Universo Primitivo

2025-2026 (1º Semestre)

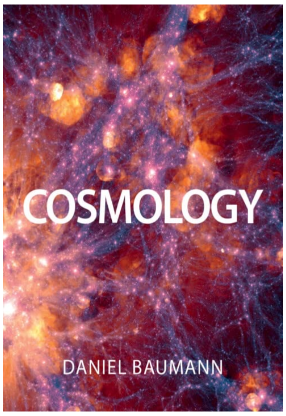
Mestrado em Física - Astronomia

Chapter 8

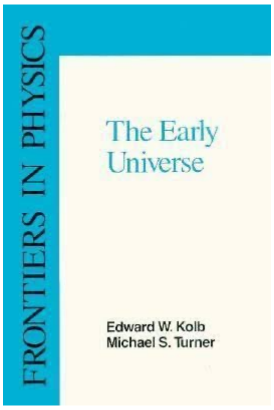
8 Inflation

- Problems of the hot Big Bang theory revisited
- Conditions for Inflation
- Distances and horizons
- Cosmological scales and horizons;
- Scalar Field Dynamics;
- Slow-roll inflation;
- Reheating

References



Ch. 4



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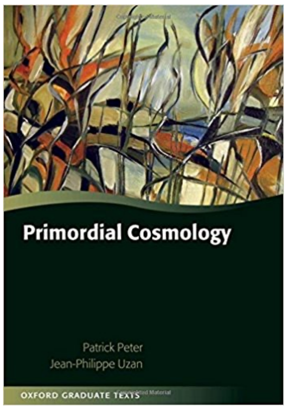
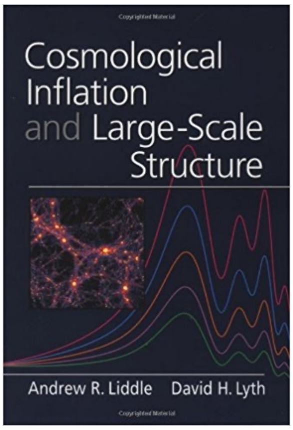


Ch. 8.1, 8.2



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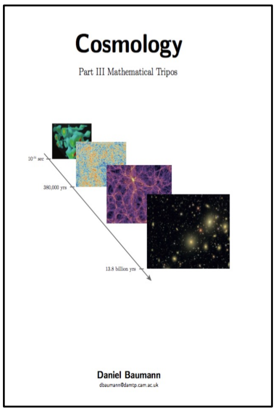


Ch. 2

Inflation

Problems of the hot Big Bang theory revisited

Friedmann-Lemaitre-Robertson-Walker **(FLRW) models** can describe the Universe expansion, but for $\Lambda = 0$ they **imply a decelerated expansion for any fluid component with** an equation state parameter $\boldsymbol{w = p/\rho c^2 > -1/3}$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \cancel{\frac{\Lambda c^2}{3}} - \frac{kc^2}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\underbrace{\left(\rho + 3\frac{p}{c^2}\right)}_{\rho(1+3w)} + \cancel{\frac{\Lambda c^2}{3}} \qquad \xrightarrow[\text{decelaration}]{1+3w>0} \qquad \boxed{w > -\frac{1}{3} \;\Rightarrow\; \ddot{a} < 0}$$

The condition $1 + 3w > 0$ is known as the **strong energy condition** (**SEC**). Since common matter and radiation all obey to SEC (have equations of state parameters with $w > -1/3$) the conclusion is that **the Universe's expansion is decelerateting!**

This leads to several difficulties known as the **hot Big Bang problems** (see next slides). **A way to solve these problems is to develop a dynamical framework where the FLRW Universes may be allowed to expand in an accelerated** way, at least during some periods of the Universe's history. **These periods are called inflationary and allows one to define inflation as any phase of the universe's expansion when**:

$$\boxed{\text{Inflation} \Leftrightarrow \ddot{a} > 0}$$

Inflation

Inflation $\Leftrightarrow \ddot{a} > 0$

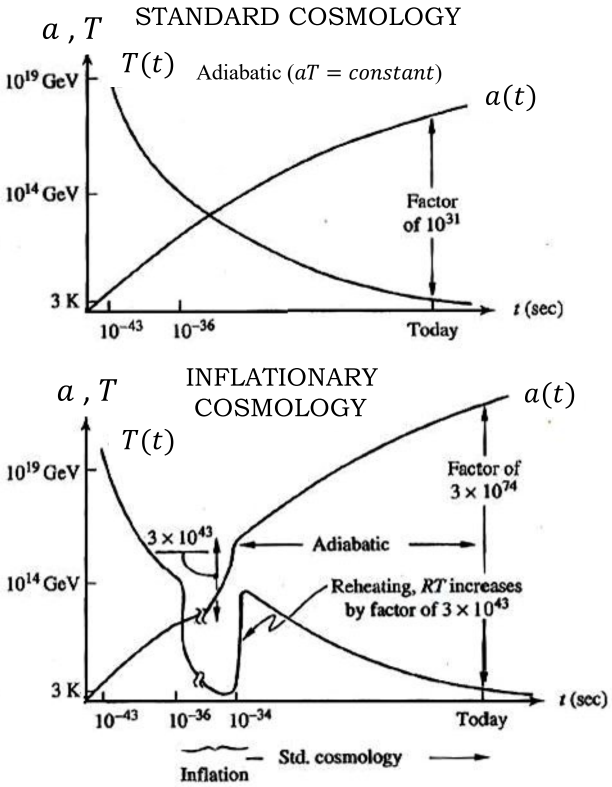


FIGURE 30.4 The evolution of the temperature of the universe and the scale factor, without and with inflation. Except for the bottom value, the temperature is given in terms of kT. (Figure adapted from Edward W. Kolb and Michael S. Turner, *The Early Universe* (page 274), ©1990 by Addison-Wesley Publishing Company, Inc., Reading, MA. Reprinted by permission of the publisher.)

Inflation

Problems of the hot Big Bang theory revisited

FLRW models with decelerated expansions are inconsistent with some important observational evidences facts and pose a number of puzzling questions:

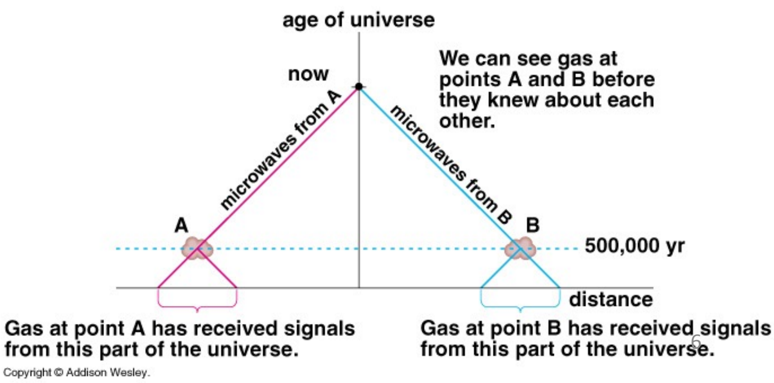
The horizon problem: The FRLW models allow one to compute the particle horizon of observer at any given time/redshift. The sky angular size of the particle horizon of an observer, θ_H, at high redshift can be approximated by:

$$\theta_H \simeq \frac{180}{\pi}\sqrt{\frac{\Omega_0}{z}}\ \mathrm{deg}$$

so an observer at $z = 1100$ (living at epoch of CMB decoupling) has a particle horizon with an angular size on our observed sky of about , $\theta_H \simeq 0.95$ deg.

This means that **there are about 54000 casual disconnect regions in the sky at CMB decoupling.**

So, why is CMB intensity spectrum so uniform temperature (2.725 °K) in all sky directions?



Inflation

Problems of the hot Big Bang theory revisited

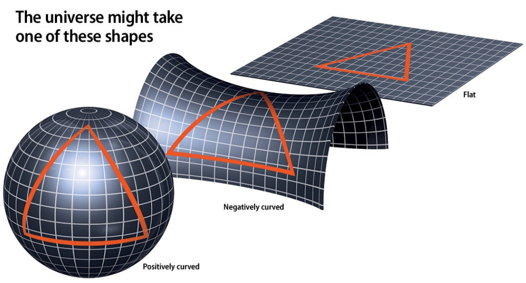
The flatness problem: At early times the Friedmann equation can be written as ($\Omega = \Omega_m + \Omega_r$):

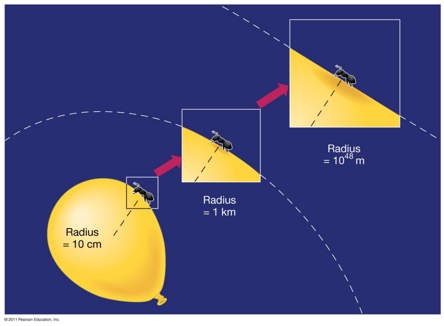
$$|\Omega(t) - 1| = \frac{|k|}{a^2(t)H^2(t)} = \frac{|k|}{\dot{a}^2(t)}$$

Since $\dot{a}(t)$ decreases with time (because $\ddot{a} < 0$) this denominator increases as $t \to 0$

So the left hand side term should approach rapidly to zero as $t \to 0$ (because, actually, $\dot{a}(t \to 0) \to \infty$). For $t \simeq 1{\times}10^{-43}$s (~Planck time) Ω should deviate no more than ~ $1{\times}10^{-60}$ from the unity.

So, why is the Universe "starting" with a energy density parameter so extremely close to 1?





Inflation

Problems of the hot Big Bang theory revisited

The monopole and other exotic particles problem:
Quantum field theories (e.g. GUT, superstring) predict that a variety of "exotic" stable particles, such as magnetic monopoles, should be produced in the early Universe and remain in measurable amounts until the present.

No such particles have yet been observed. Why?

This either implies that the predictions from particle physics are wrong, or their densities are very small and therefore there's something missing from this evolutionary picture of the Big Bang.

Inflation

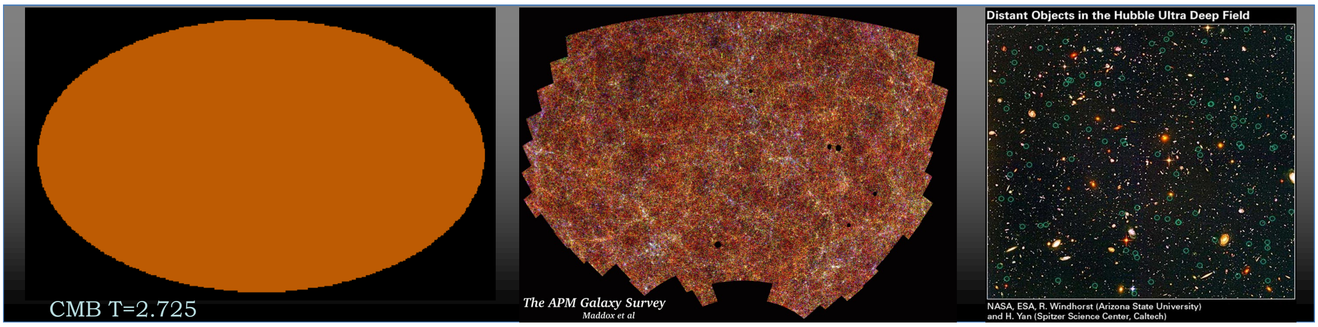
Problems of the hot Big Bang theory revisited

The origin of density fluctuations problem:
On large scales our present universe is fairly isotropic and homogeneous.

Why is that so?

At early times, that homogeneity and isotropy was even more “perfect” (due to the flattening effect effect at early times). Moreover, the FLRW universes form a very special subset of solutions of the GR equations.

So, why nature “prefers” homogeneity and isotropy from the beginning as opposed to having evolved into that stage?



Inflation

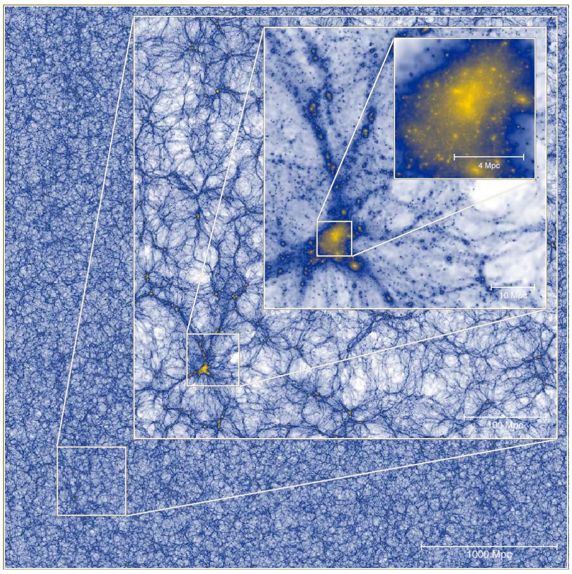
Problems of the hot Big Bang theory revisited

The origin of density fluctuations problem:

Locally the universe is not homogeneous. It displays a complex hierarchical pattern of galaxies, clusters and super clusters.

What’s the origin of cosmological structure?
Does it grow from gravitational instability?
What is the origin of the initial perturbations?

Without a mechanism to explain the existence of fluctuations one has to assume that they ``were born'' with the universe already showing the correct amplitudes on all scales, so that gravity can correctly reproduce the present-day structures?



Inflation

Conditions for Inflation

If the Universe experience periods of accelerated expansion

$$\boxed{\text{Inflation} \Leftrightarrow \ \ddot{a} > 0}$$

This requires that during these periods the Universe has to be dominated by a fluid component with an equation of state parameter $\boldsymbol{w < -1/3}$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \cancel{\frac{\Lambda c^2}{3}} - \frac{kc^2}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right) + \cancel{\frac{\Lambda c^2}{3}} \quad \xrightarrow[\text{acelaration}]{1+3w<0} \quad \boxed{w = p/\rho c^2 < -\frac{1}{3} \ \Rightarrow \ddot{a} > 0}$$

Let's us first look at the acceleration condition. For $\dot{a} \neq 0$ one has:

$$\boxed{\ddot{a} > 0} \Leftrightarrow -\frac{\ddot{a}}{\dot{a}^2} < 0 \Leftrightarrow \frac{d}{dt}\left(\dot{a}^{-1}\right) < 0 \Leftrightarrow \boxed{\frac{d}{dt}(aH)^{-1} < 0 \Leftrightarrow \frac{d}{dt}\left(cH^{-1}/a\right) < 0}$$

The quantity $\boldsymbol{d_H = cH^{-1}}$ is the **Hubble length** ($v_H = c = Hd_H$).

So, **inflation** can also be defined as **any period of the universe history when the commoving Hubble length $\boldsymbol{d_H}$ is decreasing** (shrinking).

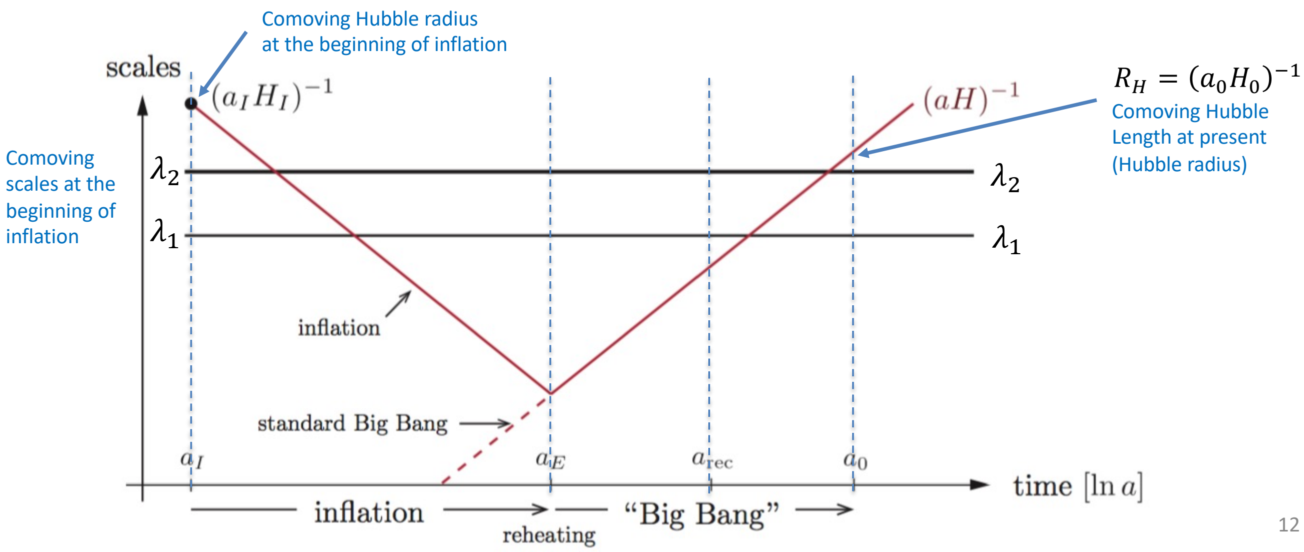
Inflation

Cosmological scales and horizons

During inflation

- any **comoving scales**, λ_1, λ_2,… **are fixed in time** because: $\lambda_{com} = \lambda_{phy}(t)/a(t)$
- but the comoving Hubble length and the particle horizon: $\chi_{ph} \sim d_H = \frac{cH^{-1}}{a} = c(aH)^{-1}$ decrease with time - when setting $c = 1$ one has $d_H = (aH)^{-1}$

So, **during inflation, physical scales** inside the horizon at a given time **grow faster and may become larger than (go beyond) the horizon.**



Inflation

Cosmological scales and horizons

Any cosmological function can be expanded in a series involving different scales. The simplest case is the 3D Fourier expressions (that can be extended for curved spaces). Examples:

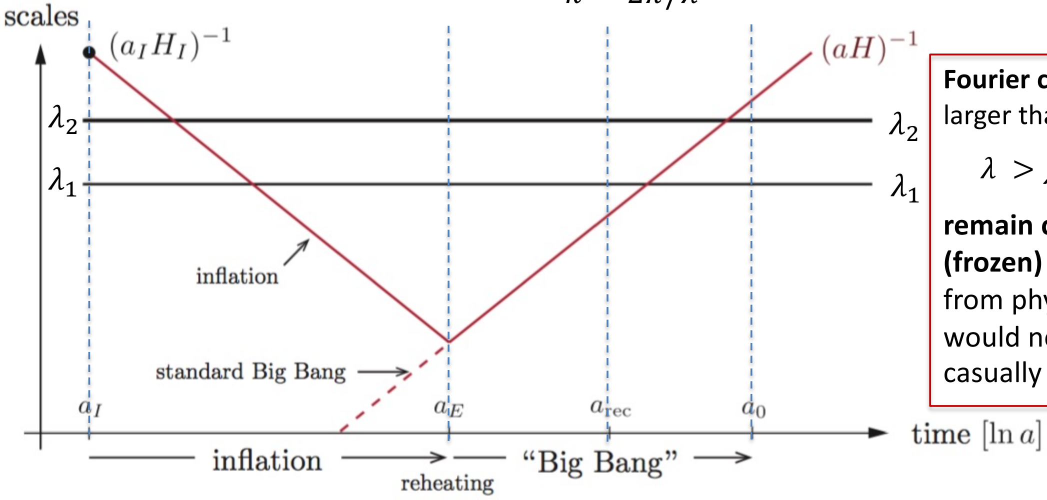
Density contrast: $\delta(\mathbf{x},t) \equiv \dfrac{\rho(\mathbf{x},t)-\rho_0(t)}{\rho_0(t)}$ $\Rightarrow$

$$\begin{cases} \delta(\mathbf{x},t) &= \sum_k \delta_k(\mathbf{k},t) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \delta_k(\mathbf{k},t) &= \dfrac{1}{V}\displaystyle\int \delta(\mathbf{x},t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x \end{cases}$$

Density Correlation function: $\xi(\mathbf{r},t) = \langle \rho(\mathbf{x},t)\rho(\mathbf{x}+\mathbf{r},t)\rangle$ $\Rightarrow$

$$\begin{cases} \xi(r,t) &= \dfrac{V}{(2\pi)^3}\displaystyle\int \left\langle |\delta_k(\mathbf{k},t)|^2 \right\rangle e^{-i\mathbf{k}\cdot\mathbf{r}} d^3k \\ P(k,t) &= \left\langle |\delta_k(\mathbf{k},t)|^2 \right\rangle = \dfrac{1}{V}\displaystyle\int \xi(r,t)\, e^{i\mathbf{k}\cdot\mathbf{r}} d^3r \end{cases}$$

$k = 2\pi/\lambda$



Fourier coefficients of scales larger than the horizon

$$\lambda > \chi_{ph} \Leftrightarrow k_\lambda < k_{ph}$$

remain constant in time (frozen) because they result from physical processes that would need to act on non-casually connected scales!

Inflation

Conditions for Inflation

The inflation conditions can be expressed in terms of other conditions. Let us first note that:

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon)\ , \qquad \text{where} \quad \varepsilon \equiv -\frac{\dot{H}}{H^2}$$

Thus:

$$\ddot{a} > 0 \Leftrightarrow \frac{d}{dt}(aH)^{-1} < 0 \Leftrightarrow -\frac{1}{a}(1-\varepsilon) < 0 \Leftrightarrow \frac{\varepsilon - 1}{a} < 0$$

So, we conclude that **inflation happens whenever**

$$\boxed{\varepsilon = -\frac{\dot{H}}{H^2} < 1}$$

ε is known as the ***slowly-varying Hubble parameter***. As long as it is smaller than 1 inflation happens. The case $\boldsymbol{\varepsilon = 0}$ is known as **perfect inflation**:

- The commoving Hubble radius is constant: $\dot{H} = 0 \Leftrightarrow H = constant$
- It implies **exponential (de Sitter) expansion**:

$$\frac{\dot{a}}{a} = H = constant \Leftrightarrow a(t) = a_i \exp(H(t - t_i))$$

Inflation

Conditions for Inflation

The inflation condition can also be written as:

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{\dot{H}/H}{\dot{a}/a} = -\frac{\mathrm{d}\ln(H)}{d\ln(a)} = -\frac{d\ln(H)}{dN} < 1$$

Where $dN = d\ln(a)$ is known as the **e-fold number**:

$$N = \int_{a_i}^{a} d\ln(a) \; = \ln\left(\frac{a}{a_i}\right)$$

The e-fold number **is used to quantify how long the inflationary period must be** to solve the Hot Big-Bang problems (usually $N \sim 40 - 70$).

During the inflationary period, ε, needs to remain small (below 1). It is then useful introduce a new parameter, $\boldsymbol{\eta}$, that **measures how fast $\boldsymbol{\epsilon}$ changes** during inflation:

$$\boxed{\eta \equiv \frac{d\ln\varepsilon}{dN} = \frac{d\ln\varepsilon}{d\ln a} = \frac{\dot{\varepsilon}}{H\varepsilon}}$$

Since ϵ needs to remain small this means that η needs to remain small, as well.
In general, one should have: $\eta < 1$ and $\varepsilon < 1$

Inflation

Conditions for Inflation

The Friedmann and the continuity equations

$$H^2 = \rho/3M_{pl}^2$$
$$\dot{\rho} = -3H(\rho + P)$$

Can be combined to relate, ε, with the equation of state parameter.
One has:

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{P}{\rho}\right) < 1 \quad \Leftrightarrow \quad w \equiv \frac{P}{\rho} < -\frac{1}{3}$$

Combining this equation with the continuity equation it is also possible to conclude that (exercise):

$$\left|\frac{d\ln\rho}{d\ln a}\right| = 2\varepsilon < 1$$

Which shows that for small ε the **energy density of the universe remains approximately constant during inflation**. Conventional matter sources would dilute with the (exponential expansion). **The energy density of whatever causes inflation needs to be an unconventional/unusual form of matter/energy**.

Inflation

Conditions for Inflation (summary table):

• *Accelerated expansion* $\ddot{a} > 0$	$\frac{d(aH)^{-1}}{dt} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0 \quad \Rightarrow \quad \boxed{\ddot{a} > 0}$
• *Slowly-varying Hubble* $\varepsilon \equiv -\frac{\dot{H}}{H^2} < 1$	$\frac{d(aH)^{-1}}{dt} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = \frac{(\varepsilon - 1)}{a} < 0 \quad \Rightarrow \quad \boxed{\varepsilon < 1}$
• *Exponential expansion* $\mathrm{d}s^2 \approx \mathrm{d}t^2 - e^{2Ht}\mathrm{d}\boldsymbol{x}^2$	$\varepsilon \ll 1 \ \rightarrow \ H = \frac{\dot{a}}{a} \approx const. \quad \Rightarrow \quad \boxed{a(t) = e^{Ht}}$
• *Negative pressure* $w \equiv \frac{P}{\rho} < -\frac{1}{3}$	$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6M_{\mathrm{pl}}^2} > 0 \quad \Rightarrow \quad \boxed{\rho + 3P < 0}$
• *Constant density* $\left\lvert\frac{d\ln\rho}{d\ln a}\right\rvert = 2\varepsilon < 1$	$H^2 \propto \rho \ \rightarrow \ 2H\dot{H} \propto \dot{\rho} \quad \Rightarrow \quad \boxed{\varepsilon = -\frac{1}{2}\frac{\dot{\rho}}{H\rho}}$

Inflation

Basic Picture

Let us now look intuitively how the inflation condition

$$\boxed{\text{inflation} \Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}(cH^{-1}/a) < 0}$$

may be used to solve the Hot Big-Bang problems

Flatness problem:

If the expansion is accelerating, $\ddot{a} > 0$, the derivative of the scale factor $\dot{a}$ **is an increasing function of time**. So, it decreases as we go back in time

$$|\Omega(t) - 1| = \frac{|k|}{a^2(t)H^2(t)} = \frac{|k|}{\dot{a}^2(t)}$$

Is an increasing function of time, so: $\dot{a}(t \rightarrow 0) \rightarrow 0$

the flatness problem is therefore solved because…

The Universe can in principle "start" with a energy density parameter far from 1.

Inflation

Basic Picture

Let us now look intuitively how the inflation condition

$$\boxed{\text{inflation} \Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}(cH^{-1}/a) < 0}$$

may be used to solve the Hot Big-Bang problems

Flatness problem:

How much inflation do we need?

Note that during inflation $\varepsilon = -\dot{H}/H^2 < 1$ is small, so if $\dot{H} \sim 0$ and $H \sim$ constant during the period of inflation $t \in [t_i, t_e]$. This means that:

$$\Omega - 1 = \frac{k}{a^2 H^2} \sim \frac{1}{a^2} \longrightarrow \frac{|\Omega - 1|_e}{|\Omega - 1|_i} = \left(\frac{a_i}{a_e}\right)^2 = e^{-2N}$$

$H \sim constant$

Since, by the end of inflation one needs to have $|\Omega - 1|_e \sim 10^{-60}$ and one wants not to have Ω arbitrarily different from 1, let's say $|\Omega - 1|_i \sim 1$, one concludes that:

$$e^{-2N} \approx 10^{-60} \Rightarrow N \approx 69.$$

Inflation

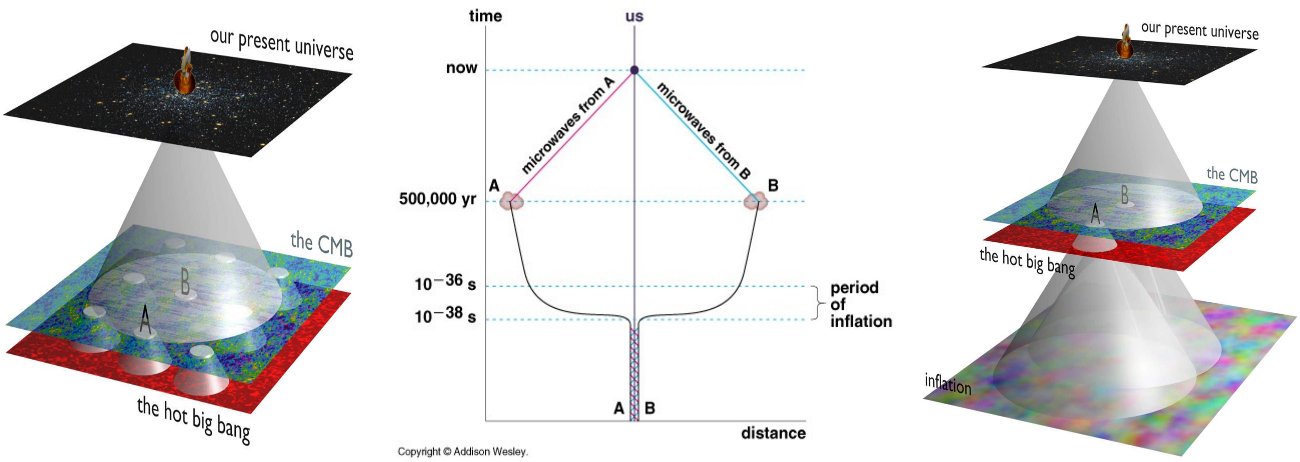
Basic Picture

Let us now look intuitively how the inflation condition

$$\boxed{\text{inflation} \Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}(cH^{-1}/a) < 0}$$

may be used to solve the SMC problems

The horizon problem: If the accelerated expansion happens in a early phase of the Universe, during a long enough period, in principle, all causally disconnected sky patches of the CMB can be put in causal contact.



Inflation

Distances and Horizons

Let us consider the travel of light along radial ($d\theta = d\phi = 0$) geodesics in a FLRW metric

$$\begin{aligned} ds^2 = &\; dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \\ = &\; dt^2 - a^2(t)\left[d\chi^2 + f_k(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right], \end{aligned}$$

written in a conformal way with the introduction of the **conformal time $d\tau = dt/a$**

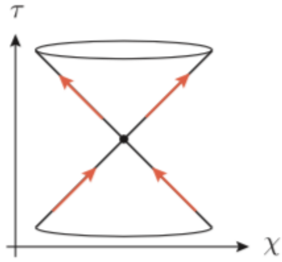
$$ds^2 = a^2(\tau)\left[d\tau^2 - d\chi^2\right]$$

(with $d\chi = dr$ for flat geometries), So light rays ($ds^2 = 0$) travel along geodesics with

$$\Delta\chi(\tau) = \pm\Delta\tau$$

From integrating this we can define the notions of:

- **Particle horizon**: $\chi_{\rm ph}(\tau) = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)}$ with $t_i = 0$
- **Event horizon**: $\chi_{\rm eh}(\tau) = \tau_f - \tau = \int_{t}^{t_f} \frac{dt}{a(t)}$ with $t_f = \infty$



Inflation

Distances and Horizons

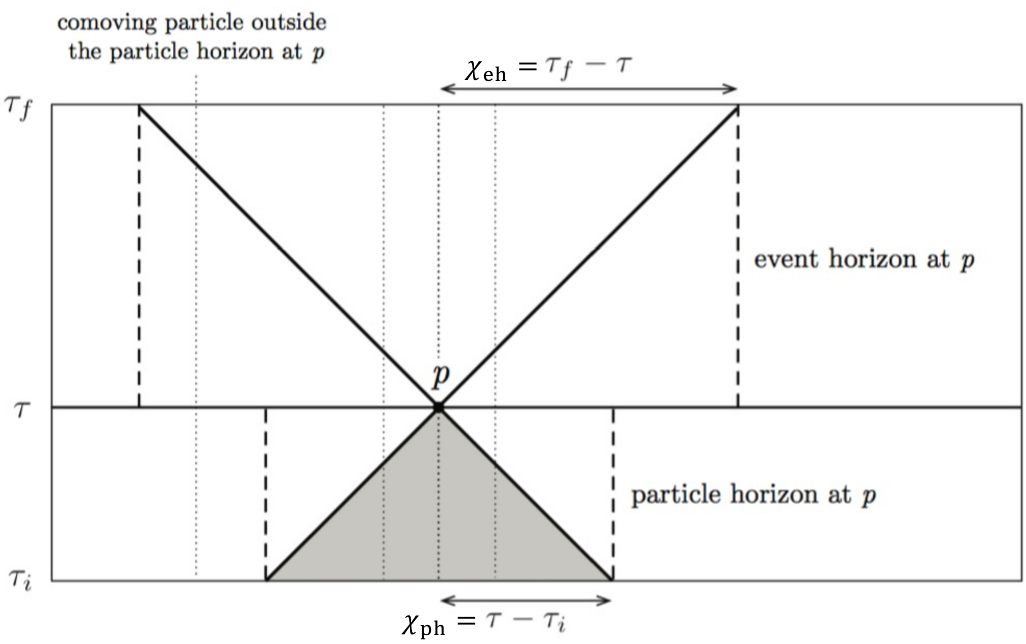


Figure 2.1: *Spacetime diagram illustrating the concept of horizons. Dotted lines show the worldlines of comoving objects. The event horizon is the maximal distance to which we can send signal. The particle horizon is the maximal distance from which we can receive signals.*

Inflation

Distances and Horizons

The particle horizon, $\chi_{\rm ph}$, the maximal commoving distance travelled by light until a time t, can be computed as follows:

$$\chi_{\rm ph}(\tau) = \int_{t_i}^{t} \frac{\mathrm{d}t}{a} = \int_{a_i}^{a} \frac{\mathrm{d}a}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1}\, \mathrm{d}\ln a$$

with $t_i = 0; a_i = 0$.

Let us evaluate this integral for a perfect (single component), with EoS w, where the scale factor evolves as $a = a_i t^{2/3(1+w)}$. The **commoving Hubble radius** inside the last integral is (Exercise):

$$\boxed{(aH)^{-1} = H_0^{-1}\, a^{\frac{1}{2}(1+3w)}}$$

For any fluid component with an equation state parameter w. All familiar matter sources have $1 + 3w > 0$ (this is an implication of the so-called **strong energy condition** (SEC)). So, in the Hot Big-Bang theory model the commoving Hubble radius is always increasing.

Using the above expressions in the integral one finds (with $t_i = 0$), see next page:

Inflation

Distances and Horizons

The particle horizon, $\chi_{\rm ph}$, will then give:

$$\chi_{\rm ph}(a) = \frac{2H_0^{-1}}{(1+3w)} \left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)} \right] \equiv \tau - \tau_i$$

Note that for **standard Friedmann evolution**, where SEC grants $1 + 3w > 0$, the second term goes to zero: $\boldsymbol{\tau_i(a_i \to 0) \to 0}$.

So, in that case

$$\boxed{\chi_{\rm ph}(t) = \frac{2H_0^{-1}}{(1+3w)}\, a(t)^{\frac{1}{2}(1+3w)} = \frac{2}{(1+3w)}\, (aH)^{-1}}$$

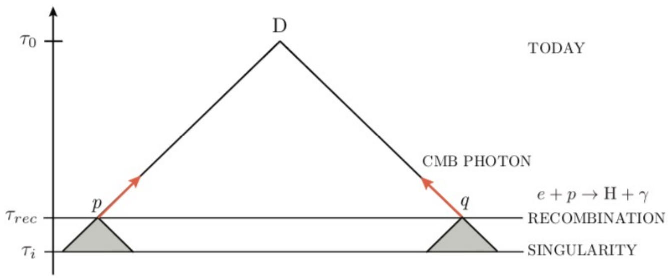
And one concludes that:

- the comoving particle horizon is proportional (and of the same order) to the comoving Hubble radius
- The comoving particle horizon / Hubble length is always increasing.

But since **during inflation, SEC is violated, $\boldsymbol{1 + 3w < 0}$** , the second term in the first equation of this slide goes to minus infinity: $\boldsymbol{\tau_i(a_i \to 0) \to -\infty}$. The first term, τ, also is negative, but less negative than τ_i and therefore and $\chi_{\rm ph} > 0$, $\chi(a_i \to 0) = \infty$.

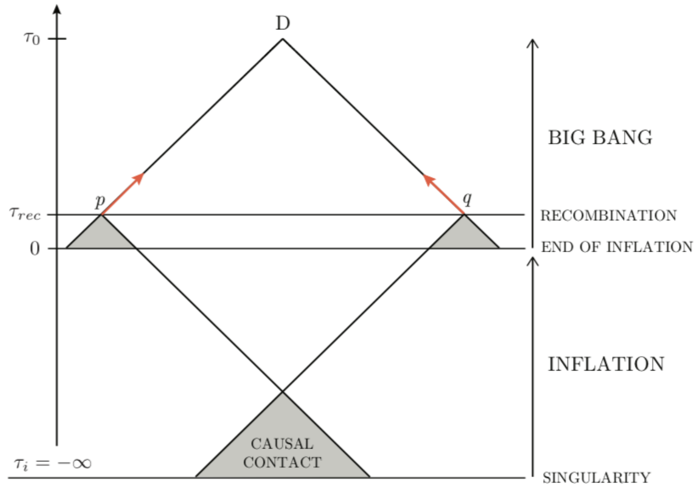
Inflation

Distances and Horizons



Hot Big-Bang evolution

$\tau_i \to 0$



$\tau_i \to -\infty$

Hot Big-Bang + inflation evolution

Inflation

Basic Picture

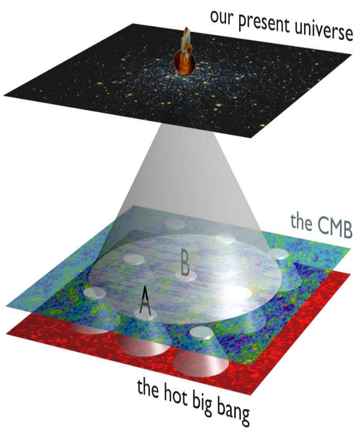
Let us now look intuitively how the inflation condition

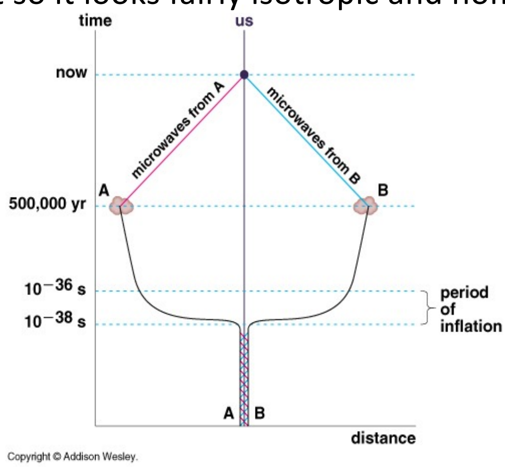
$$\boxed{\text{inflation} \Leftrightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}(cH^{-1}/a) < 0}$$

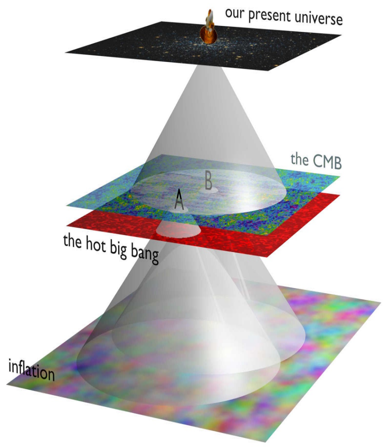
may be used to solve the SMC problems

The monopole problem: If the universe expands sufficiently after monopoles are produced their abundance can be too low to be observed.

The homogeneity problem: our visible universe comes from a causally connected region that expanded a lot so it looks fairly isotropic and homogeneous







The Theory of Inflation

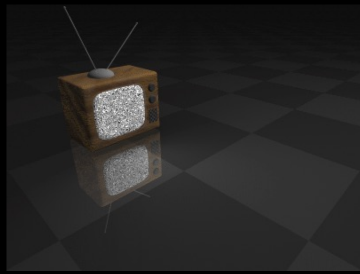
Inflation also provides a mechanism for the origin of fluctuations…

… fluctuations (density and grav. waves) are due to quantum fluctuations about the inflaton field's vacuum state $\langle | E_\phi | \rangle$

The inflation (inflaton) field ϕ has energy density fluctuations allowed by the Heisenberg uncertainty principle:

$$\Delta t \, \Delta E_\phi > \frac{\hbar}{2} \Leftrightarrow \Delta E_\phi > \frac{\hbar}{2\, \Delta t}$$

The simplest models of inflation predict random fluctuations with a "power spectrum" that has the same amplitude on all scales (scale Invariant power spectrum). Just like…



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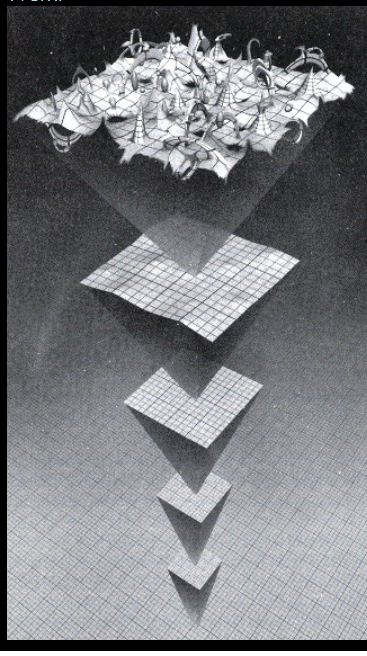


The Theory of Inflation:

The origin of fluctuations

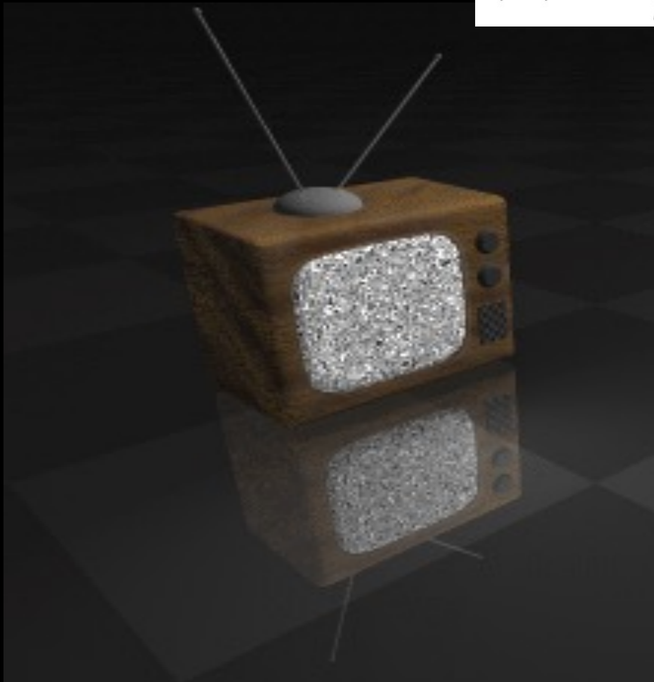
$$\delta(\mathbf{x},t) \equiv \frac{\rho(\mathbf{x},t) - \rho_0(t)}{\rho_0(t)}$$

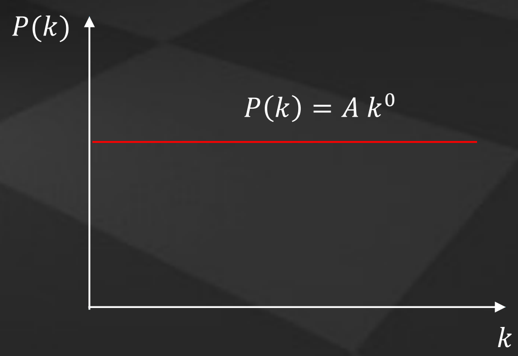
$$\delta(\mathbf{x},t) = \sum \delta_k(\mathbf{k},t) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$\delta_k(\mathbf{k},t) = \frac{1}{V}\int \delta(\mathbf{x},t) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x$$

$$\xi(r,t) = \frac{V}{(2\pi)^3}\int \langle|\delta_k(\mathbf{k},t)|^2\rangle \, e^{-i\mathbf{k}\cdot\mathbf{r}} d^3k$$

$$P(k,t) = \langle|\delta_k(\mathbf{k},t)|^2\rangle = \frac{1}{V}\int \xi(r,t)\, e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$





… static white noise!

$k = 2\pi/\lambda$

The Theory of Inflation

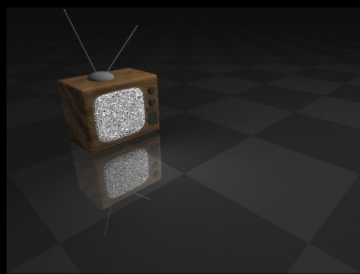
Inflation also provides a mechanism for the origin of fluctuations…

… fluctuations (density and grav. waves) are due to quantum fluctuations about the inflaton field's vacuum state

The inflation (inflaton) field has energy density fluctuations allowed by the Heisenberg uncertainty principle:

$$\Delta E_{\phi} > h/(4\pi\Delta t)$$

<u>During inflation fluctuations are "inflated" to macroscopic scales > physically connected scales become larger than the horizon scale and "freeze". Latter they will re-enter the horizon.</u>



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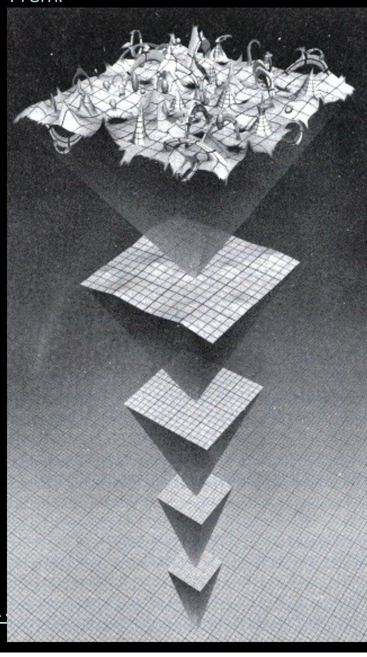


Standard Model of Cosmology (SMC)

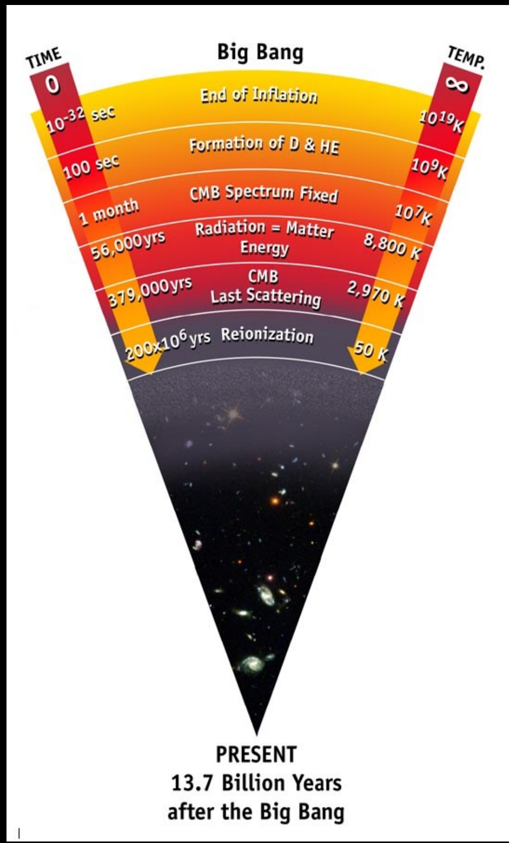
SMC = Hot Big Bang + Inflation

FLRW models provide a description for the evolution of the “background” Universe

provides a mechanism for the origin of perturbations in this “background Universe”

$$H^2(t) = \frac{8\pi G}{3}\left(\rho_\phi + \rho_r + \rho_m\right) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

ρ_ϕ is the energy density of the inflationary field. It dominates during the inflation period!



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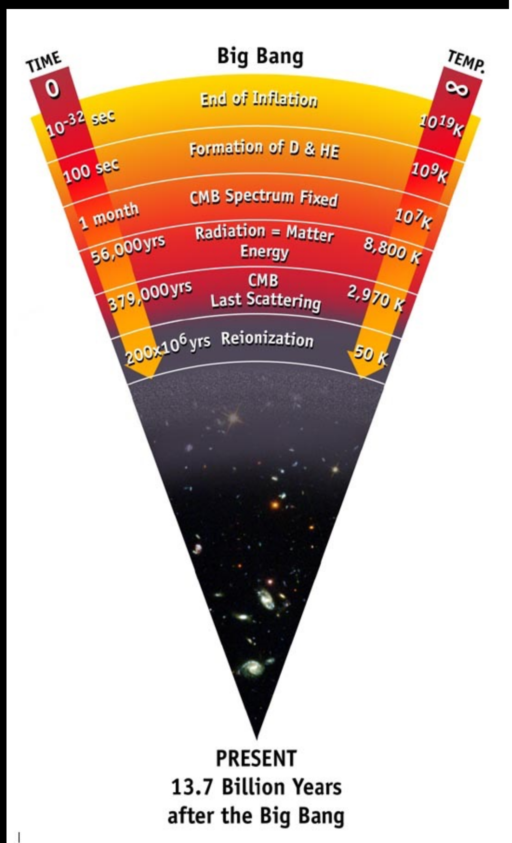
Standard Model of Cosmology (SMC)

SMC = Hot Big Bang + Inflation

After the end of inflation the universe resumes the usual Friedmann evolutionary periods

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}(\rho_r + \rho_m) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} + \cancel{\frac{8\pi G}{3}\rho_\phi}^{\,0} \\ &= H_0^2\left[\Omega_{r0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda 0}\right] \end{aligned}$$

- Background evolution is progressively dominated by:
  - Radiation
  - Matter
  - Dark Energy



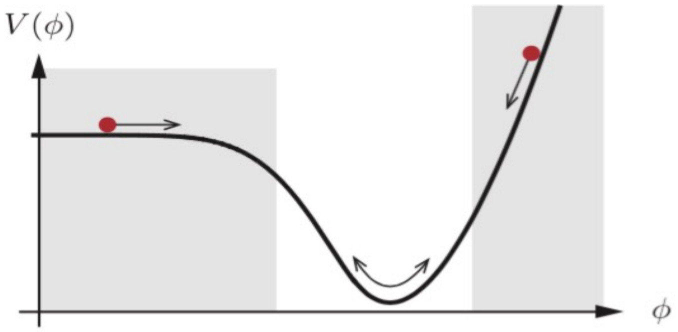
From:

Inflation

Scalar field Dynamics

Inflation is usually modelled by a scalar field $\phi = \phi(x^i, t)$, called the **inflaton field**, that can generally be a function of position and time.

Associated with each field value there's a **potential energy**, $V(\phi)$, and If the field depends on time, the field also carries **kinetic energy**.



Using the Noether's theorem one can Prove that the energy-stress tensor of any scalar Field can be computed as:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right)$$

For a homogeneous and isotropic **FLRW** universe, without perturbations (ie inhomogeneities) **the field is only a function of time**, $\phi = \phi(t)$. Computing, $T_0^0 = \rho_\phi$, and $T_j^i = -P_\phi\,\delta_j^i$ one obtains:

$$\boxed{\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)} \qquad \boxed{P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)}$$

Inflation

Scalar field Dynamics: Klein-Gordan equation

Using ρ_ϕ in the **Friedmann equation** gives:

$$\boxed{H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]}$$ (Friedmann equation)

Taking the time derivative one finds:

$$2H\dot{H} = \frac{1}{3M_{\rm pl}^2}\left[\dot{\phi}\ddot{\phi} + V'\dot{\phi}\right]$$

where $V' \equiv dV/d\phi$.

Using ρ_ϕ and P_ϕ in the **acceleration equation** and combining it with the Friedmann equation, one obtains:

$$\boxed{\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{\rm pl}^2}}$$ (Acceleration equation)

This shows that **the acceleration of the universe is sourced by the kinetic energy of the inflaton field**. Combining these two last expressions one obtains the **Klein-Gordan** equation that **describes the evolution of the inflationary field**:

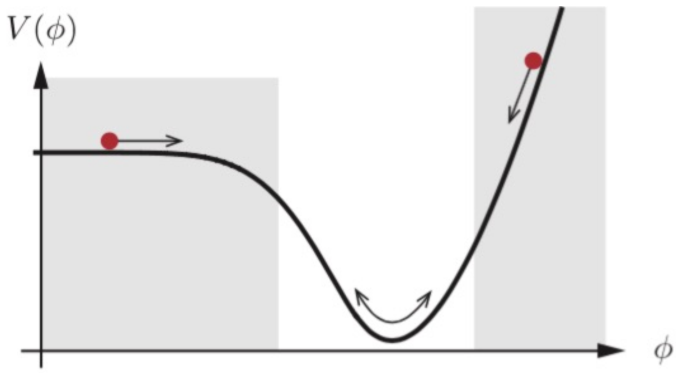
$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V' = 0}$$ (Klein-Gordan equation)

Inflation

Slow roll inflation

Combining the expressions: $\varepsilon \equiv -\frac{\dot{H}}{H^2}$ and $\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{\rm pl}^2}$ gives:

$$\boxed{\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\rm pl}^2 H^2}}$$



This means that inflation $\epsilon < 1$ only occurs if the contribution of the kinetic energy of the field to the total energy is small. When it is **very small**, the field is said to be **slow rolling…**

The time derivative of ϵ gives:

$$\dot{\varepsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_{\rm pl}^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_{\rm pl}^2 H^3}$$

Which allows us to compute the η parameter as:

$$\boxed{\eta = \frac{\dot{\varepsilon}}{H\varepsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\varepsilon - \delta)}$$

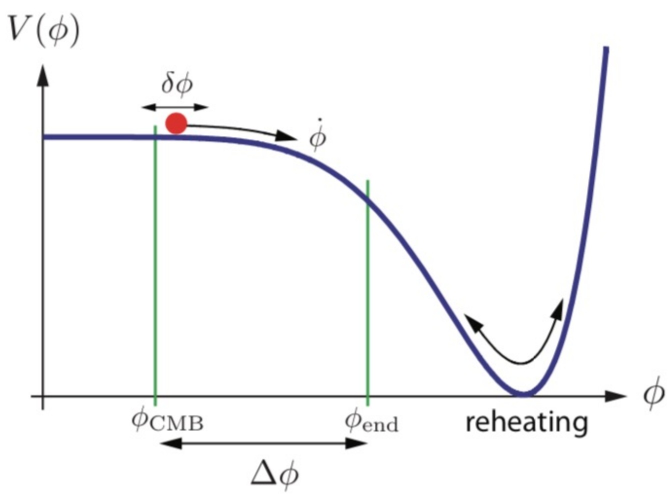
where $\delta \equiv -\ddot{\phi}/H\dot{\phi}$.

Inflation

Slow roll inflation

The conditions $\epsilon < 1$ and $|\eta| < 1$ **are a guaranty that inflation happens and persists.** Since this implies that the kinetic energy of the field is small one can assume the **slow roll inflation conditions**:

$$\boxed{\{\epsilon, |\eta|\} \ll 1}$$



and approximate the Friedmann and Klien Gordon equations as:

- Friedmann ($\dot{\phi}^2 \sim 0$): $H^2 \approx \frac{V}{3M_{\rm pl}^2}$
- Klein Gordon ($\ddot{\phi} \sim 0$): $3H\dot{\phi} \approx -V'$

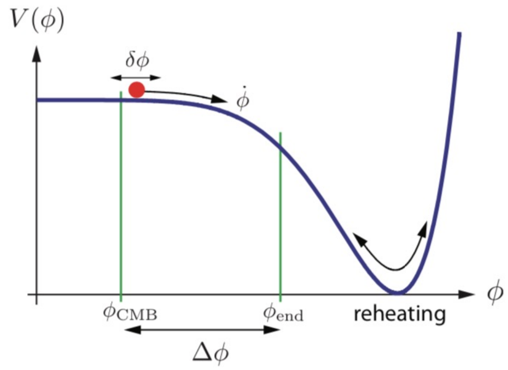
Combining these equations (plus taking the time derivative of the Klein Gordon equation) allows one to write the $\{\epsilon, |\eta|\}$ parameters as function of the potential and its derivatives (exercise):

$$\boxed{\epsilon_{\rm v} \equiv \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 \quad , \quad |\eta_{\rm v}| \equiv M_{\rm pl}^2 \frac{|V''|}{V}}$$

Inflation

Slow roll inflation

The total amount of e-folds (which gives by how much the universe expands during the inflationary period) can be derived from our knowledge of the inflationary potential.



$$N_{\rm tot} \equiv \int_{a_I}^{a_E} \mathrm{d}\ln a = \int_{t_I}^{t_E} H(t)\,\mathrm{d}t$$

Here, t_I and t_E are the times when inflation begins and ends, which happens when:

$$\varepsilon(t_I) = \varepsilon(t_E) \equiv 1$$

The integrand function above, can be approximated by (note that $\varepsilon = \frac{\frac{1}{2}\dot\phi^2}{M_{\rm pl}^2 H^2}$):

$$H\mathrm{d}t = \frac{H}{\dot\phi}\mathrm{d}\phi = \pm\frac{1}{\sqrt{2\varepsilon}}\frac{\mathrm{d}\phi}{M_{\rm pl}} \approx \frac{1}{\sqrt{2\epsilon_{\rm v}}}\frac{|\mathrm{d}\phi|}{M_{\rm pl}}$$

Which leads to:

$$\boxed{N_{\rm tot} = \int_{\phi_I}^{\phi_E} \frac{1}{\sqrt{2\epsilon_{\rm v}}}\frac{|\mathrm{d}\phi|}{M_{\rm pl}}}$$

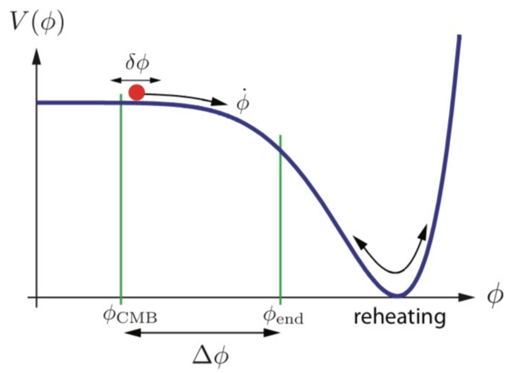
Inflation

Slow roll inflation

Using the the slow-roll expression ϵ_V,

$$\epsilon_{\rm v} \equiv \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2$$

in the number of e-folding integral one gets



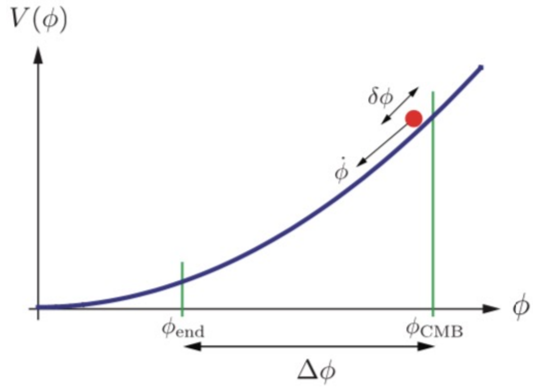
$$\boxed{N_{\rm tot} = \int_{\phi_I}^{\phi_E} \frac{1}{\sqrt{2\epsilon_V}}\frac{|d\phi|}{M_{\rm pl}}} = \int_{\phi_I}^{\phi_E} \frac{V}{M_{\rm pl}V'}\frac{|d\phi|}{M_{\rm pl}} = \boxed{\frac{1}{M_{\rm pl}^2}\int_{\phi_I}^{\phi_E} \frac{V}{V'}|d\phi|}$$

Since the number of e-foldings is counted from the moment inflation begins, **it is usual to refer to t_I as the instant "*N* e-foldings before inflation ends"**, and **ϕ_I is often expressed as ϕ_N**, the inflaton field value *N*-efoldings before the end of inflation (In fact, this instant scale is the latest to re-enter the "sound" horizon).
So one can also write:

$$N = 8\pi G \int_{\phi_e}^{\phi_N} \frac{V}{V'}\,d\phi$$

Inflation

Working example: $V(\phi) = m^2\phi^2/2$



This case belongs to an important class of potentials ($V(\phi) \propto \phi^p$) known as **Large field inflation models** (the potential evolves over super-Planckian values).

This potential allows slow-rolling. The number of e-foldings under these conditions gives:

$$N = 8\pi G \int_{\phi_e}^{\phi_N} \frac{V}{V'}\, d\phi = \frac{8\pi G}{2}\int_{\phi_e}^{\phi_N} \phi\, d\phi = \frac{8\pi G}{4}(\phi_N^2 - \phi_e^2)$$

So the value of the field at a moment N e-foldings before ϕ_e should be:

$$\phi_N^2 = \phi_e^2 + \frac{1}{2\pi G}N.$$

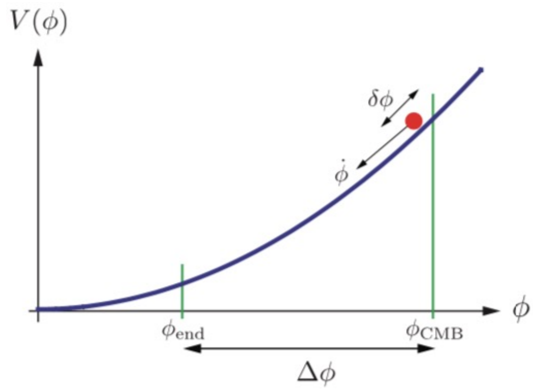
Now, we know that when inflation ends, $\epsilon_V = 1$, so using this in ϵ_V one has:

$$\frac{1}{16\pi G}\left(\frac{V'}{V}\right)_e^2 = 1 \qquad \Rightarrow \qquad \left(\frac{m^2\phi_e}{\frac{1}{2}m^2\phi_e^2}\right)^2 = 16\pi G$$

Solving for ϕ_E (see next slide)…

Inflation

Working example: $V(\phi) = m^2\phi^2/2$



This case belongs to an important class of potentials ($V(\phi) \propto \phi^p$) known as Large field inflation models (the potential evolves over super-Planckian values).

Solving for ϕ_E (continuation from previous slide), one obtains:

$$\phi_e^2 = \frac{1}{4\pi G} \approx 0.08 m_{\rm Pl}^2$$

$$\phi_N^2 = \phi_e^2 + \frac{1}{2\pi G}N = \frac{m_{\rm Pl}^2}{2\pi}\left(N + \frac{1}{2}\right) \approx \frac{m_{\rm Pl}^2}{2\pi}N,$$

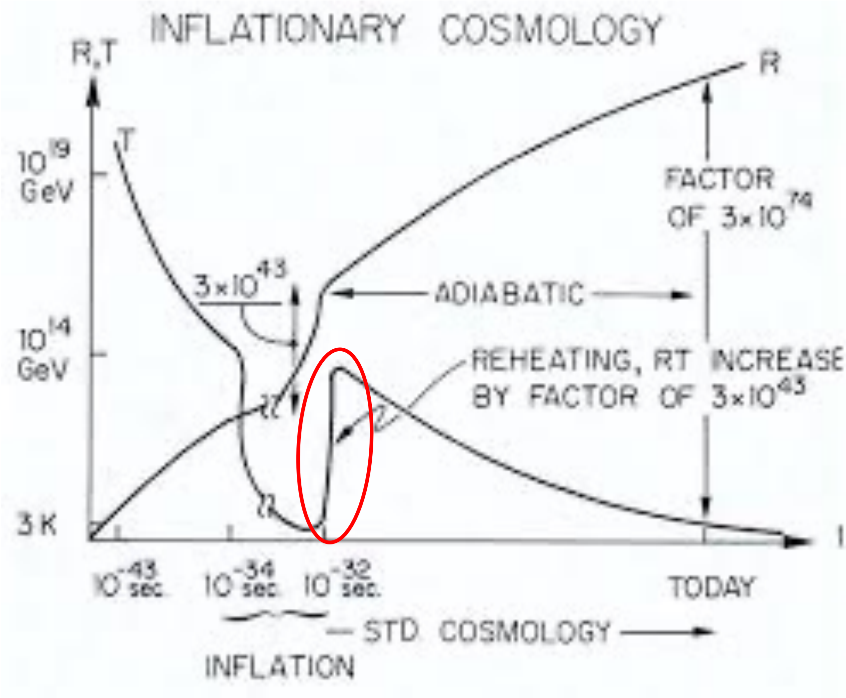
For $N \simeq 70$ these give:

$$\phi_E \approx 0.3 m_{\rm Pl}$$

$$\phi_N \approx 3.3 m_{\rm Pl}$$

Inflation

Re-heating



Inflation

Re-heating

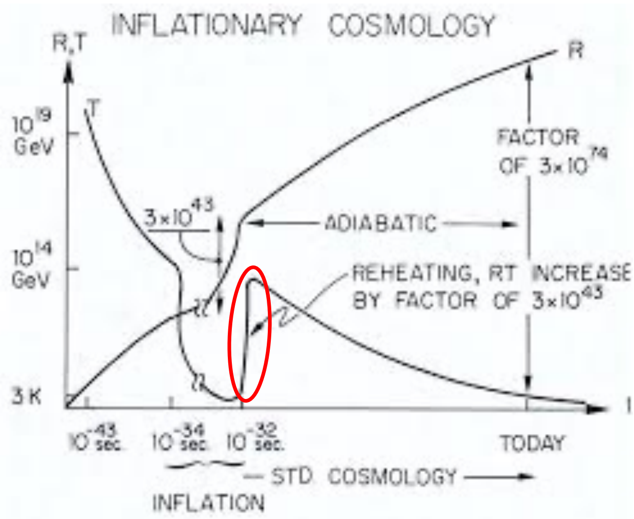
During the inflationary period most of the energy density of the Universe is given by the inflationary potential.

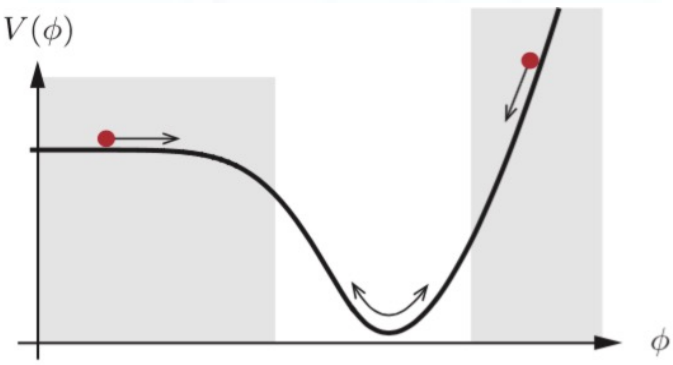
As inflation ends, the kinetic energy associated with the inflaton field is no longer negligible and the energy in the field is transferred to the matter/energy species of the fluid.

$$\dot{\rho}_\phi + 3H\rho_\phi + \Gamma\rho_\phi = 0,$$
$$\dot{\rho}_R + 3H\rho_R - \Gamma\rho_\phi = 0.$$

Where Γ is the so-called **energy width** of the inflaton decay (ρ_R is the energy density of relativistic fields).

This process is known as **reheating** and It is followed by the hot big bang evolutionary phase of the universe.





Inflation

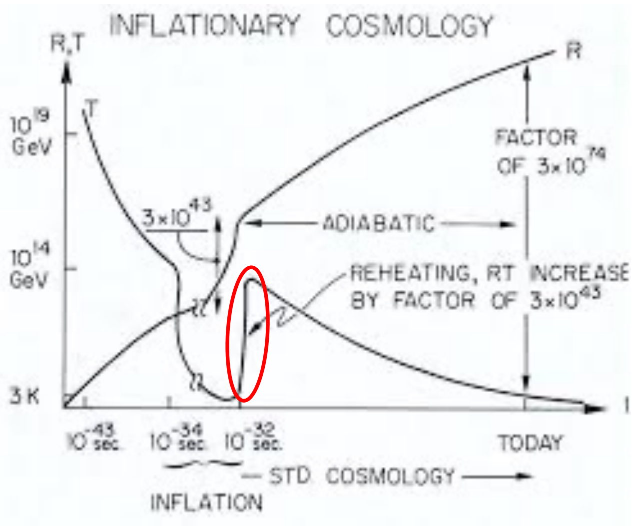
Re-heating

The basic idea behind reheating is that this period starts when ϕ begins to oscillate with a friction term about the minimum of the inflationary potential.

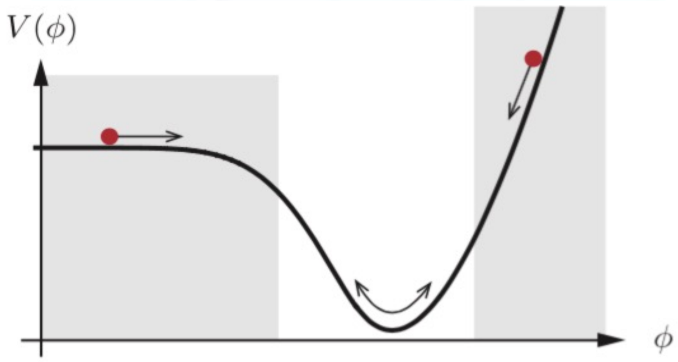
For example, taking a quadratic potential $V = m^2\phi^2/2$, the Klein-Gordon and the continuity equations give:

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi$$

$$\dot{\rho}_\phi + 3H\rho_\phi = -3HP_\phi = -\frac{3}{2}H(m^2\phi^2 - \dot{\phi}^2)$$



Oscillations decrease in amplitude due to the friction term. By the end of the process all energy of the field is transferred, leading to the beginning of the hot Big-Bang evolution.

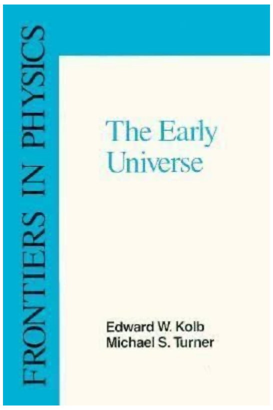


Short overview of Chapter 9

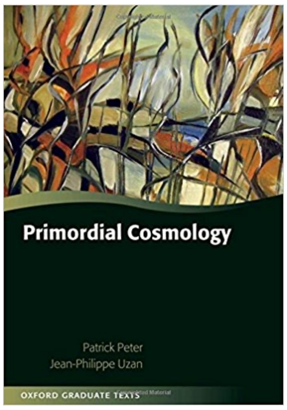
Inflation: the origin of perturbations

- The Basic Picture;
- Cosmological perturbation theory
- Quantum fluctuations in the de Sitter space;
- Primordial power spectra from inflation;
- CMB power spectrum

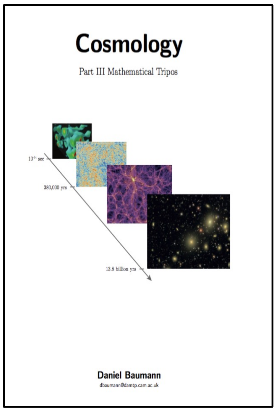
References



Ch. 8



Ch. 8



Ch. 4
Ch. 6

Inflation: the basic picture

The Inflationary phase of the Universe needs to happen at very early times. Present data is consistent with an inflationary period that lasted for about around $\Delta t \sim 10^{-36}$ at cosmic time of about $t \sim 10^{-32} - 10^{-33}$seconds

In these conditions the **inflaton field has a quantum nature** and its energy density is quantified. The **Heisenberg uncertainty principle** allows the origin of energy density fluctuations given the short timescales involved.

$$\Delta E_{\phi} > h/(4\pi\Delta t)$$

The **inflation field**, $\phi(x,t)$, therefore, **acquires a spatial dependence due to quantum fluctuations,** $\delta\phi(x,t)$, about its “background” Value, $\phi(t)$:

$$\boldsymbol{\phi(x,t) = \phi(t) + \delta\phi(x,t)}$$

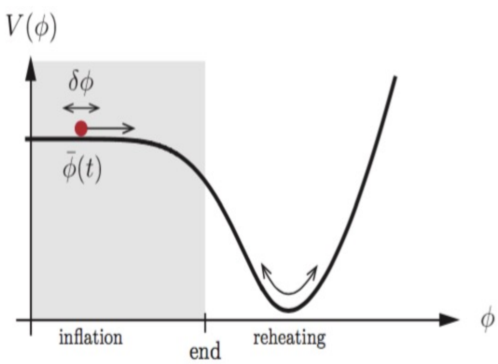


Figure 6.1: *Quantum fluctuations $\delta\phi(t, \boldsymbol{x})$ around the classical background evolution $\bar{\phi}(t)$. Regions acquiring a negative fluctuations $\delta\phi$ remain potential-dominated longer than regions with positive $\delta\phi$. Different parts of the universe therefore undergo slightly different evolutions. After inflation, this induces density fluctuations $\delta\rho(t, \boldsymbol{x})$.*

Inflation: the basic picture

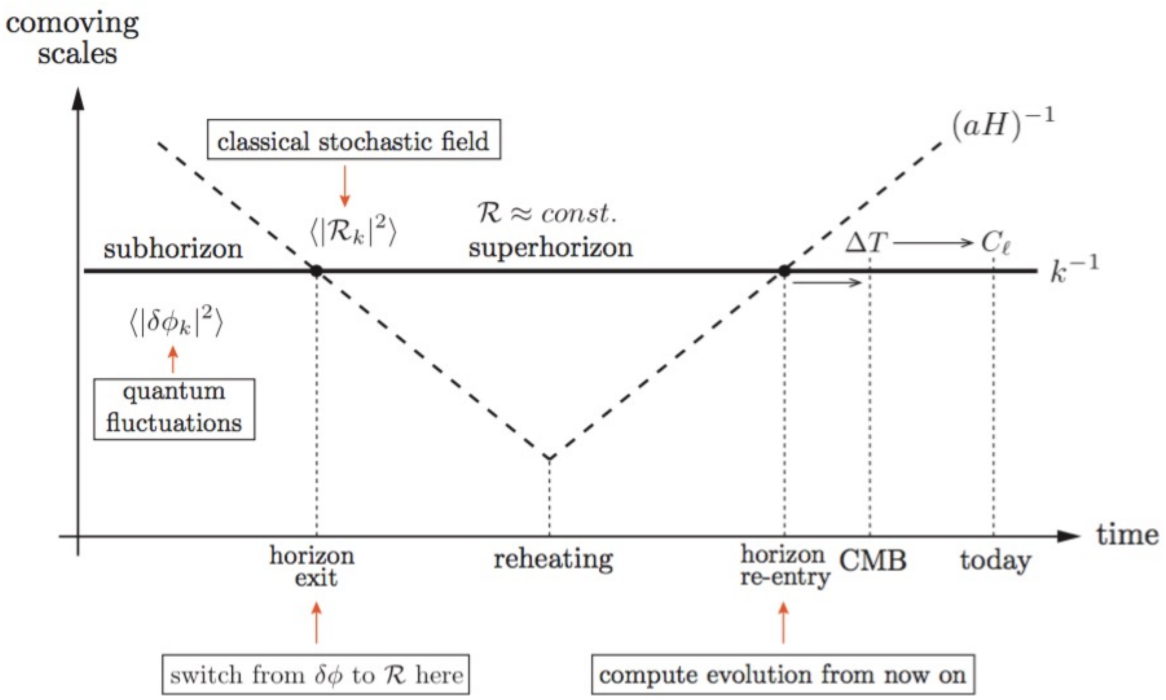


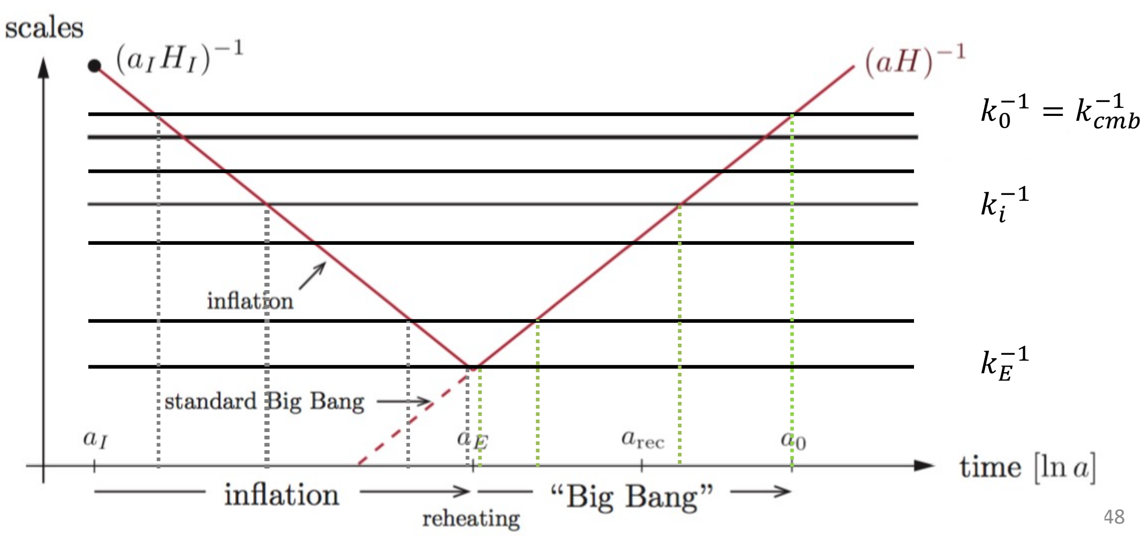
Figure 6.2: *Curvature perturbations during and after inflation: The comoving horizon $(aH)^{-1}$ shrinks during inflation and grows in the subsequent FRW evolution. This implies that comoving scales k^{-1} exit the horizon at early times and re-enter the horizon at late times. While the curvature perturbations $\mathcal{R}$ are outside of the horizon they don't evolve, so our computation for the correlation function $\langle|\mathcal{R}_k|^2\rangle$ at horizon exit during inflation can be related directly to observables at late times.*

Inflation: the basic picture

At horizon crossing of a given comoving scale $\lambda \sim 1/k$, one necessarily has:

$$k^{-1} = (aH)^{-1} \quad \Leftrightarrow \quad \boxed{k = aH}$$

So, the (comoving) Fourier mode k are simply giving (the inverse) of the comoving Hubble radius at a given epoch.



Inflation: the basic picture

Key steps to understand how perturbations are generated by inflation:

- At early time, all perturbation modes of interest are casually connected, i.e. correspond to $k \sim 1/\lambda$ larger than the horizon: $k > aH$.
- On these (small) scales perturbations in the inflaton field are described by a collection of harmonic oscillators (see Mukahnov-Sasaki Equation – chap 10)
- These perturbations have quantum nature and can be followed using quantum mechanics canonical quantification. Their amplitudes have a non-zero variance:

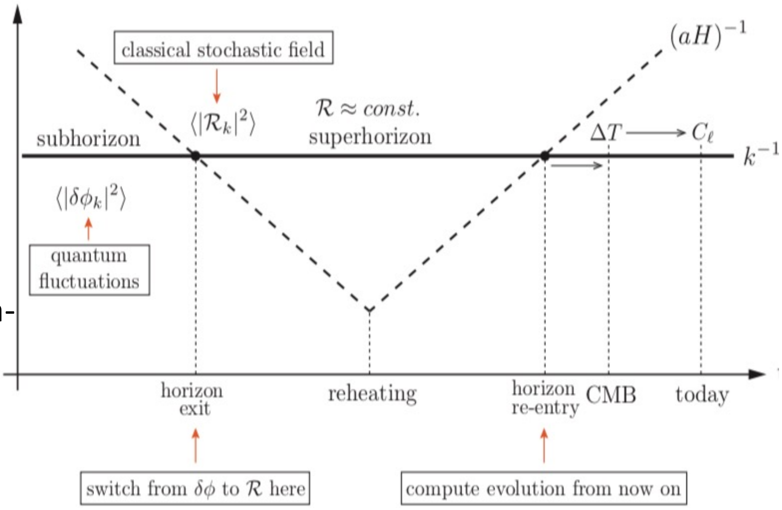
$$\langle|\delta\phi_k|^2\rangle \equiv \langle 0||\delta\phi_k|^2|0\rangle$$

- Inflaton perturbations induce comoving curvature fluctuations. In the spatially flat gauge

$$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\,\delta\phi$$

- Thus, the curvature (gauge-invariant) fluctuations have a non-zero variance:

$$\langle|\mathcal{R}_k|^2\rangle = \left(\frac{\mathcal{H}}{\bar{\phi}'}\right)^2 \langle|\delta\phi_k|^2\rangle$$



Quantum fluctuations in de Sitter space

Comoving curvature power spectrum:

The power spectra of these quantities are related via:

$$\Delta^2_{\mathcal{R}} = \frac{1}{2\varepsilon}\frac{\Delta^2_{\delta\phi}}{M^2_{\rm pl}}\,, \qquad \text{where} \quad \varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M^2_{\rm pl}H^2}$$

So, the power spectrum of the comoving curvature fluctuations is:

$$\boxed{\Delta^2_{\mathcal{R}}(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M^2_{\rm pl}}\bigg|_{k=aH}}$$

which is gauge invariant and remains constant when the wavenumber k leaves the horizon scale ($k_H = aH$) during inflation.

Since the right-hand size of the power spectra is evaluated at horizon crossing, $k = aH$, the power spectrum is purely a function of k. It is often useful to model this k dependence as:

$$\Delta^2_{\mathcal{R}}(k) \equiv A_s \left(\frac{k}{k_\star}\right)^{n_s - 1}$$

CMB observations impose constrains on $A_s = (2.196 \pm 0.060)\times 10^{-9}$ at $k_* = 0.05$ Mpc^{-1}. For the scalar spectral index constraints are $n_s = 0.9603 \pm 0.0073$.

The matter power spectrum

The observable matter perturbations at a given time (redshift) are related to the curvature perturbations at horizon re-entry:

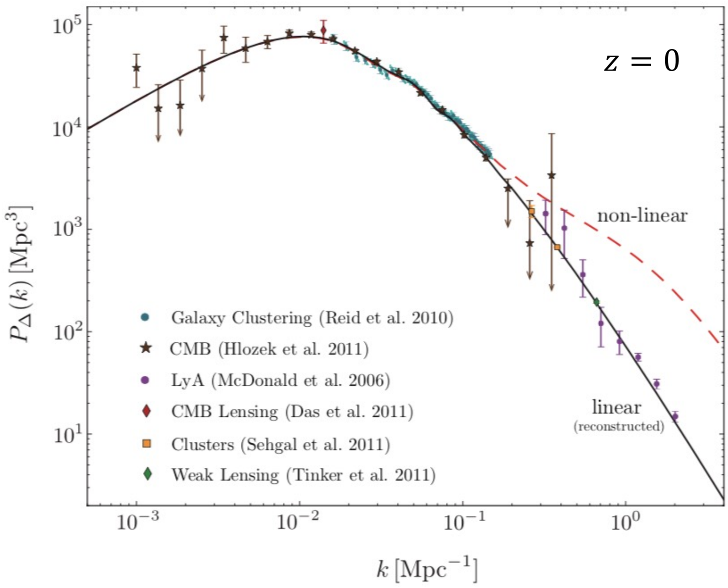
$$\Delta_{m,\boldsymbol{k}}(z) = T(k,z)\,\mathcal{R}_{\boldsymbol{k}}$$

where $T(k,z)$ is known as **transfer function** that gives the way fluctuations evolve from horizon re-entry until a given time (redshift)

The corresponding matter power spectrum is simply:

$$P_\Delta(k,z) \equiv |\Delta_{m,\boldsymbol{k}}(z)|^2 = T^2(k,z)\,|\mathcal{R}_{\boldsymbol{k}}|^2$$

To compute the transfer function one needs a Boltzmann code that Is able to properly describe the full evolution of all matter components throughout the phases of the standard Big Bang Model evolution.



Quantum fluctuations in de Sitter space

Comoving curvature power spectrum:

The spectral index one can be defined as:

$$n_s - 1 \equiv \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k}$$

This can be split in two factors:

$$\frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} = \frac{d\ln\Delta^2_{\mathcal{R}}}{dN} \times \frac{dN}{d\ln k}$$

The derivative with respect to e-folds is

$$\frac{d\ln\Delta^2_{\mathcal{R}}}{dN} = 2\frac{d\ln H}{dN} - \frac{d\ln\varepsilon}{dN}\,. \tag{6.5.63}$$

The first term is just -2ε and the second term is $-\eta$ (see Chapter 2). The second factor in (6.5.62) is evaluated by recalling the horizon crossing condition $k = aH$, or

$$\ln k = N + \ln H\,. \tag{6.5.64}$$

Hence, we have

$$\frac{dN}{d\ln k} = \left[\frac{d\ln k}{dN}\right]^{-1} = \left[1 + \frac{d\ln H}{dN}\right]^{-1} \approx 1 + \varepsilon\,. \tag{6.5.65}$$

To first order in the Hubble slow-roll parameters, we therefore find

$$\boxed{n_s - 1 = -2\varepsilon - \eta}\,. \tag{6.5.66}$$