**.

(e.g., Martinelli et al 2020, https://arxiv.org/pdf/2007.16153.pdf)

Let us now insert the result in the expression for the surface brightness:

$$S = \frac{L}{D^2}\left(\frac{d_A}{d_L}\right)^2 = \frac{L}{D^2}\left(\frac{1}{f(1+z(r))}\right)^2$$

We confirm that the brightness is no longer distance-independent, but becomes redshift-dependent:

$$S = \frac{L}{D^2}\left(\frac{d_A}{d_L}\right)^2 = \frac{L}{D^2}\frac{1}{(1+z)^4}$$

This extra factor of $(1+z)^4$ solves Olbers' paradox, since the flux no longer diverges in the small flux limit:

$$F_{\rm tot} \sim \int_{F_{\rm min}}^{F_{\rm max}} F^2 F^{-5/2} dF \sim F_{\rm max}^{1/2} - F_{\rm min}^{1/2}$$

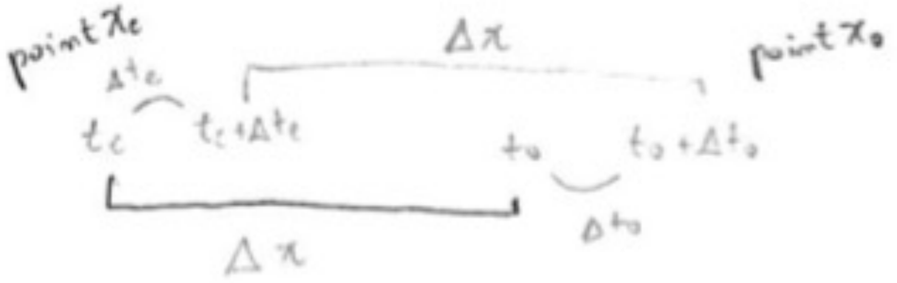
(remember that the earlier result was $F^{-1/2}$)

Expansion - cosmic flow – redshift

We saw that the universal redshift is capable of explaining why the cosmological sky is not bright.

Now, is there a plausible mechanism that can produce this type of universal redshift?

Hypothesis: *one possible mechanism is to consider that the* ***universe evolves (expands) as opposed to being static.***



Let us then consider an expanding Universe,

and at the same time let us try to find an expression for z(r)

To be in agreement with the cosmological principle, let us consider the Universe as a **homogeneous** sphere that expands **isotropically**.

Radial expansion is the only expansion model that keeps the homogeneity (note however that it is not the only possibility to ensure isotropy).

Consider the following:

Let us write $\vec{r} = a(t)\,\vec{x}$ → homogeneity allows to separate time and space.

position of matter element

$\vec{x}$: comoving coordinate is constant in time

it is also the physical distance at t : $a(t) = 1$

(this is t_0 by definition, i.e. today) $0 < a < 1$

$\Downarrow$

$r < x$

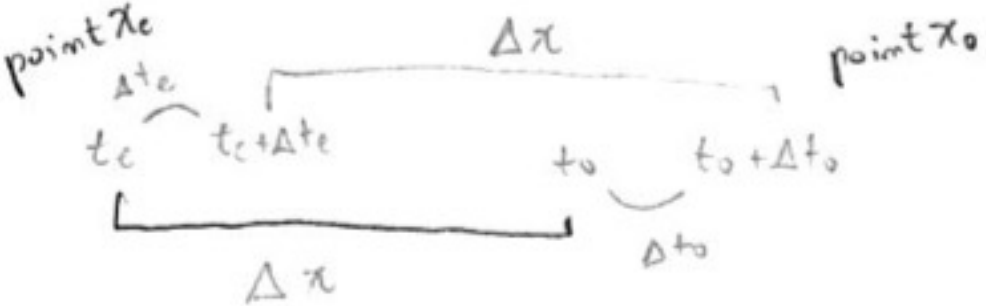
$a(t)$ cosmic scale factor

Some quantities appear naturally with the expansion

x - comoving coordinate - the absolute reference frame → a particle that is comoving with the expansion keeps a constant value of its comoving coordinate.
a(t) - scale factor
r(t) - proper coordinate

convention: $0<a<1$ → $r < x$

Let us consider the emission of a lightwave (2 pulses) in the expanding Universe



x_e, x_0 comoving coordinates

Δt_e and Δt_0 may be different.

(due to the expansion)

the 2 pulses around Δt_e define the frequency of the emitted signal

The " " Δt_0 " " " received "

$$\nu_e = \frac{1}{\Delta t_e} \quad , \quad \nu_0 = \frac{1}{\Delta t_0}$$

$$\lambda_e = c\Delta t_e \quad , \quad \lambda_0 = c\Delta t_0$$

Redshift is defined as $z = \dfrac{\lambda_0 - \lambda_e}{\lambda_e} = \dfrac{\Delta t_0 - \Delta t_e}{\Delta t_e}$ (if $z<0 \Rightarrow$ blueshift)

We want to derive an expression for the universal redshift z(r) created by the expansion

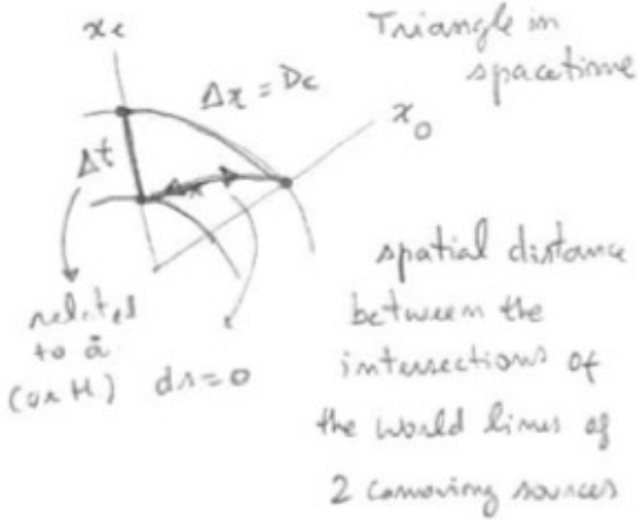
Since we are dealing with non-instantaneous light propagation, we will need to use (special) relativity (no need for GR, no dynamics involved) (so flatness is assumed).

Let us compute Δt_e and Δt_o using the metric:

$$ds^2 = -dt^2 + a^2\left(d\chi^2 + f_K^2(\chi)\, d\Omega^2\right)$$

$ds^2 = 0$ null geodesic $\rightarrow$ $dt = a\, d\chi$

Note that this relation is space-time trigonometry!



Knowing the "hypotenuse"(ds^2=0) and one side of the triangle (dx^2), we get the other side (dt^2).

$$\Delta\chi = \int_{\chi_e}^{\chi_o} d\chi = \int_{t_e}^{t_o} \frac{dt}{a(t)} = \text{Constant}$$

The “space-time triangle” allows us to find a relation between the time ratios and the scale factor:

$$\Delta\chi = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} \longrightarrow = \int_{t_e+\Delta t_e}^{t_0} \frac{dt}{a} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a} =$$

$$= \int_{t_e}^{t_0} \frac{dt}{a} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a} = \Delta\chi - \frac{\Delta t_e}{a(t_e)} + \frac{\Delta t_0}{a(t_0)}$$

assume a(t) is cte over this interval

$$\Rightarrow \boxed{\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)}}$$

$$\rightarrow z = \frac{\Delta t_0}{\Delta t_e} - 1 \quad \Leftrightarrow \quad \boxed{z = \frac{a(t_0)}{a(t_e)} - 1}$$

(definição)

with $a(t_0) = 1$, today

$$\Rightarrow z = \frac{1}{a} - 1 \quad \Leftrightarrow \quad \boxed{a = \frac{1}{1+z}}$$

This is the result: $1 + z(r) = 1 / a(r)$

In fact, we did not find an explicit solution for $z(r)$ but only a relation $z(a)$ (Note that this well-known relation is the result of a derivation, it is not the definition of redshift).

The model for the expansion is characterized by $a(r)$, where a varies from $a(r=\infty) = 0$ to $a(r=0) = 1$,
and thus the expansion indeed creates a universal redshift with the required properties → z increases with r, and $z(r=0)=0$

(Note: the monotonic behavior $z(a)$ is the reason why the redshift can be used as a time variable in the evolution of the Universe)

$a(r)$ is a central quantity that characterizes the cosmological model at the homogeneous level. It is determined from the equations of the theory of which the expansion model is a solution (a theory of gravity). Similarly, the behaviours of $a(t)$, $z(r)$, or inversely $r(z)$, should all be predictions of the theory.

Measurements of these functions, especially $r(z)$, i.e. $d_L(z)$ and $d_A(z)$, are widely used to test the cosmological model.

Hubble's law - Hubble radius

The observation of the dark night sky is a quite indirect hint of the expansion of the Universe! **Eventually a more direct observation was made.**
Universal redshift observed: it was observed that the redshift of all observed galaxies increased with their distance (linearly).

Hubble law (local)

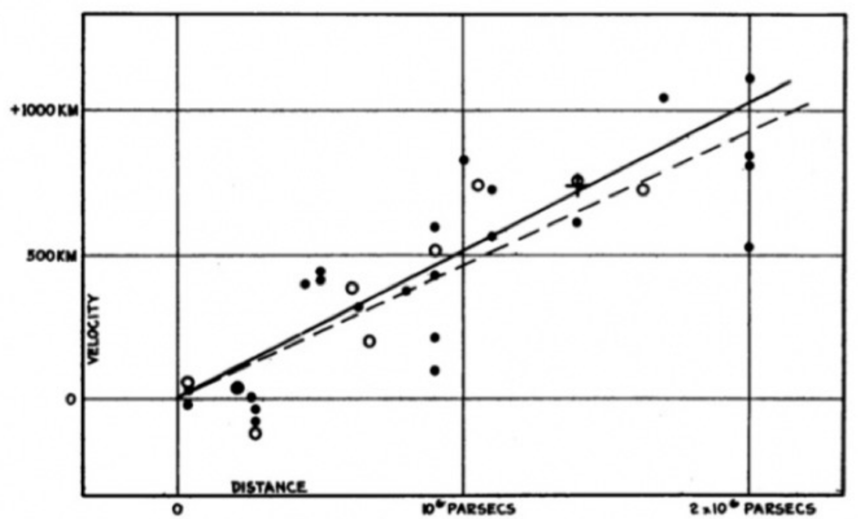


FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.

This correlation was interpreted as a universal recession of the galaxies because the redshift was interpreted as a velocity through Doppler's effect:
$v/c = z$

The observations support the expanding model that had already been proposed as possible solutions of Einstein equations: prediction of a universal redshift, expansion dynamics (**Friedmann** 1922), derivation of z(r) (**Lemaitre** 1927).

Vesto Slipher (Obs. Lowell)
measured redshifts in galaxy spectra (1912-1922)
41 galaxies, most with $z > 0$

Milton Humason (Mt. Wilson)
measured redshifts in galaxy spectra (1920)

Edwin Hubble (Mt Wilson)
measured galaxy distances (1923-1926)

Georges Lemaître combined the 2 types of measurements and found a linear relation $v = \text{constant} \cdot d$ (1927) with a slope H_0 = 625 Km/s/Mpc [Ann. Soc. Sci. Brux.]

Hubble e Humason
new distance measurements using Cepheids (1927-1929)

Hubble combined the 2 types of measurements and found a linear relation v(d) (1929) with a slope H_0 = 530 Km/s/Mpc

Lemaître published the english translation of his paper in MNRAS (1931), but his results about the linear correlation and the H_0 value were not included.

Until recently the reason for the non-inclusion of the main results of Lemaitre in the MNRAS paper was a mystery. Was it a conspiracy made by Hubble?

In 2011, Mario Lívio researching the letters between Lemaitre, the translator and the editor, found out that Lemaitre himself has asked to not include the results that he considered were already “old news”.

More recently, this issue was debated in the annual meeting of the IAU (2018) and there was a voting open to the worldwide research community, to propose the change of the naming of Hubble’s law.

78% of the votes approved the change: since N**ovember 2018, Hubble’s law is now named Hubble-Lemaitre’s law.**

These results introduced the idea of a recession velocity v(t)

The linear relation is consistent with an expansion r = ax. Indeed,

$$r = ax \rightarrow v = \mathring{a}\, x = \mathring{a}/a\ ax = H_0\, r$$

(the linear relation tells us that the meaning of the constant slope is $\mathring{a}/a$)

(note: $\mathring{a} = da/dt$)

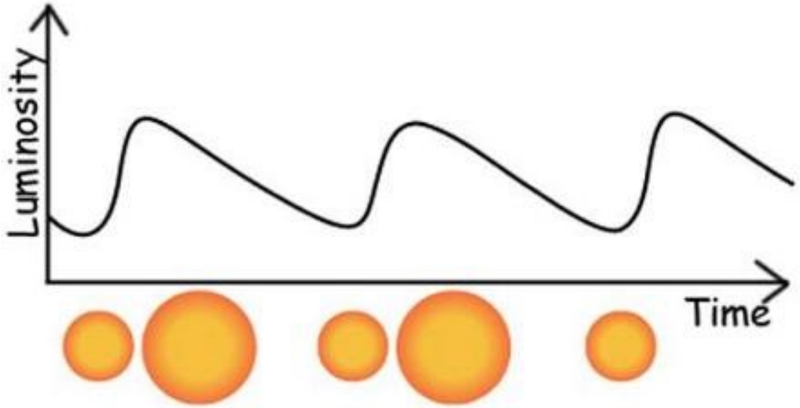
This defines the Hubble constant $H_0 = \mathring{a}/a$

$$H_0 = 100\ h\ \mathrm{Km/s/Mpc}$$

$h \sim 0.7 \rightarrow H_0 \sim 70$ Km/s/Mpc $\rightarrow$ a galaxy that is 1 Mpc more distant than one closer to the observer, recedes with a velocity 70 Km/s faster than the one that is closer.

The distance measurements were made by identifying Cepheid stars on the observed galaxies.

These are variable stars (pulsating radially)



They are very bright stars (10 000 times more than the Sun).

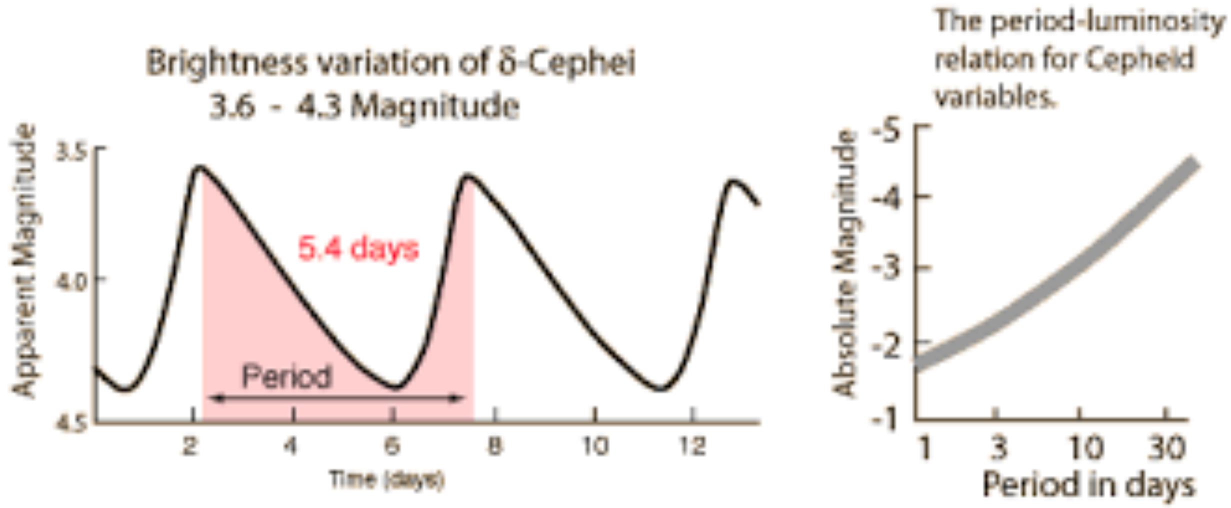
They are in a thermodynamical unstable state: they have a layer of compressed ionized He at a depth such that it traps heat produced by the star making the star to expand. The expansions makes it to cool down and He recombines (neutral gas) => energy escapes and the star falls inward, compressing He again → a cycle is produced.

The main point is that the Period of oscillation is related with the luminosity.

$\uparrow P \Rightarrow \uparrow L \Rightarrow \downarrow M$

P → L

L + F → D



The period increses with the luminosity peak.

So, the period is the proxy for the distance (subject to calibration)

The absolute values of the period-luminosity relation are calibrated with observations (more reliable than to calibrate from a theoretical model for the astrophysics of these stars) → need to observe other Cepheids with known distances (D + F → L). Those are Cepheids in our galaxy (eg: polaris or δ Ceph)

The distances to these nearby Cepheids are obtained by parallax.
(Earth-Moon eclipse, Earth-Sun baseline, 1pc = 1arcsec)

These are the first steps of the so-called distance ladder.

Methods and distance ladder

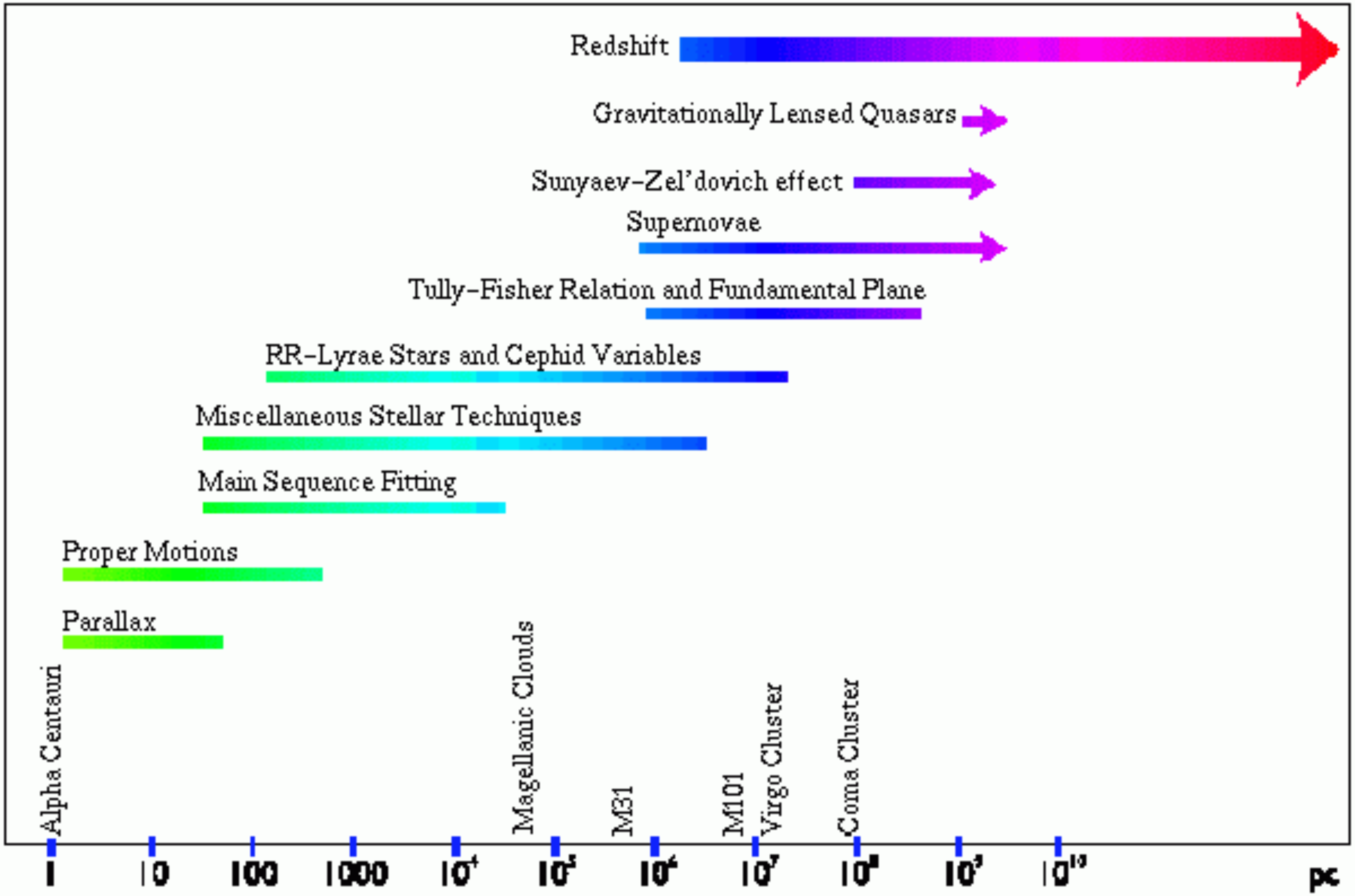


Figure 3.2: *The different distance estimators. This seemingly simple plot shows a grand overview of our efforts to measure distances in the Universe. Adapted from [Rowan-Robinson, 1985] and [Roth and Primack, 1996].*

www.astro.gla.ac.uk/ users/kenton/C185/

Cumulative errors in the intermediate steps of the ladder introduce large uncertainties in the final result.

The result from Hubble is H_0 = 530 Km/s/Mpc

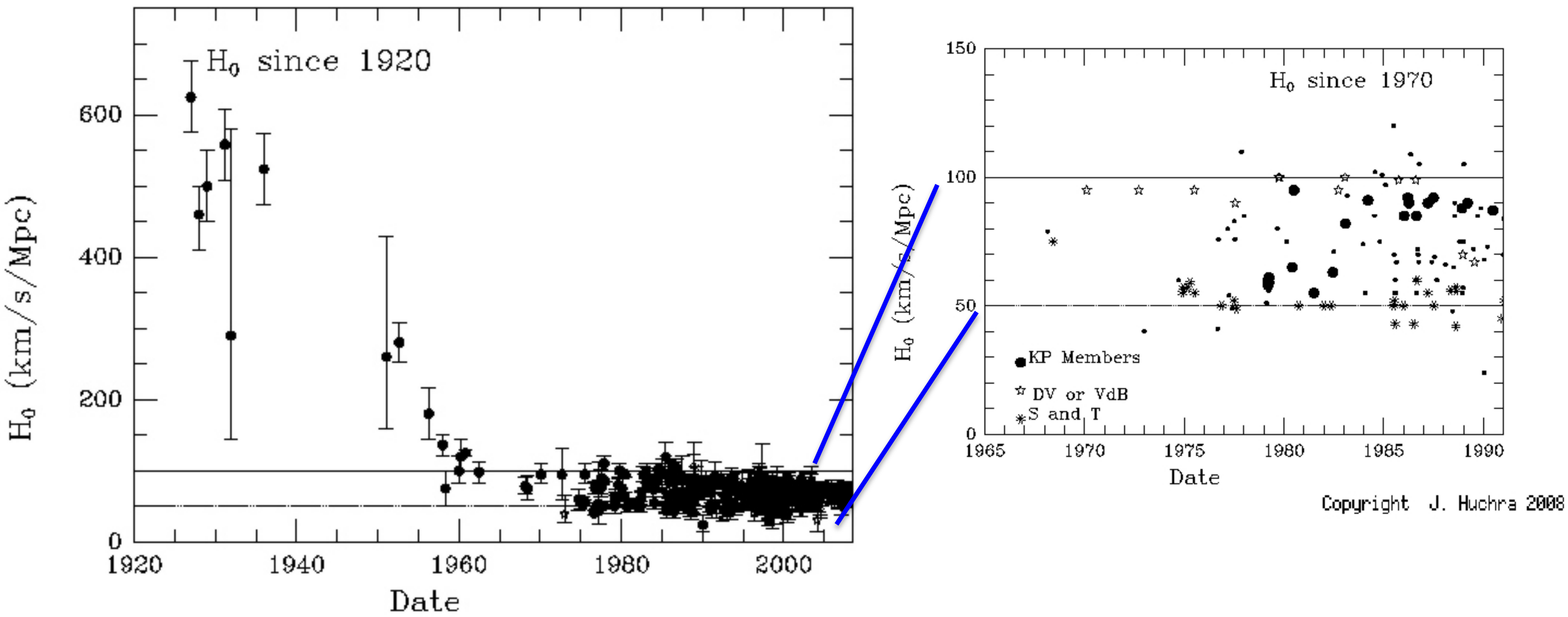
In 1929, Hubble published a result of $H_0 = 530$ Km/s Mpc^{-1}, already using Cepheids.

But in fact there are two types of Cepheids (ones are Population I stars, and are brightest, and the others are Population II stars). He was observing both types without knowing, and using a P-L relation valid only for Type II.

Furthermore, for farther away galaxies where Cepheids were not seen, he was observing very bright stars (like Novae), but some of them were in fact not stars but H II regions.

The determination of the Hubble constant has dominated observational cosmology throughout all the XXth century!

Only in the XXIth century did other cosmological parameters start to be measured with higher precision and using a great variety of methods → CMB, galaxy clustering, BAO, weak lensing, etc. → precision cosmology

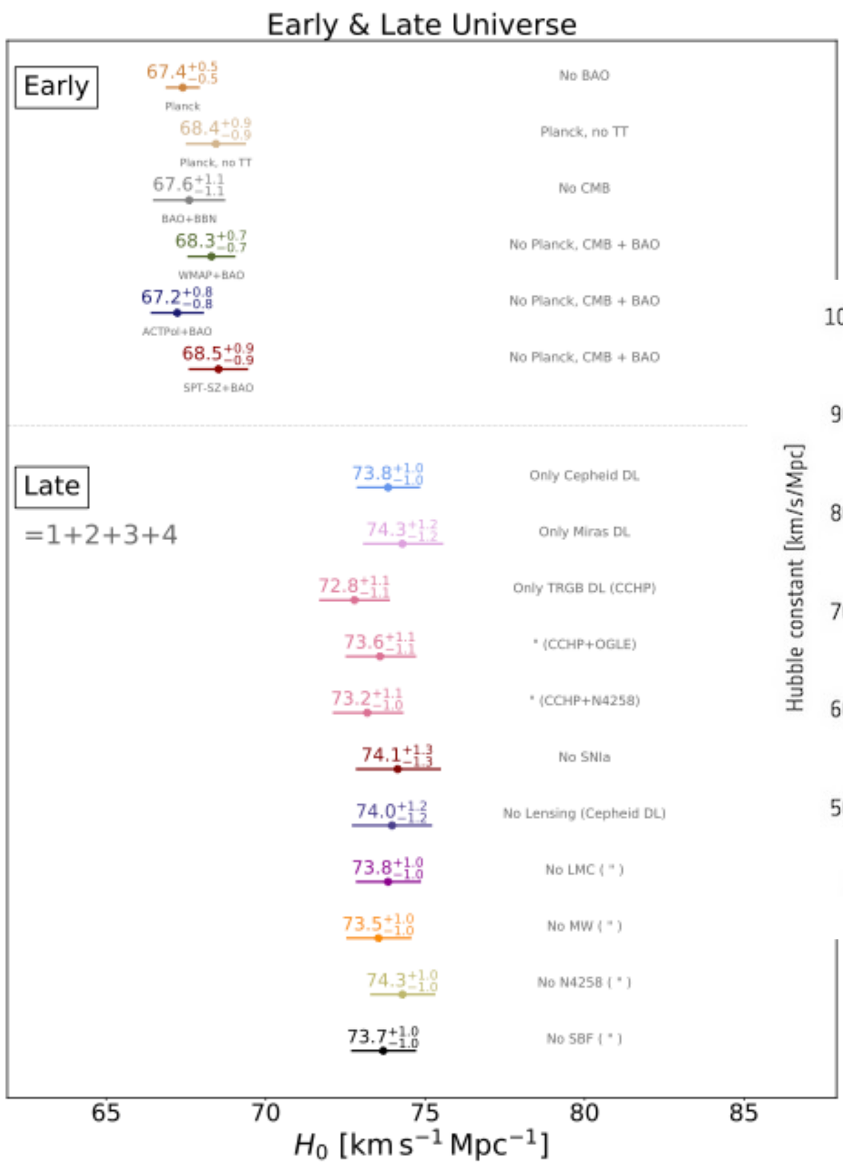


Long discrepancy between 2 groups: h ~1 e h ~0.5 due to issues with evolution and calibration on sources with peculiar velocities.

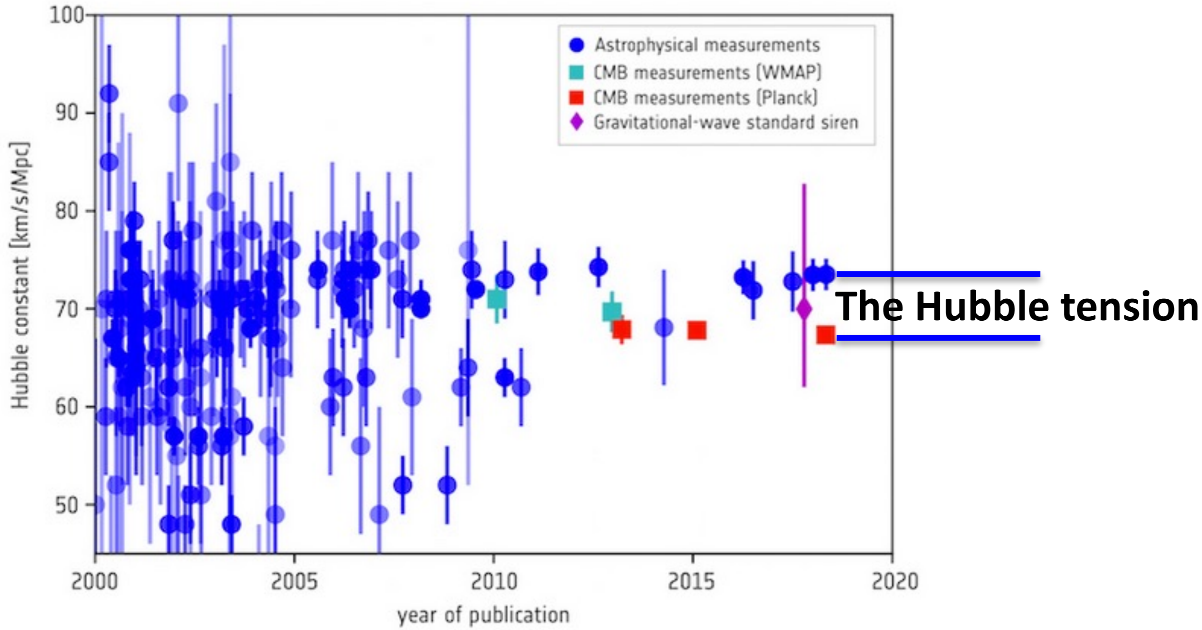
The polemic only ended in 2001 with the HST Key Project:

by observing Cepheids and supernovae in the same galaxy → h = 0.72 ± 5%

However the debate re-opened in the last decade with the H_0 measurements made by the CMB Planck mission (and other surveys of the early Universe) finding lower values → 0.67 ± 1%



There is a systematic separation between the higher (lower) values found with late (early) universe data.



Further reading:

Riess et al 2020, https://arxiv.org/pdf 2001.03624.pdf

Knox & Millea 2020, https://arxiv.org/pdf/1908.03663.pdf

Note that the **Hubble law is also a relation between the scale factor and redshift** (and so it is a direct solution of Olbers' paradox):

Hubble law:

$$v = H_0 r \Leftrightarrow z = \frac{H_0}{c} r$$ (assuming Doppler effect z = v/c)

Now, the observations were made at a = 1 and are valid for the local Universe.

In the local Universe (a ~1) we have 1-a = Δa = Δt (da/dt)_t_0 (Taylor expansion)

→ $$\frac{1-a}{\Delta t} = \frac{\dot{a}}{a}$$ (using $a(t_0) = 1$)

$$H_0 \, \Delta t = 1 - a$$

Considering Δt = r/c, we get : z = 1- a

→ **so Hubble law tells us that z = 1 - a**

This means that the assumption of the Doppler effect, plus that the linear relation z vs r translates into a linear relation v vs r → implies a linear relation z vs a.

This is not the expression we found before.
But note it is a linear approximation to our expression (Taylor expansion) :

$$a = \frac{1}{1+z} = 1 - z + \mathcal{O}(z^2)$$

This means that only in the local Universe ('a' close to 1, z close to 0) can the redshift be interpreted as a Doppler effect

and the relation redshift vs scale factor (or redshift vs distance) is linear.

This relation - **(local) Hubble law -** was the one observed by Hubble.

We see that in general, the relation between redshift and scale factor is not linear **and the interpretation of the redshift as a Doppler effect leads to an inconsistency.**

However the relation v vs r can be written in a (apparently) linear form defining the

Hubble function H(t) $$H(t) = \frac{\dot{a}}{a}(t)$$

instead of the Hubble constant H_0 → a **generalized Hubble law.**

The velocity of a particle is given by

$$\vec{v}(\vec{r},t) = \dot{\vec{r}} = \dot{a}\,\vec{x} = \frac{\dot{a}}{a}\,a\,\vec{x} = \frac{\dot{a}}{a}\,\vec{r} \Rightarrow \vec{v}_{(t)} = H(t)\,\vec{r}(t)$$

where $H(t) = \frac{\dot{a}(t)}{a(t)}$ is the <u>expansion rate</u>

$$H_0 = \frac{\dot{a}}{a}(t=t_0) = \dot{a}(t_0) \rightarrow \underline{\text{Hubble parameter}}$$

We can also define a **Hubble length**, since the inverse of the Hubble function has dimensions of time (or length considering c = 1 dimensionless):

r_H (t) - Hubble radius = c/H

Hubble radius today is = 3000 Mpc/h

At $r = r_H \rightarrow v = c \rightarrow$ Beyond the Hubble radius, recession velocities are larger than the speed of light. (This is not a problem since the interpretation of the recession as a Doppler efffect is only valid in the local Universe).

Note however that

since a(t) grows → H(t) decreases in time → $r_H(t)$ grows
(e.g. power law expansion $a(t) \sim t^n \rightarrow \dot{a}(t) \sim t^{n-1} \rightarrow r_H(t) \sim t$)

if a(t) grows decelerating (slope n < 1) → **$r_H(t)$ grows faster than a(t)**

This explains (an apparent paradox) why in a decelerating Universe, **we can observe objects beyond the Hubble radius**, i.e., we detect light coming from points with "recession velocity" larger than c:

Those photons start by being dragged away by the expansion and their proper distance to the observer initially increases. But to the increase of r_H, those points even though farther away have a decreasing recession velocity and they end up being caught by the growing Hubble radius reaching regions where v<c. From that point on, their proper distance starts to decrease, until reaching the observer.

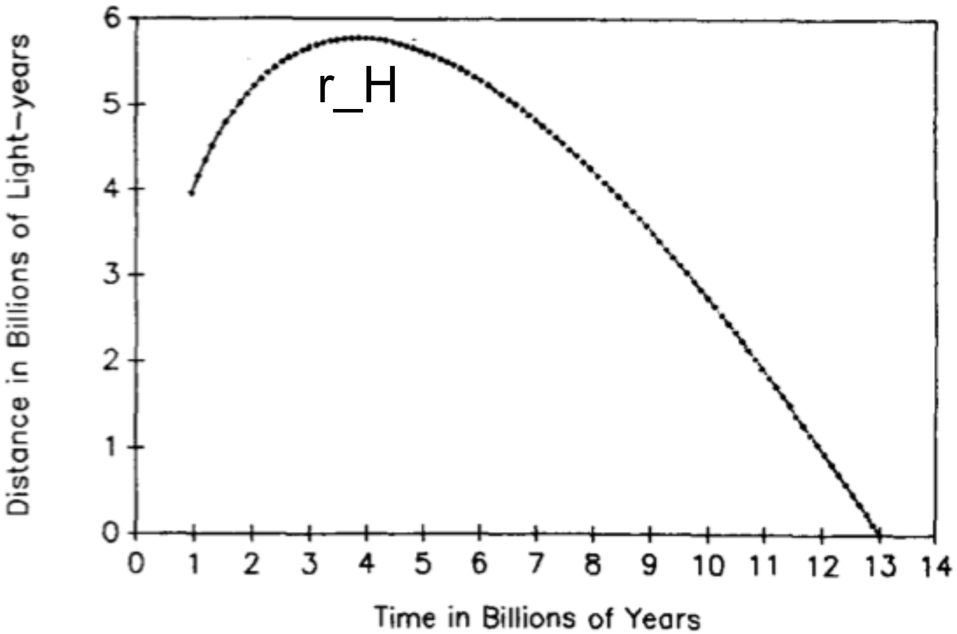


Fig. 1. The photon is emitted from a quasar at $t_e = 0.95$ Gyrs. The quasar (and thus the photon) is 3.8 Gcyrs away from us at time of emission. Initially the photon is "dragged away from us" by the cosmological expansion to a distance of 5.8 Gcyrs at $t = 3.9$ Gyrs. At this time gravity has slowed the expansion rate of space such that the photon is at a coordinate position with recessional velocity of c. The photon then begins to approach us and arrives at $t_0 = 13$ Gyrs.

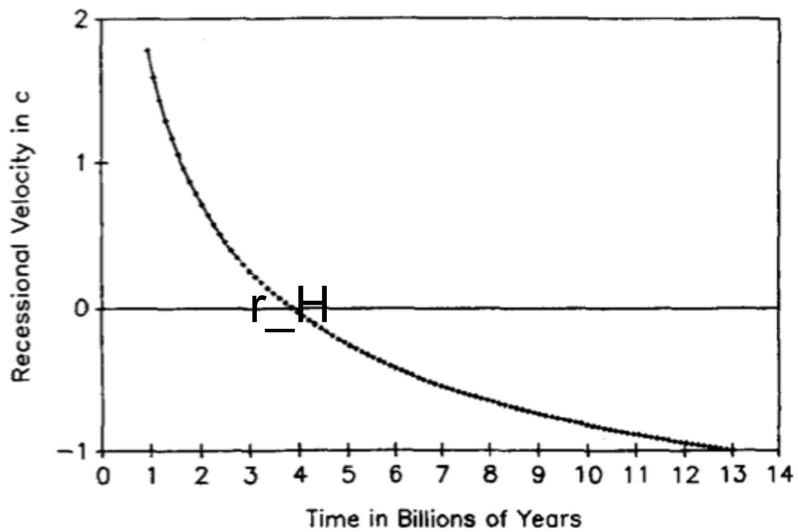
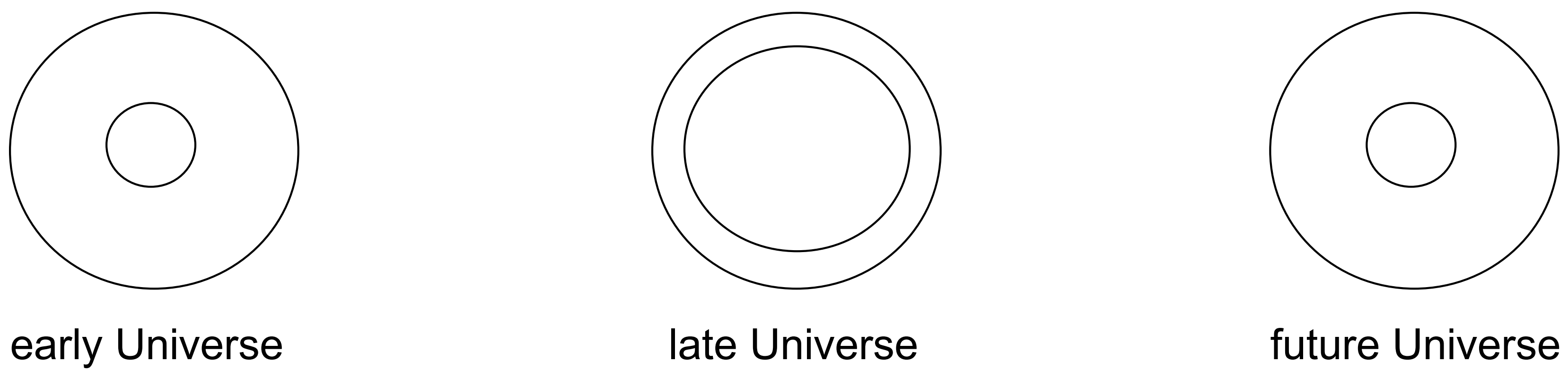


Fig. 2. Because the photon is being "dragged away from us" initially, its recessional velocity is initially positive. At emission the photon is receding at 1.8c. $\dot{r}_p = 0$ corresponds to the time when gravity has slowed the recessional velocity of space at the photon's position to c. After $t = 3.9$ Gyrs, the amount of expanding space between the photon and receiver is decreasing and the photon approaches the receiver at an increasing rate until the recessional velocity at reception is $-c$.

The fact that the **Hubble radius is not comoving** with the expansion provides a natural way of introducing a feature (a scale) in the homogeneous Universe → r_H is a purely kinematical characteristic scale.

This means that the purely homogeneous Universe at different times is more than just an expanded version of itself.

(by purely homogeneous, I mean a completely empty homogeneous Universe, with no structure evolution, or thermal evolution, or radiation emission and peculiar movements, which of course are physical characteristics that allow us to infer there is an evolution).



The Hubble radius also provides the weak/strong gravity threshold

Consider **a sphere with Hubble radius**.

Its potential is $V = G\, M / r \sim \rho\, r^{2}$:

$GM/r_H = 6.67 \times 10^{-11}\ 4/3\ \pi\ 9 \times 10^{-27}\ (3000/0.7)^2\ (3.1 \times 10^{22})^2$ N m/Kg (i.e. m^2/s^2)

using $G = 6.67 \times 10^{-11}$ N m^2 / Kg^2

$\rho = 9 \times 10^{-27}$ Kg / m^3

$r_H = 3000/0.7$ Mpc

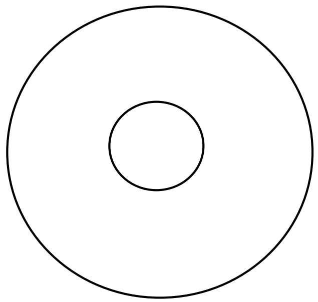
1 pc = 3.1×10^{16} m

This means that $GM/r_H = 4.44 \times 10^{16}$ N m / Kg

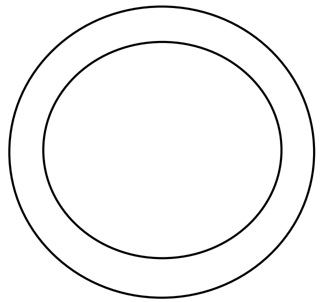
Note that [N m /kg] = [$(m/s)^2$] → the corresponding velocity is

$$v = 2.1 \times 10^5 \text{ Km/s} \sim c$$

and so the Hubble radius marks the limit of Newtonian gravity
(also called the "space-time curvature radius of the Universe", beyond which GR gravity is mandatory and new gravitation effects may arise)



early Universe (more relativistic)



late Universe (more Newtonian)

→ for $r < r_H$ we can use Newtonian mechanics (plus special relativity) and consider a flat space-time,

and indeed **we saw that fundamental concepts such as the cosmological principle, the expanding Universe and the redshift do not arise from general relativity.**

Note also that from $V = G M / r \sim \rho r^2$ we conclude that

→ for the same density, a larger region is a stronger gravitational field (hence "more relativistic") → GR effects are important on very large-scales (Newtonian gravity breaks down)

On the contrary on smaller (and also large, up to the Hubble radius) regions of the Universe, Newtonian gravity is still a good description. This includes galaxies, clusters, and even many large-scale structures (LSS) and filaments.

→ for the same size, a denser region is a stronger gravitational field (hence "more relativistic") → GR effects are important on very dense small-scales (like black holes)

This is another strong gravity regime, the **small-distance limit**, where interactions are strong even with smaller masses. This gives rise to the strong gravity effects for black holes and the PPN corrections in solar system mechanics.

So GR is very relevant for very large scales cosmology and black holes physics, but less used in astrophysics.