Universo Primitivo
2024-2025 (1º Semestre)

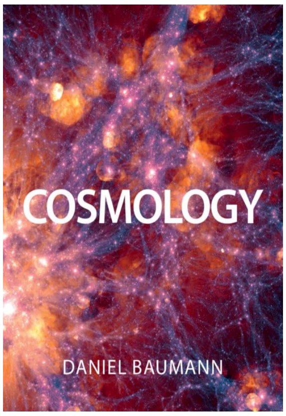
Mestrado em Física - Astronomia

Chapter 9

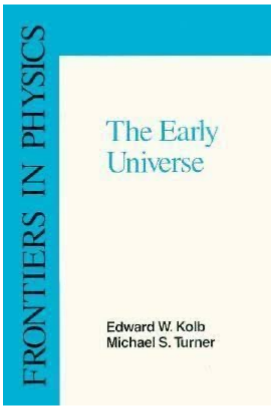
9 Inflation: the origin of perturbations

- The Basic Picture;
- Cosmological perturbation theory
- Quantum fluctuations in the de Sitter space;
- Primordial power spectra from inflation;
- CMB power spectrum

References



Ch. 4



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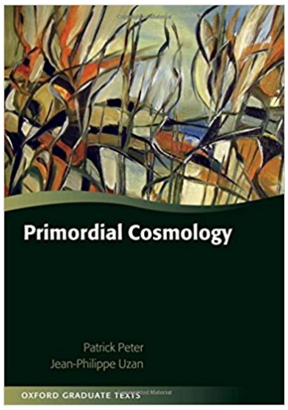
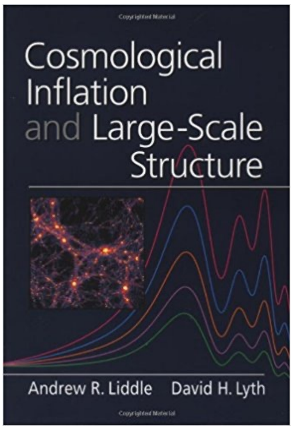


Ch. 8.1, 8.2



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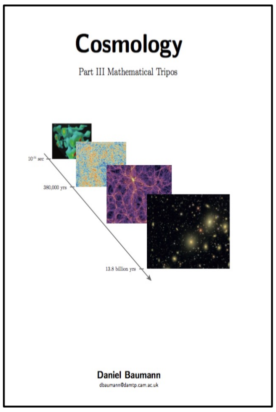

Ch. 2

Inflation: the basic picture

The Inflationary phase of the Universe needs to happen at very early times. Present data is consistent with an inflationary period that lasted for about around $\Delta t \sim 10^{-36}$ at cosmic time of about $t \sim 10^{-32} - 10^{-33}$seconds

In these conditions the **inflaton field has a quantum nature** and its energy density is quantified. The **Heisenberg uncertainty principle** allows the origin of energy density fluctuations given the short timescales involved.

$$\Delta E_{\phi} > h/(4\pi\Delta t)$$

The **inflation field**, $\phi(x,t)$, therefore **acquires a spatial dependence due to quantum fluctuations**, $\delta\phi(x,t)$, about its "background" Value, $\phi(t)$:

$$\boldsymbol{\phi(x,t) = \phi(t) + \delta\phi(x,t)}$$

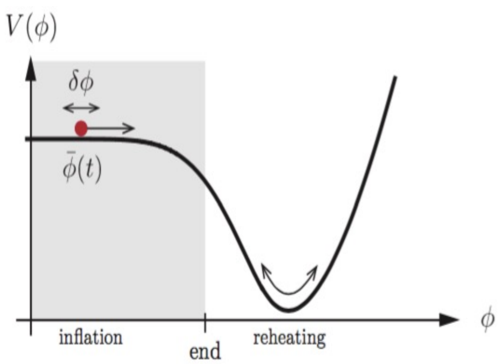

Figure 6.1: *Quantum fluctuations $\delta\phi(t,\boldsymbol{x})$ around the classical background evolution $\bar{\phi}(t)$. Regions acquiring a negative fluctuations $\delta\phi$ remain potential-dominated longer than regions with positive $\delta\phi$. Different parts of the universe therefore undergo slightly different evolutions. After inflation, this induces density fluctuations $\delta\rho(t,\boldsymbol{x})$.*

Inflation: the basic picture

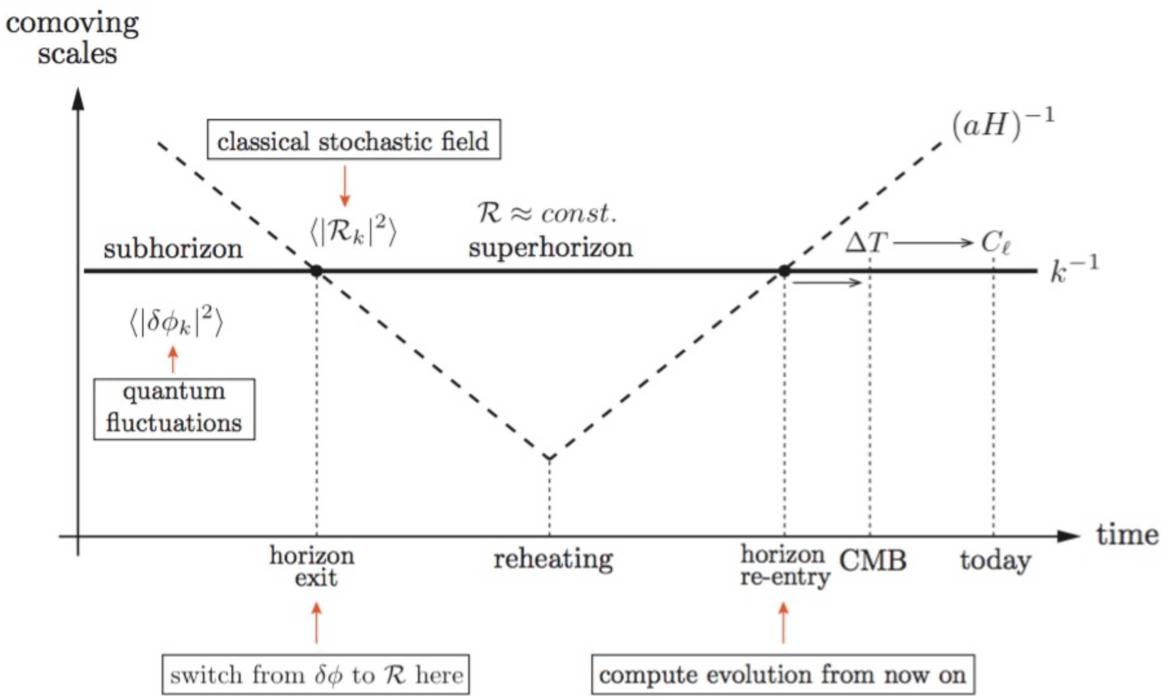

Figure 6.2: *Curvature perturbations during and after inflation: The comoving horizon $(aH)^{-1}$ shrinks during inflation and grows in the subsequent FRW evolution. This implies that comoving scales k^{-1} exit the horizon at early times and re-enter the horizon at late times. While the curvature perturbations $\mathcal{R}$ are outside of the horizon they don't evolve, so our computation for the correlation function $\langle|\mathcal{R}_k|^2\rangle$ at horizon exit during inflation can be related directly to observables at late times.*

Relativistic (GR) perturbation theory

Metric perturbations:

Metric perturbations can be described as:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

Let us assume the unperturbed metric $\bar{g}_{\mu\nu}$ is FLRW, written in a conformal way,

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]$$

The perturbed metric, $\delta g_{\mu\nu}$, can be written in a general way as,

$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2A)\mathrm{d}\tau^2 - 2B_i\mathrm{d}x^i\mathrm{d}\tau - (\delta_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right]$$

Which is symmetric and A, B_i and h_{ij} are functions of time and space. In total these encapsulate 10 independent functions (degrees of freedom, d.o.f.):

$$g_{\mu\nu} = a^2(\tau)\begin{pmatrix} 1+2A & -2B_1 & -2B_2 & -2B_3 \\ -2B_1 & -(1+h_{11}) & -h_{12} & -h_{13} \\ -2B_2 & -h_{12} & -(1+h_{22}) & -h_{23} \\ -2B_3 & -h_{13} & -h_{23} & -(1+h_{33}) \end{pmatrix}$$

Relativistic (GR) perturbation theory

Scalar, Vector Tensor (SVT) decomposition

The perturbation variables can be decomposed into their scalar, vector and tensor dependences. **This is useful because these dependences do not mix at linear order**:

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{vector}}$$

$$h_{ij} = \underbrace{2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{2\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{2\hat{E}_{ij}}_{\text{tensor}}$$

with,

$$\partial_{\langle i}\partial_{j\rangle}E \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E\ ,$$
$$\partial_{(i}\hat{E}_{j)} \equiv \frac{1}{2}\left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right)\ .$$

where:

SVT d.o.f.

- 4 • *scalars:* $A,\ B,\ C,\ E$
- 4 • *vectors:* $\hat{B}_i,\ \hat{E}_i$
- 2 • *tensors:* $\hat{E}_{ij}$

$\partial^i\hat{B}_i = 0$

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$\partial^i\hat{E}_i = 0$ and $\partial^i\hat{E}_{ij} = 0$

Relativistic (GR) perturbation theory

Gauge freedom

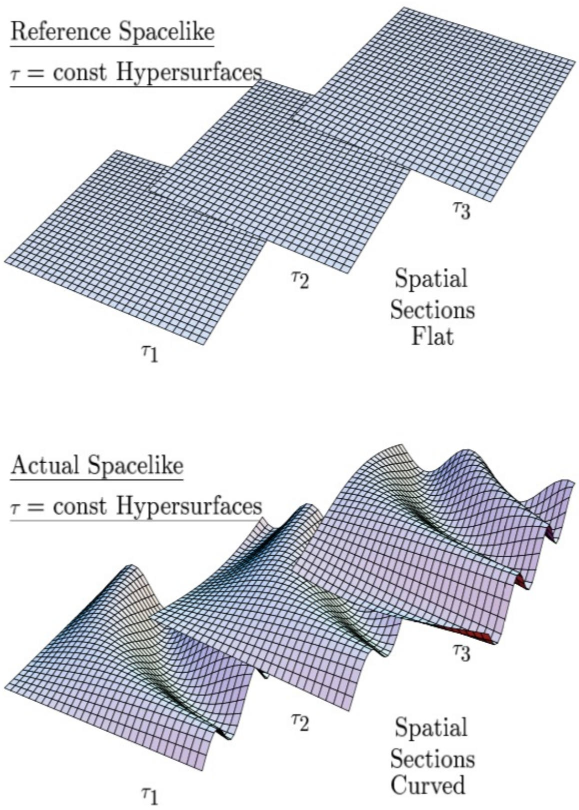
GR is a gauge theory where the gauge transformations are generic coordinate transformations.

$$\mathrm{d}s^2 = g_{\mu\nu}(X)\mathrm{d}X^\mu\mathrm{d}X^\nu = \tilde{g}_{\alpha\beta}(\tilde{X})\mathrm{d}\tilde{X}^\alpha\mathrm{d}\tilde{X}^\beta$$

$$g_{\mu\nu}(X) = \frac{\partial\tilde{X}^\alpha}{\partial X^\mu}\frac{\partial\tilde{X}^\beta}{\partial X^\nu}\tilde{g}_{\alpha\beta}(\tilde{X})$$

A gauge choice is a way of choosing the (time) slicing and (spatial) threading of spacetime.

GAUGE CHOICE $\Longleftrightarrow$ SLICING AND THREADING



How to treat Perturbations?

- Either find **gauge invariant variables** to describe perturbations. These variables are called **real spacetime perturbations**.
- Or **fix a gauge** choice and **keep track of all perturbations and check how quantities transform**.

Relativistic (GR) perturbation theory

Gauge-invariant perturbation variables

One avoids gauge problems by defining special combinations of the SVT perturbations that do not change under coordinate transformations. These are known as the **Bardeen potentials** (or Bardeen Variables)

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')' , \qquad \hat{\Phi}_i \equiv \hat{E}_i' - \hat{B}_i , \qquad \hat{E}_{ij}$$

$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E .$$

where ´ is derivative with respect to conformal time, τ, and $\mathcal{H} \equiv a'/a$ is the Hubble parameter in conformal time.

Useful Gauge fixing choices

The gauge freedom can be used to conveniently set some of the above variables to zero:

- **Newtonian Gauge**: $E = B = 0$
  The metric simply becomes:
  $$\mathrm{d}s^2 = a^2(\tau)\left[(1 + 2\Psi)\mathrm{d}\tau^2 - (1 - 2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]$$
  where the remaining non-zero variables were renamed to $A \equiv \Psi$, $C \equiv -\Phi$

Relativistic (GR) perturbation theory

Useful Gauge fixing choices

(continuation)

- **Spatially flat gauge** : $C = E = 0$
  This is a convenient gauge choice for the calculation of the inflationary perturbations.

- **Uniform density gauge**: consists in choosing the time-slicing in a way that the total density perturbation (see perturbed stress-energy tensor subsection) is set to zero: $\delta\rho = 0$

- **Comoving gauge**: consists in choosing coordinates in a way that the total momentum density vanishes (see perturbed stress-energy tensor subsection): $q_i = (\bar{\rho} + \bar{P})v_i = 0$. One has that $q_i = B_i = 0$.
  This choice is naturally connected to the inflationary initial conditions

Relativistic (GR) perturbation theory

Perturbed Stress-Energy Tensor

For small perturbations the perturbed stress-energy tensor can be written as:

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu$$

where the unperturbed stress-energy tensor is

$$\bar{T}^\mu{}_\nu = (\bar{\rho} + \bar{P})\bar{U}^\mu \bar{U}_\nu - \bar{P}\,\delta^\mu_\nu$$

and one has that, $\bar{U}_\mu = a\delta^0_\mu,\ \bar{U}^\mu = a^{-1}\delta^\mu_0$, for a comoving observer.
The perturbation to the stress-energy tensor can be written as:

$$\delta T^\mu{}_\nu = (\delta\rho + \delta P)\bar{U}^\mu \bar{U}_\nu + (\bar{\rho} + \bar{P})(\delta U^\mu \bar{U}_\nu + \bar{U}^\mu \delta U_\nu) - \delta P\,\delta^\mu_\nu - \Pi^\mu{}_\nu$$

where Π^μ_ν **is the anisotropic stress tensor** and the perturbed density, pressure and four-velocity vectors generally depend on space and time.
To 1st order one has (**see eg Baumann**):

$$\delta U^\mu = a^{-1}\left[-A, v^i\right]; \qquad \delta U_\nu = a\left[A, -(v^i + B_i)\right]$$

and

$$U^\mu = a^{-1}\left[1 - A, v^i\right]; \qquad U_\nu = a\left[1 + A, -(v^i + B_i)\right]$$

Relativistic (GR) perturbation theory

Perturbed Stress-Energy Tensor

(continuation)
Using these expressions of U^μ and U_ν in δU^μ_ν one gets

$$\begin{aligned}
\delta T^0{}_0 &= \delta\rho\ , \\
\delta T^i{}_0 &= (\bar{\rho} + \bar{P})v^i\ , \\
\delta T^0{}_j &= -(\bar{\rho} + \bar{P})(v_j + B_j)\ , \\
\delta T^i{}_j &= -\delta P \delta^i_j - \Pi^i{}_j\ .
\end{aligned}$$

The quantity $q_i = (\bar{\rho} + \bar{P})v_i$ is called the **momentum density three-vector**. Note that the perturbed (peculiar) velocity $\delta U^i \equiv v^i/a$ is not additive quantity, but q_i is additive. If there are several fluid components all the quantities bellow are additive:

$$\delta\rho = \sum_I \delta\rho_I\ , \quad \delta P = \sum_I \delta P_I\ , \quad q^i = \sum_I q^i_I\ , \quad \Pi^{ij} = \sum_I \Pi^{ij}_I$$

And the stress-energy tensor is also additive: $T_{\mu\nu} = \sum_I T^I_{\mu\nu}$

The **SVT decomposition** can also be applied to the perturbed stress-energy tensor: $\delta\rho$ and δP only have scalar parts; $q_i = \partial_i q + \hat{q}_i$ has a scalar and a vector part; Π_{ij} has scalar, vector and tensor parts: $\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi + \partial_{(i}\hat{\Pi}_{j)} + \hat{\Pi}_{ij}$

Relativistic (GR) perturbation theory

Gauge-invariant perturbation quantities

Comoving-gauge density perturbation: The quantity :

$$\bar{\rho}\Delta \equiv \delta\rho + \bar{\rho}'(v+B)$$

Where v is a scalar velocity function such that $v_i = \partial_i v$, is gauge-invariant. It is very useful to study density perturbations .

Comoving Curvature perturbation: In a arbitrary gauge, the intrinsic curvature of hypersurfaces of constant time can be computed using the spacial part of the perturbed metric. Since this is a scalar it only receives contributions from the scalar variables of the spatial part of metric ($E_{ij} \equiv \partial_{\langle i}\partial_{j\rangle}E$) :

$$\gamma_{ij} \equiv a^2\left[(1+2C)\delta_{ij} + 2E_{ij}\right]$$

After some long calculations (**see Baumann**) the intrinsic curvature is given by:

$$a^2 R_{(3)} = -4\nabla^2\left(C - \frac{1}{3}\nabla^2 E\right)$$

The comoving curvature perturbation

$$\mathcal{R} = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v)$$

Is gauge-invariant and it is defined as the comoving curvature computed in the comoving gauge ($q_i = B_i = 0$). In the Newtonian gauge this is $\mathcal{R} = -\Phi + \mathcal{H}v$.

Relativistic (GR) perturbation theory

Adiabatic versus Isocurvature perturbations

Density perturbations are said to be **adiabatic** if

$$\delta\rho_I(\tau,\boldsymbol{x}) \equiv \bar{\rho}_I(\tau+\delta\tau(\boldsymbol{x})) - \bar{\rho}_I(\tau) = \bar{\rho}_I'\delta\tau(\boldsymbol{x})$$

for all fluid components, I. This implies:

$$\delta\tau = \frac{\delta\rho_I}{\bar{\rho}_I'} = \frac{\delta\rho_J}{\bar{\rho}_J'} \qquad \text{for all species } I \text{ and } J$$

If fluid components obey to independent continuity equations, $\bar{\rho}_I' = -3\mathcal{H}(1+w_I)\bar{\rho}_I$ one gets:

$$\frac{\delta_I}{1+w_I} = \frac{\delta_J}{1+w_J} \qquad \text{for all species } I \text{ and } J$$

This also implies that the total density density of the fluid is perturbed and is given simply by

$$\delta\rho_{\text{tot}} = \bar{\rho}_{\text{tot}}\delta_{\text{tot}} = \sum_I \bar{\rho}_I\delta_I$$

Relativistic (GR) perturbation theory

Adiabatic versus Isocurvature perturbations

(continuation)

Isocurvature perturbations are perturbation in the different fluid components in a way that conserves the total energy density. This implies that different fluid components have fluctuations such as the quantity:

$$S_{IJ} \equiv \frac{\delta_I}{1+w_I} - \frac{\delta_J}{1+w_J}$$

is different from zero.

Linear perturbation GR equations & conservation laws

Once the perturbed stress-energy tensor and perturbed metric are defined one proceeds with the calculation of the:

- Perturbed metric connections;
- The conservation laws of the perturbed stress-energy tensor;
- The Einstein equations involving the perturbed quantities up to linear order of the perturbed quantities (higher order calculations are more complex or impossible to do). (e.g. **Ch.4 Baumann**)
- Solve the resulting equations to derive the evolution of perturbations (e.g. **Ch.5 Baumann**)

Relativistic (GR) perturbation theory

Linear perturbation GR equations & conservation laws (Newton. gauge)

$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2\Psi)\mathrm{d}\tau^2 - (1-2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] \; . \tag{4.4.168}$$

In these lectures, we won't encounter situations where anisotropic stress plays a significant role, so we will always be able to set $\Psi = \Phi$.

- The Einstein equations then are

$$\begin{aligned} \nabla^2\Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) &= 4\pi G a^2\,\delta\rho \; , & (4.4.169)\\ \Phi' + \mathcal{H}\Phi &= -4\pi G a^2(\bar\rho + \bar P)v \; , & (4.4.170)\\ \Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi &= 4\pi G a^2\,\delta P \; . & (4.4.171)\end{aligned}$$

  The source terms on the right-hand side should be interpreted as the sum over all relevant matter components (e.g. photons, dark matter, baryons, etc.). The Poisson equation takes a particularly simple form if we introduce the comoving gauge density contrast

$$\nabla^2\Phi = 4\pi G a^2 \bar\rho\,\Delta \; . \tag{4.4.172}$$

- From the conservation of the stress-tensor, we derived the relativistic generalisations of the continuity equation and the Euler equation

$$\begin{aligned} \delta' + 3\mathcal{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar P}{\bar\rho}\right)\delta &= -\left(1 + \frac{\bar P}{\bar\rho}\right)(\boldsymbol{\nabla}\cdot\boldsymbol{v} - 3\Phi') \; , & (4.4.173)\\ \boldsymbol{v}' + 3\mathcal{H}\left(\frac{1}{3} - \frac{\bar P'}{\bar\rho'}\right)\boldsymbol{v} &= -\frac{\boldsymbol{\nabla}\delta P}{\bar\rho + \bar P} - \boldsymbol{\nabla}\Phi \; . & (4.4.174)\end{aligned}$$

Inflation: the basic picture

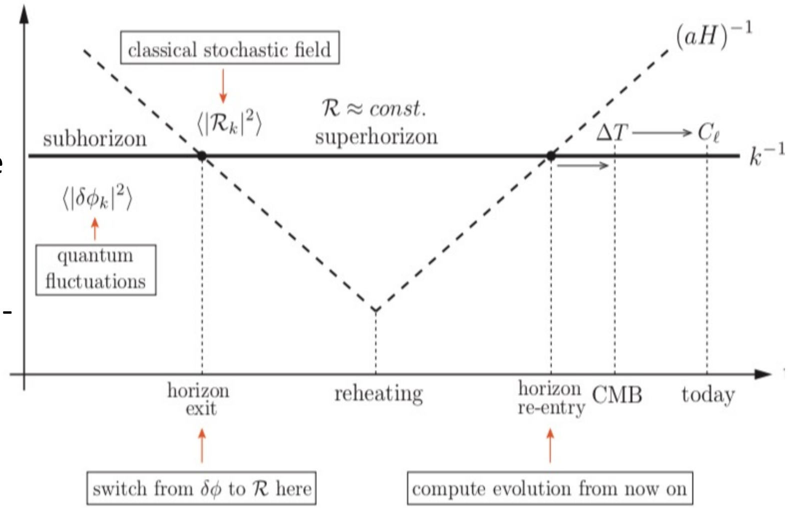
Key steps to understand how perturbations are generated by inflation:

- At early time all perturbation modes of interest are casually connected, i.e. correspond to $k = 1/\lambda$ larger then the horizon: $k > aH$.
- On these (small) scales perturbations in the inflaton field are described by a collection of harmonic oscillators
- These perturbations have quantum nature and can be followed using quantum mechanics canonical quantification. Their amplitudes have a non-zero variance: $\langle|\delta\phi_k|^2\rangle \equiv \langle 0||\delta\phi_k|^2|0\rangle$
- Inflaton perturbations induce comoving curvature fluctuations. In the spatially flat gauge

$$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\,\delta\phi$$

- Thus the curvature (gauge-invariant) fluctuations have a non-zero variance:

$$\langle|\mathcal{R}_k|^2\rangle = \left(\frac{\mathcal{H}}{\bar{\phi}'}\right)^2 \langle|\delta\phi_k|^2\rangle$$



Inflation: the basic picture

Relation between curvature and inflaton field perturbations

The relation between the inflaton field perturbation and the curvature perturbations is the simplest if one computes it using the *spatially flat gauge.* This is given by:

$$\boxed{\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\,\delta\phi}$$

$\delta\phi \to \mathcal{R}$.—From the gauge-invariant definition of $\mathcal{R}$, eq. (4.3.159), we get

$$\mathcal{R} = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v) \xrightarrow{\text{spatially flat}} \mathcal{H}(B+v)\ . \qquad (6.1.3)$$

We recall that the combination $B+v$ appeared in the off-diagonal component of the perturbed stress tensor, cf. eq. (4.2.76),

$$\delta T^0{}_j = -(\bar{\rho}+\bar{P})\partial_j(B+v)\ . \qquad (6.1.4)$$

We compare this to the first-order perturbation of the stress tensor of a scalar field, cf. eq. (2.3.26),

$$\delta T^0{}_j = g^{0\mu}\,\partial_\mu\phi\,\partial_j\delta\phi = \bar{g}^{00}\,\partial_0\bar{\phi}\,\partial_j\delta\phi = \frac{\bar{\phi}'}{a^2}\partial_j\delta\phi\ , \qquad (6.1.5)$$

to get

$$B+v = -\frac{\delta\phi}{\bar{\phi}'}\ . \qquad (6.1.6)$$

Substituting (6.1.6) into (6.1.3) we obtain (6.1.2).

Inflation: the basic picture

Relation between curvature and inflaton field perturbations

The relation between the inflaton field perturbation and the curvature perturbations is the simplest if one computes it using the *spatially flat gauge.* This is given by:

$$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi$$

Therefore the variance of the curvature and the inflaton field perturbations are also related in a simple way,

$$\langle|\mathcal{R}\ |^2\rangle = \left(\frac{\mathcal{H}}{\bar{\phi}'}\right)^2 \langle|\delta\phi\ |^2\rangle$$

Expending both perturbations in Fourier series, taking each k mode independently, one obtains a similar relation between the coefficients of the Fourier expansions (i.e. the perturbations in Fourier space)

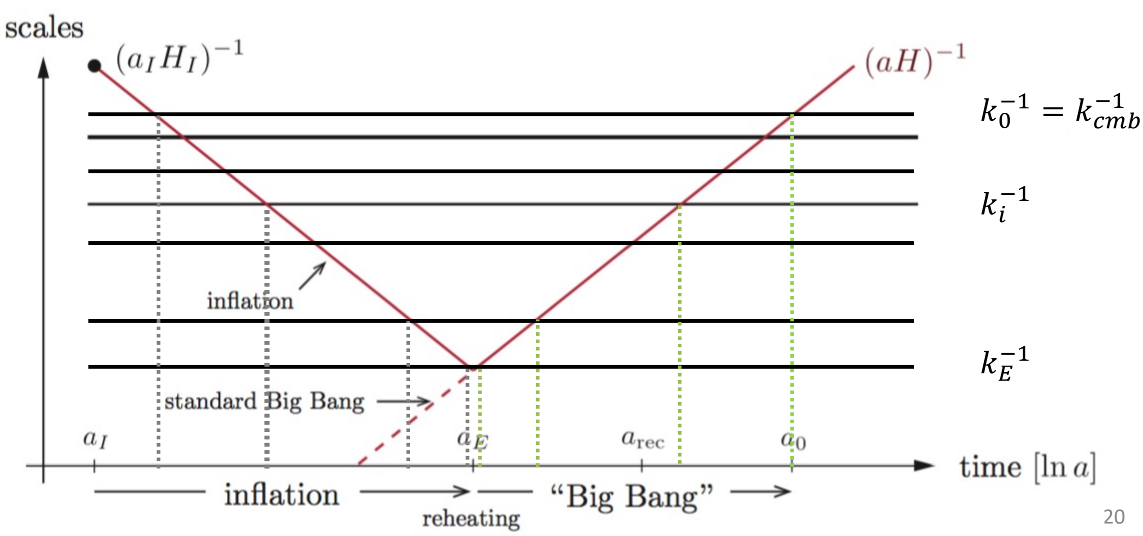
$$\langle|\mathcal{R}_k|^2\rangle = \left(\frac{\mathcal{H}}{\bar{\phi}'}\right)^2 \langle|\delta\phi_k|^2\rangle$$

Inflation: the basic picture

At horizon crossing of a given comoving scale $\lambda = 1/k$, one necessarily has:

$$k^{-1} = (aH)^{-1} \quad \Leftrightarrow \quad k = aH$$

So the (comoving) Fourier mode k are simply giving (the inverse) of the comoving Hubble radius at a given epoch.



Mukahnov-Sasaki equation

Classical inflaton field fluctuations:

Let us first see how the **inflaton field action** can be used to derive the inflaton perturbations. The action is:

$$S = \int \mathrm{d}\tau\, \mathrm{d}^3x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)\right]$$

(the integrand function is the Lagrangian density). Evaluating for a **unperturbed FLRW** metric one gets (exercise: prove this):

$$S = \int \mathrm{d}\tau \mathrm{d}^3x \left[\frac{1}{2} a^2 \left((\phi')^2 - (\nabla\phi)^2\right) - a^4 V(\phi)\right]$$

To introduce perturbations, it is convenient to write them in the following way:

$$\boxed{\phi(\tau, \boldsymbol{x}) = \bar{\phi}(\tau) + \frac{f(\tau, \boldsymbol{x})}{a(\tau)}}$$

To derive an equation of motion for the perturbation $f(\tau, x)$ one usually does:

- Assume $\phi(\tau, x)$ in the action S.
- Expand the action up to 2nd order in the fluctuations f
- Collect all 1st order and 2nd order action terms in 2 separate actions: $S^{(1)}$ and $S^{(2)}$.
- Apply the Euler-Lagrange equations to both actions.

Mukahnov-Sasaki equation

Classical inflaton field fluctuations:

The result for using the action, $S^{(1)}$, gives the Kein-Gordan equation for the background field:

$$\bar{\phi}'' + 2\mathcal{H}\bar{\phi}' + a^2 V_{,\phi} = 0$$

From the $S^{(2)}$, which can be approximated by (see Baumann Sect. 6.2),

$$S^{(2)} \approx \int \mathrm{d}\tau \mathrm{d}^3x \, \frac{1}{2} \left[(f')^2 - (\nabla f)^2 + \frac{a''}{a} f^2\right]$$

the Euler-Lagrange equation gives the so called **Mukahnov-Sassak**i equation

$$f'' - \nabla^2 f - \frac{a''}{a} f = 0 \qquad \text{(real space-time)}$$

$$\boxed{f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0} \qquad \text{(fourier space-time)}$$

This has an exact solution of the form:

$$f_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right)$$

Mukahnov-Sasaki equation

Classical inflaton field fluctuations:

where $\boldsymbol{\alpha}$**, and** $\boldsymbol{\beta}$ **are set by imposing as initial conditions a plane-wave solution at early times,** $\boldsymbol{\tau \to 0}$**.** Assuming a pure de Sitter space ($a = e^{Ht}$) one has:

$$\tau = \int^t e^{-Ht}\, dt = -H^{-1} e^{-Ht} = -\frac{1}{aH} \quad ; \qquad \frac{a''}{a} = \frac{2}{\tau^2}$$

The solution is then

$$f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)$$

On **sub-horizon scales**, $k^2 \gg a''/a \approx 2\mathcal{H}^2$, the M-S equation becomes

$$f_{\boldsymbol{k}}'' + k^2 f_{\boldsymbol{k}} \approx 0$$

which is a classical harmonic oscillator with spatial frequency $\omega(k) = k$.

However we expect these fluctuations to be of quantum mechanics (QM) nature. To treat this one applies the canonical formalism of QM to the classical harmonic oscillator.

Quantum fluctuations in de Sitter space

Canonical quantization of the inflaton fluctuations:

One proceeds as for the harmonic oscillator theory in QM. The relevant classical quantities in the action $S^{(2)}$ are the:

- Inflaton fluctuation: $f = a\delta\phi$
- Momentum conjugate of f: $\pi \equiv \frac{\partial \mathcal{L}}{\partial f'} = f'$

One then **promotes the fields** $f(\tau, x)$ **and** $\pi(\tau, x)$ **to quantum operators** that satisfy the following commutation rules:

$$[\hat{f}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{x}')] = i\delta(\boldsymbol{x} - \boldsymbol{x}')$$

$$[\hat{f}_{\boldsymbol{k}}(\tau), \hat{\pi}_{\boldsymbol{k}'}(\tau)] = \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3x'}{(2\pi)^{3/2}} \underbrace{[\hat{f}(\tau, \boldsymbol{x}), \hat{\pi}(\tau, \boldsymbol{x}')]}_{i\delta(\boldsymbol{x} - \boldsymbol{x}')} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}'}$$

$$= i \int \frac{\mathrm{d}^3x}{(2\pi)^3} e^{-i(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}}$$

$$= i\delta(\boldsymbol{k} + \boldsymbol{k}') \ ,$$

i.e. they commute in real and fourier spaces for $x \neq x'$ and $k \neq -k'$, respectively

Quantum fluctuations in de Sitter space

Canonical quantization of the inflaton fluctuations:

The inflaton perturbation operator can then be written in terms of the creation and annihilation operators:

$$\hat{f}_{\boldsymbol{k}}(\tau) = f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)a_{\boldsymbol{k}}^\dagger$$

where $f_k(\tau)$ and $f_k^*(\tau)$ are the solution of the M-S equation,

$$f_k'' + \omega_k^2(\tau)f_k = 0\ , \qquad \text{where} \quad \omega_k^2(\tau) \equiv k^2 - \frac{a''}{a}$$

The creation and annihilation operators verify

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger] = \delta(\boldsymbol{k} + \boldsymbol{k}')$$

The quantum states (in the Hilbert space) are constructed by defining a **vacuum state** $|0>$ via the condition $\hat{a}_k|0> = 0$.

Excited states of the inflaton perturbation are created using the usual creation rule:

$$|m_{\boldsymbol{k}_1}, n_{\boldsymbol{k}_2}, \cdots\rangle = \frac{1}{\sqrt{m!n!\cdots}}\left[(a_{\boldsymbol{k}_1}^\dagger)^m(a_{\boldsymbol{k}_2}^\dagger)^n\cdots\right]|0\rangle$$

Quantum fluctuations in de Sitter space

Quantum fluctuations about the zero point (vacuum state):

Finally one can obtain inflaton perturbation operator spectrum by computing the mean and variance expectation values about the vacuum state $|0>$. One has:

$$\hat{f}(\tau, \boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\left[f_k(\tau)\hat{a}_{\boldsymbol{k}} + f_k^*(\tau)a_{\boldsymbol{k}}^\dagger\right]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\ .$$

The expectation value for $<\hat{f}> = 0$ naturally, but the variance does not. One has:

$$\begin{aligned}\langle|\hat{f}|^2\rangle &\equiv \langle 0|\hat{f}^\dagger(\tau,\mathbf{0})\hat{f}(\tau,\mathbf{0})|0\rangle \\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\langle 0|\big(f_k^*(\tau)\hat{a}_{\boldsymbol{k}}^\dagger + f_k(\tau)\hat{a}_{\boldsymbol{k}}\big)\big(f_{k'}(\tau)\hat{a}_{\boldsymbol{k}'} + f_{k'}^*(\tau)\hat{a}_{\boldsymbol{k}'}^\dagger\big)|0\rangle \\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}f_k(\tau)f_{k'}^*(\tau)\langle 0|[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}^\dagger]|0\rangle \\ &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}|f_k(\tau)|^2 \\ &= \int \mathrm{d}\ln k\ \frac{k^3}{2\pi^2}|f_k(\tau)|^2\ .\end{aligned}$$

Quantum fluctuations in de Sitter space

Quantum fluctuations about the zero point (vacuum state):

One defines the dimensionless power spectrum of the inflaton fluctuations as

$$\Delta_f^2(k,\tau) \equiv \frac{k^3}{2\pi^2}|f_k(\tau)|^2$$

This means that the classical solution $f_k(\tau)$ determines the variance of the quantum fluctuations. Given the relation between the fluctuation f and the inflaton field, $\delta\phi = f\ /\ a$ one has:

$$\Delta_{\delta\phi}^2(k,\tau) = a^{-2}\Delta_f^2(k,\tau) = \left(\frac{H}{2\pi}\right)^2 \left(1 + \left(\frac{k}{aH}\right)^2\right) \xrightarrow{\text{superhorizon}} \left(\frac{H}{2\pi}\right)^2$$

So at horizon crossing one can use the following approximation:

$$\Delta_{\delta\phi}^2(k) \approx \left(\frac{H}{2\pi}\right)^2 \Bigg|_{k=aH}$$

Going back to the relation between the inflaton fluctuation and the curvature fluctuations,

$$\langle|\mathcal{R}_k|^2\rangle = \left(\frac{\mathcal{H}}{\bar{\phi}'}\right)^2 \langle|\delta\phi_k|^2\rangle$$

Quantum fluctuations in de Sitter space

Comoving curvature power spectrum:

The power spectra of these quantities is related via:

$$\Delta_{\mathcal{R}}^2 = \frac{1}{2\varepsilon}\frac{\Delta_{\delta\phi}^2}{M_{\rm pl}^2}\ , \qquad \text{where} \quad \varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\rm pl}^2 H^2}$$

So the power spectrum of the comoving curvature fluctuations is:

$$\Delta_{\mathcal{R}}^2(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M_{\rm pl}^2}\Bigg|_{k=aH}$$

which is gauge invariant and remains constant when the wavenumber k leaves the horizon scale ($k_H = aH$) during inflation.

Since the right-hand size of the power spectra is evaluated at horizon crossing, $k = aH$, the power spectrum is purely a function of k. It is often useful to model this k dependence as:

$$\Delta_{\mathcal{R}}^2(k) \equiv A_s \left(\frac{k}{k_\star}\right)^{n_s - 1}$$

CMB observations impose constrains on $A_s = (2.196 \pm 0.060)\times 10^{-9}$ at $k_* = 0.05$ $\mathrm{M}pc^{-1}$. For the scalar spectral index constraints are $n_s = 0.9603 \pm 0.0073$.

Quantum fluctuations in de Sitter space

Comoving curvature power spectrum:

The spectral index one can be defined as:

$$n_s - 1 \equiv \frac{d\ln \Delta^2_{\mathcal{R}}}{d\ln k}$$

This can be split in two factors:

$$\frac{d\ln \Delta^2_{\mathcal{R}}}{d\ln k} = \frac{d\ln \Delta^2_{\mathcal{R}}}{dN} \times \frac{dN}{d\ln k}$$

The derivative with respect to e-folds is

$$\frac{d\ln \Delta^2_{\mathcal{R}}}{dN} = 2\,\frac{d\ln H}{dN} - \frac{d\ln \varepsilon}{dN} \,. \tag{6.5.63}$$

The first term is just -2ε and the second term is $-\eta$ (see Chapter 2). The second factor in (6.5.62) is evaluated by recalling the horizon crossing condition $k = aH$, or

$$\ln k = N + \ln H \,. \tag{6.5.64}$$

Hence, we have

$$\frac{dN}{d\ln k} = \left[\frac{d\ln k}{dN}\right]^{-1} = \left[1 + \frac{d\ln H}{dN}\right]^{-1} \approx 1 + \varepsilon \,. \tag{6.5.65}$$

To first order in the Hubble slow-roll parameters, we therefore find

$$\boxed{n_s - 1 = -2\varepsilon - \eta} \,. \tag{6.5.66}$$

The matter power spectrum

The observable matter perturbations at a given time (redshift) are related to the curvature perturbations at horizon re-entry:

$$\Delta_{m,\boldsymbol{k}}(z) = T(k,z)\,\mathcal{R}_{\boldsymbol{k}}$$

where $T(k,z)$ is known as **transfer function** that gives the way fluctuations evolve from horizon re-entry until a given time (redshift)

The corresponding matter power spectrum is simply:

$$P_\Delta(k,z) \equiv |\Delta_{m,\boldsymbol{k}}(z)|^2 = T^2(k,z)\,|\mathcal{R}_{\boldsymbol{k}}|^2$$

To compute the transfer function one needs a Boltzmann code that Is able to properly describe the full evolution of all matter components throughout the phases of the standard Big Bang Model evolution.