UNIVERSO PRIMITIVO

Mestrado em Física Astronomia 2024-2025

Exercise Sheet 1

1. In a FRLW universe, fundamental observers experience no external forces and have fixed coordinates in the comoving coordinate system. The proper distance between two of such observers scales as $\boldsymbol{r}(t) = a(t)\,\boldsymbol{x}$, where $a(t)$ is the scale factor and $\boldsymbol{x}$ is their comoving separation.
   1.1. Derivative this expression to obtain the Hubble law, $\boldsymbol{v}(t) = H(t)\,\boldsymbol{r}(t)$, where $H = \dot{a}/a$.
   1.2. Derive a similar expression for a pair of non-fundamental observers that have a relative peculiar velocity, $\boldsymbol{v_p} = \dot{\boldsymbol{x}}$, in the commoving coordinate system.

2. Consider the energy-stress tensor with mixed indexes, T^{μ}_{ν}, for a homogeneous and isotropic perfect fluid.
   2.1. Apply the conservation law $T^{\mu}_{\nu;\mu} = 0$ to the $\nu = 0$ component to obtain the energy conservation equation $\dot{\rho} = -3H(\rho + p)$, where $H = \dot{a}/a$ is the *Hubble constant*.
   2.2. Use this equation to prove that $dE = -pdV$, where $dE = d(\rho a^3 L^3)$ is the energy inside a volume element, $dV = d(a^3 L^3)$, where L^3 is an arbitrary comoving volume.
   2.3. Integrate the energy conservation equation in 2.1 to prove that $\rho(t) = \rho_i \left(\frac{a(t)}{a_i}\right)^{-3(1+w)}$ where ρ_i, a_i are integration constants and w is the equation of state (EoS) parameter for a given fluid component.
   2.4. Use the expression in 2.3 to derive the time dependence of the scale factor for the following components: radiation ($w = 1/3$); collisionless matter ($w = 0$); and cosmological constant ($w = -1$) assuming the conditions (1), (2) and (3) at the bottom of slide 12 of chapter 2 of the course notes, respectively.

3. Consider the FLRW dynamic equations discussed in class.
   3.1. Use the Friedman equation and the acceleration equations to derive the energy conservation equation in 2.1.
   3.2. Use the definition of the cosmological density parameters to re-write Friedmann equation in the following form (the subscript 'm' refers to all forms of matter, i.e. baryon and dark matter):

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}(\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \\ &= H_0^2 \left[\Omega_{r0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda 0}\right] \end{aligned}$$

   3.3. Consider the concordance model with: $\Omega_{\mathrm{r0}} = 9.4 \times 10^{-5}$, $\Omega_{\mathrm{m0}} = 0.32$, $\Omega_{\mathrm{k0}} = 0$, $\Omega_{\Lambda 0} = 0.68$. Derive approximate values for the redshift at radiation-matter equality and matter-dark energy equality epochs.

4. Use the Friedmann equation in 3.2 to compute the Age of the universe for a:
   4.1. Critical density universe ($\Omega_{\mathrm{r0}} = 0$, $\Omega_{\mathrm{m0}} = 1$, $\Omega_{\mathrm{k0}} = 0$, $\Omega_{\Lambda 0} = 0$), with $H_0 = 70\ km\ s^{-1}\ Mpc^{-1}$
   4.2. Flat, Λ −Universe with $\Omega_{\mathrm{r0}} \simeq 0$, $\Omega_{\mathrm{m0}} = 0.32$, $\Omega_{\mathrm{k0}} = 0$, $\Omega_{\Lambda 0} = 0.68$, $H_0 = 70\ km\ s^{-1}\ Mpc^{-1}$
   [Hint: integrate the Friedmann equation with respect to the scale factor]