UNIVERSO PRIMITIVO

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Exercise Sheet 4

1. Use the relation $T_\nu = (4/11)^{1/3}\, T_\gamma$ derived in classroom to obtain the cosmic neutrino background number and energy density expressions in slide 24 of Chapter 4. Compute these densities at present assuming an effective number of neutrino families equal to $N_{eff} = 3.046$ and a CMB present-day temperature $T_\gamma = 2.725$ K.

2. Recall the Riccati equation derived in classroom for weakly interactive massive particles (WIMPs) written as ($Y \equiv N_X$):

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}\left[Y^2 - Y_{eq}^2\right]$$

where $x = M_X/T$, M_X is the mass of the WIMP particles, T is the photon temperature and λ can be treated as a constant. Let $\Delta \equiv Y - Y_{eq}$ be the variable that measures the deviation of Y from its equilibrium value.

2.1. Prove that:

$$\frac{d\Delta}{dx} = -\frac{dY_{eq}}{dx} - \frac{\lambda}{x^2}\left[\Delta^2 + 2Y_{eq}\Delta\right]$$

2.2. Simplify this equation using the approximation $Y \simeq Y_{eq} \Rightarrow \Delta \simeq 0$ and $d\Delta/dx \simeq 0$, valid for the temperature range $1 < x < x_f$, where $x_f = M_X/T_f$ is the freeze-out temperature. [Hint: note that under these approximations, the first term inside the square brackets is smaller than the second term]

2.3. Derive an expression for Δ assuming $Y_{eq} \approx e^{-x}$. How does it depend on x and λ?

2.4. Re-derive Δ, now using $Y_{eq} = N_X^{eq} = n_X^{eq}/s$ [Hint: assume that the WIMP particles are already non-relativistic and write their equilibrium density, n_X^{eq}, and the specific entropy of the fluid, s, as a function of x].

3. Considering the equilibrium number density of protons, neutrons and a nuclear species with Z protons and $A - Z$ neutrons (where A is the nuclear atomic mass and Z the charge of the nucleus) can be written as:

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right)$$

where $i = \{p, n, A\}$, show that the number density of the nucleus A is given by:

$$n_A = \frac{g_A}{2^A} A^{3/2} \left(\frac{m_B T}{2\pi}\right)^{3(1-A)/2} n_p^Z n_n^{A-Z} \exp\left(\frac{B_A}{T}\right)$$

where $B_A = Zm_p + (A - Z)m_n - m_A$ is the biding energy of the nucleon A. [Hint: Note that the chemical potential of the nucleus, μ_A, is related with the chemical potentials of the protons, μ_p, and neutrons, μ_n, by $\mu_A = Z\mu_p + (A - Z)\mu_n$. Use also the approximations $m_A = Am_B$, with $m_B = m_p \approx m_n$].

4. The action of a real scalar field ϕ in curved spacetime is $S = \int d^4x\, \mathcal{L}$, where the Lagrangian density (metric signature $(+,-,-,-)$) is

$$\mathcal{L} = \sqrt{-g}\, \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)\right]$$

Here $V(\phi)$ is an arbitrary potential and $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor. (See, e.g., Section 2.1.1.3 of *Primordial Cosmology* by P. Peter & J.-P. Uzan for an introduction to Lagrangian field theory in curved spacetimes). Consider a *flat* FLRW spacetime with line element, $ds^2 = dt^2 - a(t)^2\, \delta_{ij} dx^i dx^j$, and a *homogeneous* scalar field $\phi = \phi(t)$.

4.1. Evaluate the scalar-field Lagrangian density for $\phi = \phi(t)$ in this spacetime.

4.2. Use the Euler-Lagrange equation to derive the equation of motion for the scalar field (the Klein-Gordon equation).

4.3. Use the Friedmann and acceleration equations together with $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and $p_\phi = \dot{\phi}^2/2 - V(\phi)$ to prove that $\dot{H} = -\dot{\phi}^2/2M_{PL}$. Determine the equation-of-state (EoS) parameter, $w = p_\phi/\rho_\phi$, and the scale factor, $a(t)$, assuming that the field is slow-rolling, $\dot{\phi} \sim 0$.

5. The discovery of a Higgs-like scalar particle at the LHC raises the question of whether the Higgs field itself could be responsible for driving cosmological inflation. To investigate this possibility, consider the Higgs potential $V(\phi) = \lambda\,(\phi^2 - v^2)^2$ where λ is the Higgs self-coupling constant and $v = 246$ GeV is the electroweak vacuum expectation value. (See the Chapter 8 course slides or Sec. 4.3.2 of *Cosmology* by D. Baumann for a review of inflation and the definition of the inflationary slow-roll parameters.)

5.1. Sketch the potential and compute the slow-roll parameters, $\epsilon_V = M_{PL}^2 (V'/V)^2/2$, and $\eta_V = M_{PL}^2 V''/V$ (note that $'$ denotes derivative with respect to ϕ).

5.2. Discuss if the slow-roll conditions can be satisfied simultaneously inside the field range $0 < \phi < v$. Is slow-roll inflation possible inside this range?

5.3. Now look at the regime, $\phi \gg v$. Show that $\epsilon_V(\phi)$ and $\eta_V(\phi)$ become independent of v. For what field values does inflation occur? Determine the field values at the end of inflation, ϕ_E, and at a number of e-foldings $N_* = 60$ before, ϕ_* (assume that $\phi_* \gg \phi_E$).

5.4. Knowing that the power spectrum of scalar perturbations generated by slow-roll inflation is given by

$$\Delta_R^2 \simeq \left(\frac{1}{8\pi^2 \epsilon_V} \frac{H^2}{M_{PL}^2}\right)_{k=aH}$$

compute the amplitude of the power spectrum of scalar fluctuations, Δ_R^2, at ϕ_* ($\Delta_*^2 = \Delta_R^2(\phi_*)$. Express your answer in terms of N_* and the mass of the Higgs boson defined as $m_H^2 = V''(\phi = v)$.

5.5. Estimate the value of m_H required to match the observed scalar power amplitude $\Delta_*^2 = 2 \times 10^{-9}$. Is this consistent with the LHC measurement of $m_H = 125$ GeV?