Aula 16

Melhorando a solução de equações diferenciais com aproximações de 2ª e 4ª ordem

De volta à lei de Newton do arrefecimento

$$\frac{dT}{dt} = -\alpha(T - T_{ar})$$

Solução exata:

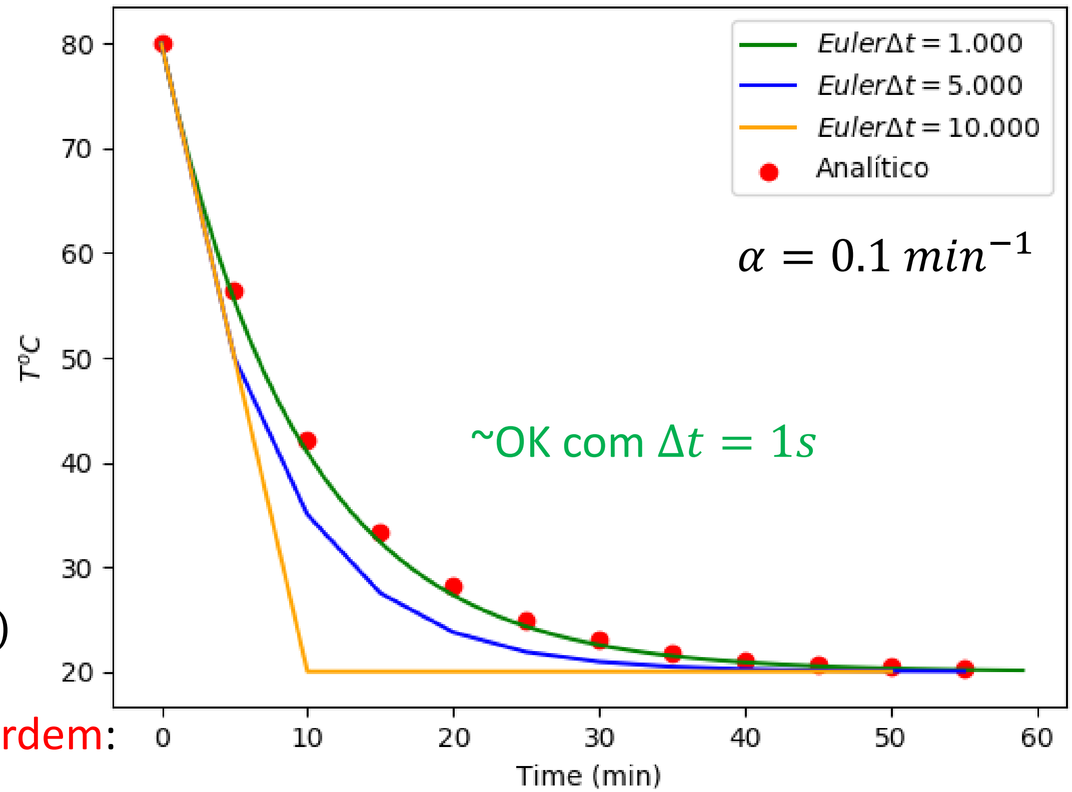
$$T = (T_0 - T_{ar})e^{-\alpha t}$$

Método de Euler:

$$T(t + \Delta t) = T(t) - \alpha \Delta t(T(t) - T_{ar})$$

Trata-se de uma aproximação de 1ª ordem:

$$\left(\frac{dT}{dt}\right)_{t=t} \approx \frac{T(t + \Delta t) - T(t)}{\Delta t} + E(\Delta t)$$



Diferenças centradas (2ª ordem)

$$\left(\frac{dy}{dt}\right)_{t=a} \approx \frac{y(a+\Delta t) - y(a-\Delta t)}{2\Delta t} + E(\Delta t^2)$$

Que se pode escrever:

$$\left(\frac{dy}{dt}\right)_{t=a+\Delta t/2} \approx \frac{y(a+\Delta t/2) - y(a-\Delta t/2)}{\Delta t} + E(\Delta t^2)$$

ou

$$T(t+\Delta t) = T(t) - \alpha\Delta t\left(T\left(t+\frac{\Delta t}{2}\right) - T_{ar}\right)$$

$$T(t+\Delta t) = T(t) - \alpha\Delta t\left(\frac{T(t+\Delta t) + T(t)}{2} - T_{ar}\right)$$

$$T(t+\Delta t)\left(1+\frac{\alpha\Delta t}{2}\right)=T(t)-\alpha\Delta t\left(\frac{T(t)}{2}-T_{ar}\right)$$

ou

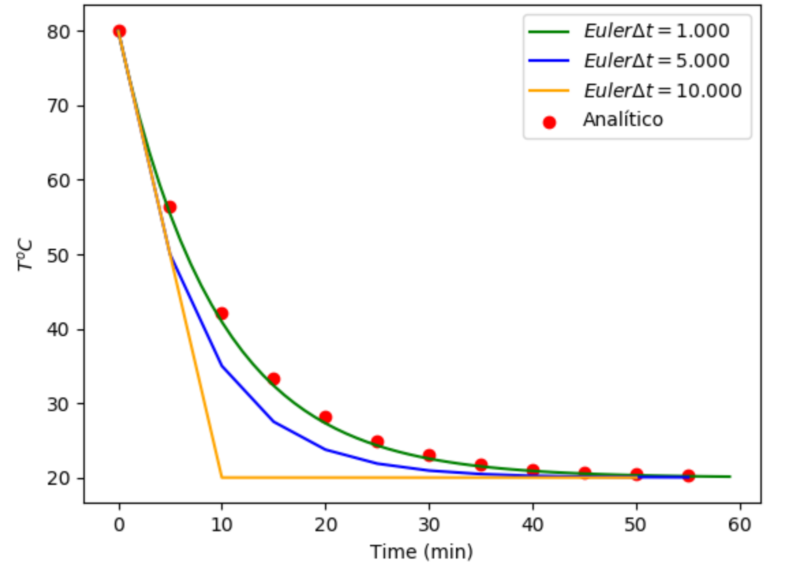
$$T(t+\Delta t)=\left[T(t)-\alpha\Delta t\left(\frac{T(t)}{2}-T_{ar}\right)\right]\left(1+\frac{\alpha\Delta t}{2}\right)^{-1}$$

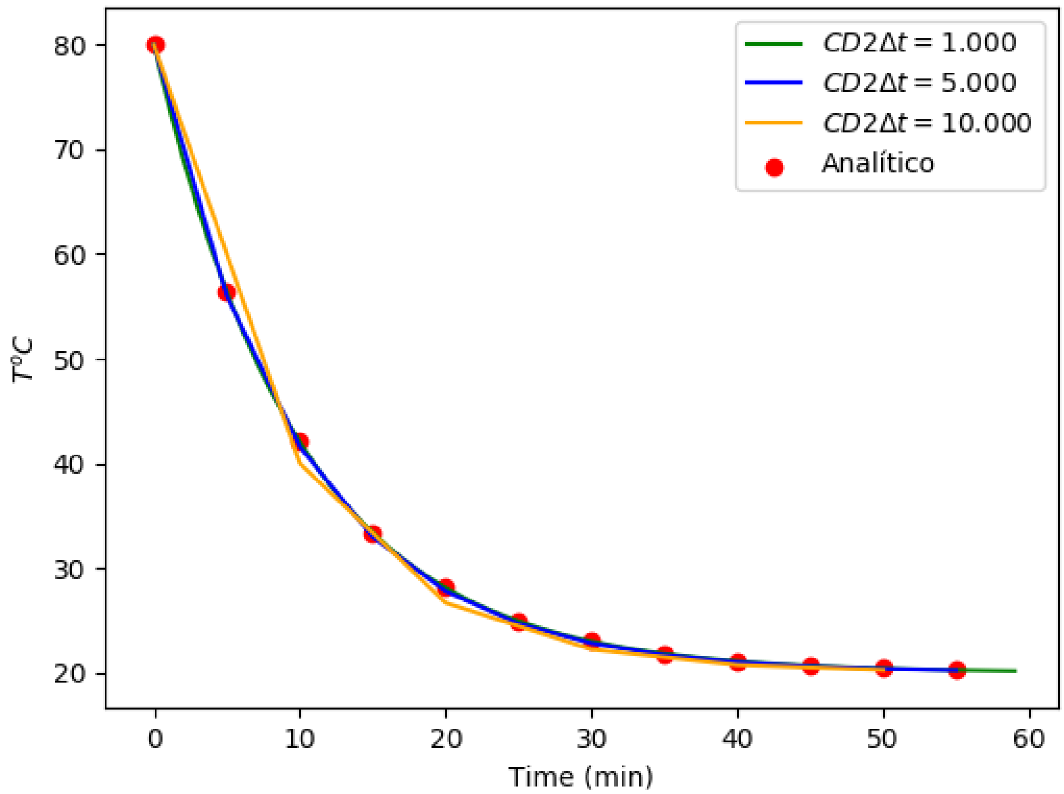
Em vez de (Euler):

$$T(t+\Delta t)=T(t)-\alpha\Delta t(T(t)-T_{ar})$$

```
kc=0
for dt in dts:
    tempo=np.arange(0.,timemax,dt)
    n=len(tempo)
    T=np.zeros(tempo.shape)
    T[0]=Tinicial
    for k in range(1,n):
        T[k]=(T[k-1]-alpha*(0.5*T[k-1]-Tar)*dt)\
              /(1+0.5*alpha*dt)
    plt.plot(tempo,T,color=cores[kc],\
              label=r'$CD2 \Delta t=%6.3f$'%(dt))
    plt.xlabel('Time (min)')
    plt.ylabel(r'$T ^{o} C$')
    kc=kc+1
```

Dif Centrada vs Euler





Perfeito com $\Delta t = 5s$

Aproximação de 4ª ordem de Runge-Kutta

$$\frac{\partial \phi}{\partial t} = f(\phi, t), \Delta t = t^{n+1} - t^n$$

$$k_1 = \Delta t\, f(\phi^n, t^n)$$

$$k_2 = \Delta t\, f\left(\phi^n + \frac{k_1}{2}, t^n + \frac{\Delta t}{2}\right)$$

$$k_3 = \Delta t\, f\left(\phi^n + \frac{k_2}{2}, t^n + \frac{\Delta t}{2}\right)$$

$$k_4 = \Delta t\, f(\phi^n + k_3, t^n + \Delta t)$$

$$\phi^{n+1} = \phi^n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$$

Runge-Kutta

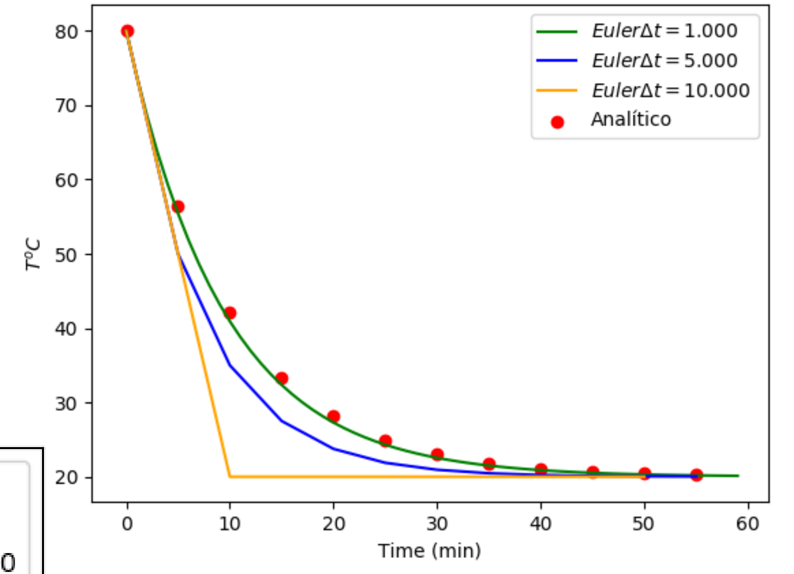
```
def RK4x(x,dxdt,dt):
    k1=dxdt(x)*dt
    k2=dxdt(x+k1/2)*dt
    k3=dxdt(x+k2/2.)*dt
    k4=dxdt(x+k3)*dt
    xP=x+1./6.*(k1+2*k2+2*k3+k4)
    return xP
def dTdt(T):
    Tar=20.
    alpha=0.1
    c=-alpha*(T-Tar)
    return c
```

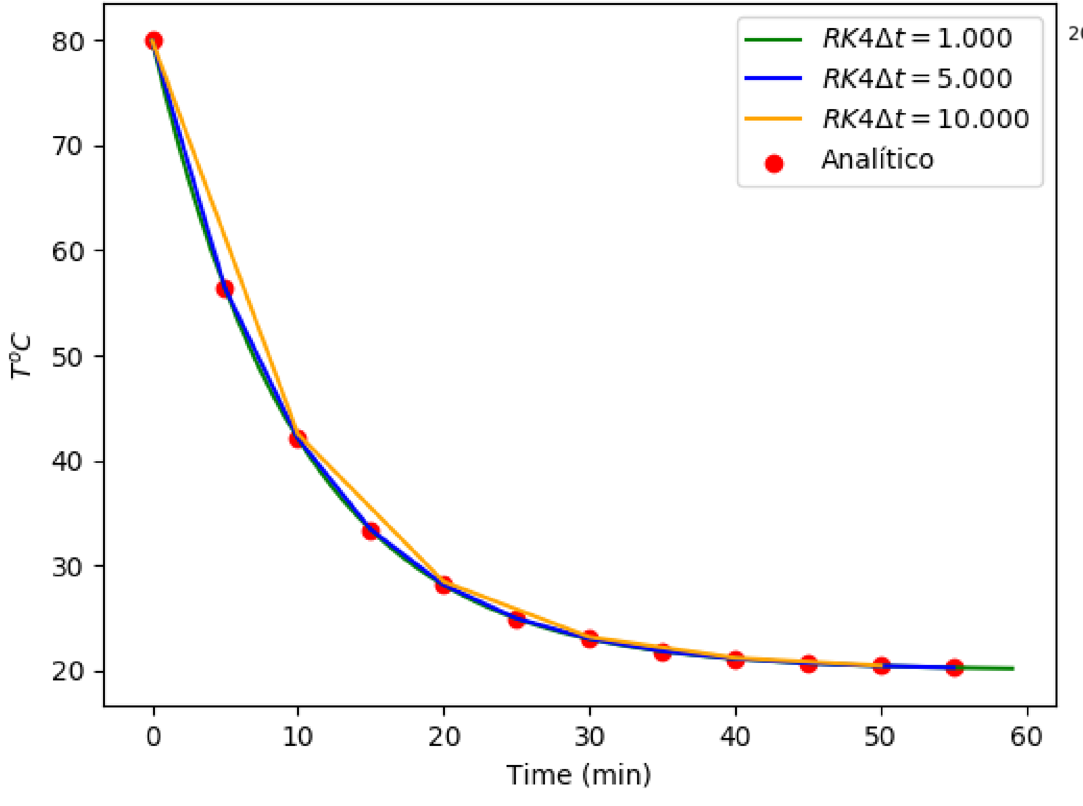
$$\frac{\partial\phi}{\partial t} = f(\phi, t)$$

Não depende de t

```
kc=0
for dt in dts:
    …
    T[0]=Tinicial
    for k in range(1,n):
        T[k]=RK4x(T[k-1],dTdt,dt)
    …
```

RK4 vs Euler





Perfeito com $\Delta t = 10s$

Equações mais complicadas

Vamos voltar ao pêndulo gravítico:

$$\frac{dv}{dt} = L\frac{d\omega}{dt} = L\frac{d^2\theta}{dt^2} = -g\sin\theta$$

São duas equações acopladas de 1ª ordem:

$$\frac{d\omega}{dt} = -\frac{g}{L}\sin\theta$$

$$\frac{d\theta}{dt} = \omega$$

Pêndulo gravítico

$$\frac{d\omega}{dt} = -\frac{g}{L}\sin\theta$$

$$\frac{d\theta}{dt} = \omega$$

O acoplamento resulta do facto de a taxa de variação de cada variável depender da outra.

Runge-Kutta[4]

$$\frac{d\omega}{dt} = -\frac{g}{L}\sin\theta \; ; \; \frac{d\theta}{dt} = \omega$$

$$\frac{\partial\omega}{\partial t} = f(\theta) \qquad \frac{\partial\theta}{\partial t} = f(\omega)$$

$$m_1 = \Delta t\, f(\theta^n) \qquad k_1 = \Delta t\, f(\omega^n)$$

$$m_2 = \Delta t\, f\left(\theta^n + \frac{k_1}{2}\right) \qquad k_2 = \Delta t\, f\left(\omega^n + \frac{m_1}{2}\right)$$

$$m_3 = \Delta t\, f\left(\theta^n + \frac{k_2}{2}\right) \qquad k_3 = \Delta t\, f\left(\omega^n + \frac{m_2}{2}\right)$$

$$m_4 = \Delta t\, f(\theta^n + k_3) \qquad k_4 = \Delta t\, f(\omega^n + m_3)$$

$$\theta^{n+1} = \theta^n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \qquad \omega^{n+1} = \omega^n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$$

RK4 para o pêndulo gravítico

```
def domegadt(theta):
    g=9.8065;L=1.
    dfdt=-g/L*np.sin(theta)
    return dfdt
def dthetadt(omega):
    dfdt=omega
    return dfdt
def rk4S(x,u,dxdt,dudt,dt):
    k1=dxdt(u)*dt
    m1=dudt(x)*dt
    k2=dxdt(u+m1/2)*dt
    m2=dudt(x+k1/2)*dt
    k3=dxdt(u+m2/2.)*dt
    m3=dudt(x+k2/2.)*dt
    k4=dxdt(u+m3)*dt
    m4=dudt(x+k3)*dt
    xP=x+1./6.*(k1+2*k2+2*k3+k4)
    uP=u+1./6.*(m1+2*m2+2*m3+m4)
    return xP,uP
```

$$\frac{d\omega}{dt} = -\frac{g}{L}\sin\theta$$

$$\frac{d\theta}{dt} = \omega$$

$$\frac{\partial x}{\partial t} = f(u) \qquad x \equiv \theta \quad u \equiv \omega$$

$$k_1 = \Delta t\, f(u^n, t^n)$$

$$k_2 = \Delta t\, f\left(u^n + \frac{m_1}{2}, t^n + \frac{\Delta t}{2}\right)$$

$$k_3 = \Delta t\, f\left(u^n + \frac{m_2}{2}, t^n + \frac{\Delta t}{2}\right)$$

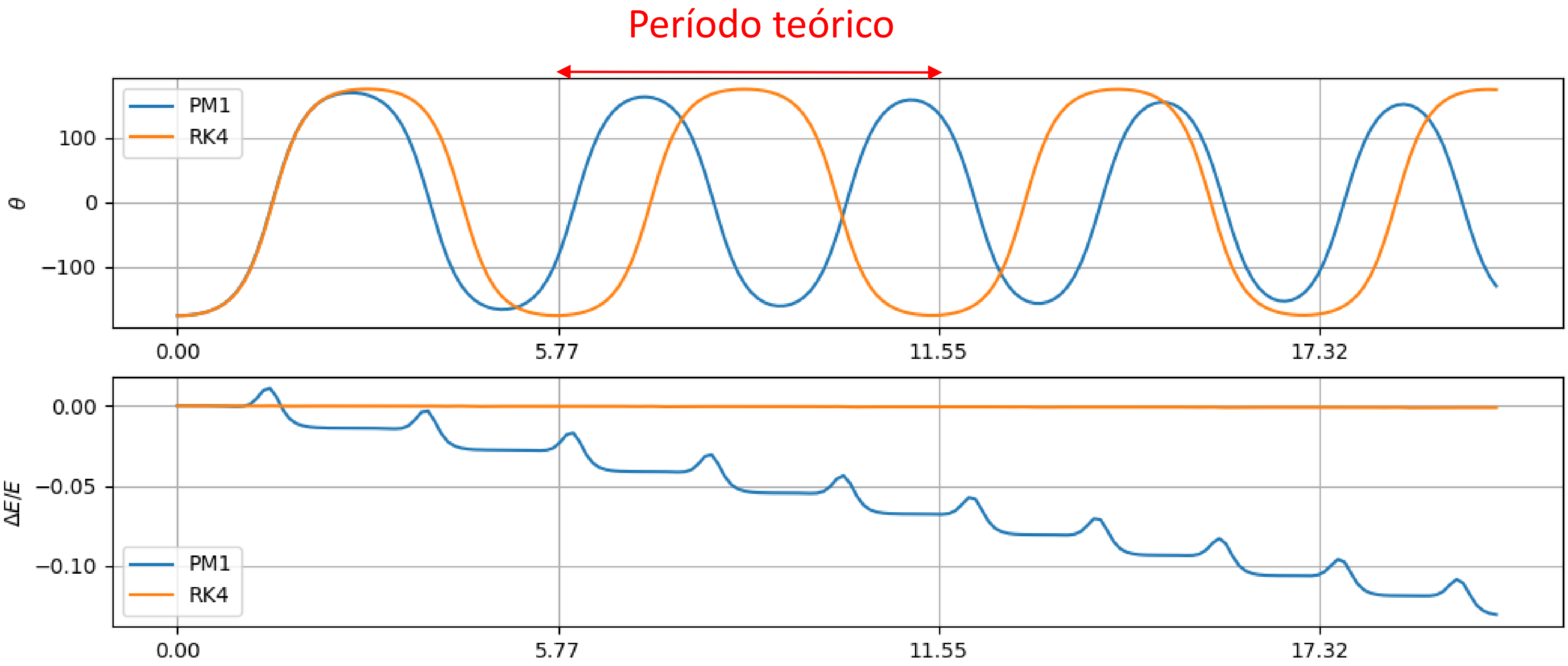
$$k_4 = \Delta t\, f(u^n + m_3, t^n + \Delta t)$$

$$x^{n+1} = x^n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$$

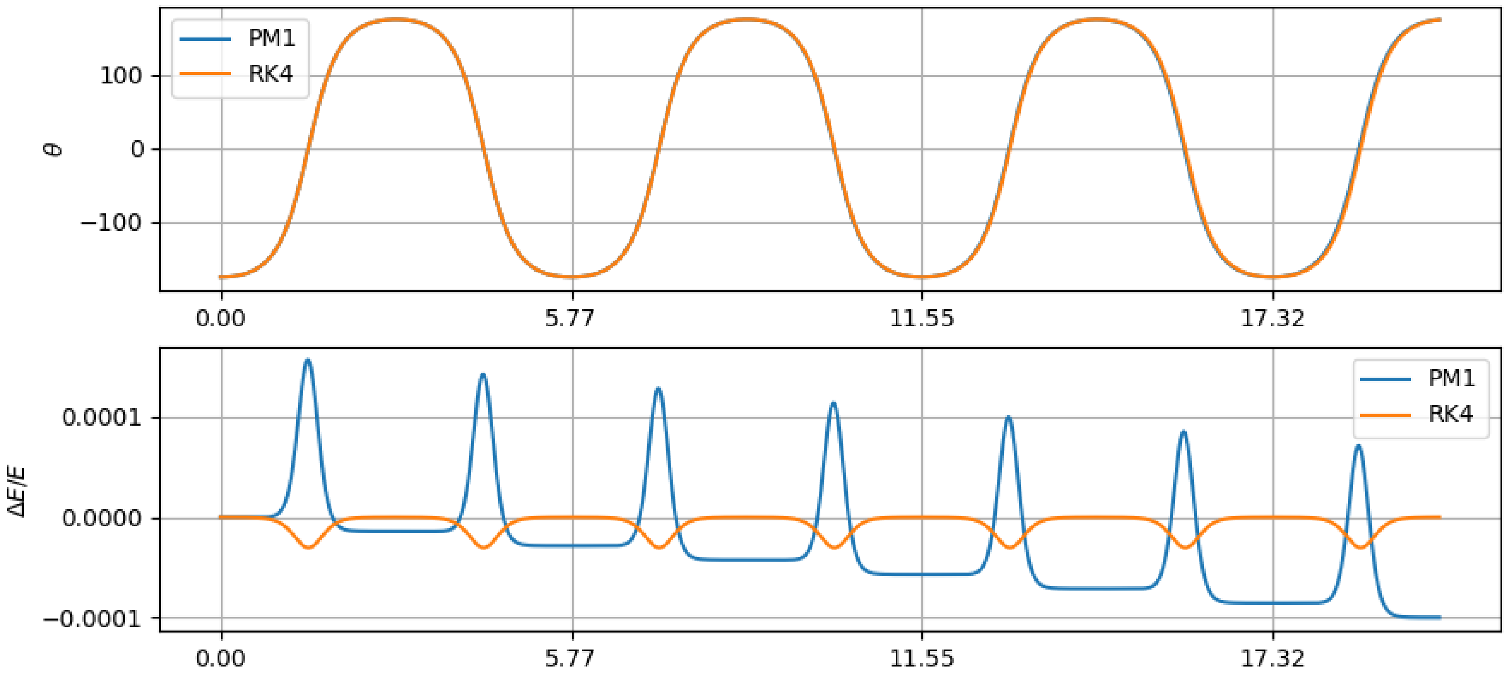
Pêndulo

```
dt=0.1 #passo de tempo
theta0=-175./180.*np.pi #amplitude em radianos
omega0=0. #velocidade angular inicial
T=2*np.pi*np.sqrt(L/g)
t=np.arange(0.,10*T,dt) #vetor de tempos (10T)
n=len(t)
theta=np.zeros(t.shape); omega=np.copy(theta)
theta[0]=theta0; omega[0]=omega0
for kt in range(1,n):
    omega[kt],theta[kt]=\
    rk4S(omega[kt-1],theta[kt-1],\
    domegadt,dthetadt,dt)
```

$\Delta t = 0.1s$


Período teórico
PM1
RK4
100
0
−100
θ
0.00
5.77
11.55
17.32
0.00
−0.05
−0.10
ΔE/E
PM1
RK4
0.00
5.77
11.55
17.32

$\Delta t = 0.01s$



Como esperado converge quando $\Delta t \longrightarrow 0$, mas RK4 mantém-se melhor (energia)