

Exercises on the Poincaré Group and Scalar Field Transformations

Lorentz Transformations

Exercise 1

- 1 Show that Lorentz transformations satisfy

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}.$$

Poincaré Transformations

Exercise 2

A general transformation:

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

- 1 Show that (Λ, a) forms a group.
- 2 Derive the composition rule:

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2).$$

Lie Algebra

Exercise 3Generators: $P^\mu, M^{\mu\nu}$

$$[P^\mu, P^\nu] = 0$$
$$[M^{\mu\nu}, P^\rho] = i(\eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu)$$

- 1 Verify these commutation relations.

Scalar Field Transformation

Exercise 4

A scalar field transforms as:

$$\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}(x - a)).$$

- 1 Show this preserves scalar character.
- 2 Derive infinitesimal translation transformation.
- 3 Show that for infinitesimal Lorentz transformations:

$$\delta\phi(x) = -\frac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x).$$

Klein–Gordon Lagrangian

Exercise 5

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

- 1 Show translation invariance.
- 2 Show Lorentz invariance.

Casimir Operators

Exercise 6

- 1 Show that

$$P^2 = P_\mu P^\mu$$

is invariant.

- 2 Show that for scalar fields:

$$P^2 \phi = m^2 \phi.$$

- 3 Explain why mass labels irreducible representations.

Finite Transformations

Exercise 7

- 1 Exponentiate generators to obtain finite translations.
- 2 Exponentiate boost generator in x -direction.
- 3 Show scalar field transformation under finite boost.

Casimir Operators and Classification

Exercise 8

- 1 Show that

$$P^2 = P_\mu P^\mu$$

commutes with all Poincaré generators.

- 2 Show that

$$W^2 = W_\mu W^\mu$$

is a Casimir operator of the Poincaré group.

- 3 Evaluate W^2 in the rest frame and show that it reduces to

$$W^2 = -m^2 s(s + 1)$$

for massive representations.