Schrödinger to Klein–Gordon

Rui Santos

FCUL & CFTC

2026

The Schrödinger Equation

The time-dependent Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(t,\mathrm{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathrm{x})\right)\psi(t,\mathrm{x})$$

- First order in time
- Second order in space
- Probability density: $\rho = |\psi|^2$

Plane Wave Solutions

For $V = 0$, consider:

$$\psi(t, \mathrm{x}) = e^{i(\mathrm{p}\cdot\mathrm{x} - Et)/\hbar}$$

Schrödinger dispersion relation:

$$E = \frac{\mathrm{p}^2}{2m}$$

Problems:

- Not Lorentz invariant
- Uses non-relativistic energy-momentum relation
- Instantaneous spreading of wave packets

Relativistic Energy-Momentum Relation

Special relativity:

$$E^2 = \mathrm{p}^2 + m^2$$

Promote E and p to operators

$$E \to i\partial_t, \qquad \mathrm{p} \to -i\nabla$$

Applying the operators to the relativistic relation:

$$E^2\psi = (\mathrm{p}^2 + m^2)\psi$$

leads to:

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\psi = 0$$
$$\partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$$

Manifestly Lorentz invariant form

$$(\Box + m^2)\phi(x) = 0$$

Plane Wave Solutions

Assume:

$$\phi(x) = e^{-ip_\mu x^\mu}$$

Then:

$$p_\mu p^\mu = m^2$$

$$E = \pm\sqrt{\mathrm{p}^2 + m^2}$$

New feature

Both positive and negative energy solutions

Probability Density Problem

Define current:

$$j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

$$\partial_\mu j^\mu = 0$$

But:

$$j^0 \not\geq 0$$

ϕ cannot be interpreted as a single-particle wavefunction

- Klein–Gordon equation describes a *field*
- $\phi(x)$ is an operator
- Particles emerge as field excitations

Historical resolution

Birth of Quantum Field Theory

Summary

- Schrödinger equation: non-relativistic QM
- Relativity requires $E^2 = p^2 + m^2$
- Leads naturally to Klein–Gordon equation
- Negative energies and probability issues $\Rightarrow$ QFT

Schrödinger to Klein–Gordon

Rui Santos

FCUL & CFTC

2026

Exercise 1: Dispersion Relations

The free Schrödinger equation admits plane-wave solutions:

$$\psi(t, \mathrm{x}) = e^{i(\mathrm{p}\cdot\mathrm{x} - Et)}.$$

1. Derive the dispersion relation.
2. Show explicitly that it is not Lorentz invariant.
3. Compare with the relativistic dispersion relation and identify the regime of validity.

Exercise 2: Relativistic Operator Substitution

Consider the relativistic energy-momentum relation:

$$E^2 = \mathrm{p}^2 + m^2.$$

1. Promote E and p to operators.
2. Derive the Klein–Gordon equation.
3. Write it in manifestly Lorentz-covariant form.

Exercise 3: Order of Time Derivatives

1. Explain why a relativistic wave equation cannot be first order in time and second order in space.
2. What is the physical consequence of the Klein–Gordon equation being second order in time?

Exercise 4: Probability Density

The conserved current for the Klein–Gordon field is:

$$j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*).$$

1. Show that $\partial_\mu j^\mu = 0$.
2. Explain why j^0 cannot be interpreted as a probability density.

Exercise 5: Non-relativistic Consistency Check

Assume the field decomposition:

$$\phi(t, \mathrm{x}) = e^{-imt}\psi(t, \mathrm{x}).$$

1. Show that the Klein–Gordon equation reduces to the Schrödinger equation.
2. Explain which physical degrees of freedom are removed in this limit.

Solution 1: Dispersion Relations

Inserting the plane wave into Schrödinger's equation:

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi$$

gives:

$$E = \frac{\mathrm{p}^2}{2m}.$$

This relation is not invariant under Lorentz transformations.
Relativistic energy:

$$E = \sqrt{\mathrm{p}^2 + m^2} = m + \frac{\mathrm{p}^2}{2m} + \cdots$$

Validity

$|\mathrm{p}| \ll m$

Solution 2: Klein–Gordon Equation

Operator substitutions:

$$E \to i\partial_t, \qquad \mathrm{p} \to -i\nabla$$

Lead to:

$$(\partial_t^2 - \nabla^2 + m^2)\phi = 0.$$

Covariant form:

$$(\partial_\mu \partial^\mu + m^2)\phi = 0,$$

which is Lorentz invariant.

Solution 3: Time Derivatives

Lorentz invariance requires space and time to appear symmetrically.
Second-order spatial derivatives imply second-order time derivatives.

Consequence

Two independent solutions:

$$E = \pm\sqrt{\mathrm{p}^2 + m^2}$$

corresponding to particles and antiparticles.

Solution 4: Probability Density

Using the Klein–Gordon equation and its conjugate:

$$\partial_\mu j^\mu = 0.$$

However:

$$j^0 = i(\phi^* \partial_t \phi - \phi \partial_t \phi^*)$$

is not positive definite.

Conclusion

ϕ cannot be interpreted as a single-particle wavefunction.

Solution 5: Non-relativistic Limit

Substitute:

$$\phi = e^{-imt}\psi$$

Neglecting higher-order time derivatives yields:

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi.$$

Physical meaning

Negative-energy (antiparticle) modes are removed.