Chapter 2
The Integral

Le vrai est simple et clair; et quand notre manière d'y arriver est embarrassée et obscure, on peut dire qu'elle mène au vrai et n'est pas vraie.

Fontenelle

2.1 The Cauchy Integral

The *Lebesgue integral* is a positive linear functional satisfying the property of monotone convergence. It extends the *Cauchy integral*.

Definition 2.1.1. Let Ω be an open subset of $\mathbb{R}^N$. We define

$$\mathcal{C}(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is continuous}\},$$

$$\mathcal{K}(\Omega) = \{u \in \mathcal{C}(\mathbb{R}^N) : \text{supp } u \text{ is a compact subset of } \Omega\}.$$

The support of u, denoted by spt u, is the closure of the set of points at which u is different from 0.

Let $u \in \mathcal{K}(\mathbb{R}^N)$. By definition, there is $R > 1$ such that

$$\text{spt } u \subset \{x \in \mathbb{R}^N : |x|_\infty \leq R - 1\}.$$

Let us define the *Riemann sums* of u:

$$S_j = 2^{-jN} \sum_{k \in \mathbb{Z}^N} u(k/2^j).$$

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The factor 2^{-jN} is the volume of the cube with side 2^{-j} in $\mathbb{R}^N$. Let $C = [0,1]^N$ and let us define the *Darboux sums* of u:

$$A_j = 2^{-jN} \sum_{k \in \mathbb{Z}^N} \min\{u(x) : 2^j x - k \in C\}, \quad B_j = 2^{-jN} \sum_{k \in \mathbb{Z}^N} \max\{u(x) : 2^j x - k \in C\}.$$

Let $\varepsilon > 0$. By uniform continuity, there is j such that $\omega_u(1/2^j) \leq \varepsilon$. Observe that

$$B_j - A_j \leq (2R)^N \varepsilon, A_{j-1} \leq A_j \leq S_j \leq B_j \leq B_{j-1}.$$

The *Cauchy integral* of u is defined by

$$\int_{\mathbb{R}^N} u(x)dx = \lim_{j\to\infty} S_j = \lim_{j\to\infty} A_j = \lim_{j\to\infty} B_j.$$

Theorem 2.1.2. *The space $\mathcal{K}(\mathbb{R}^N)$ and the Cauchy integral*

$$\mathcal{K}(\mathbb{R}^N) \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^N} u\,dx$$

are such that

(a) for every $u \in \mathcal{K}(\mathbb{R}^N)$, $|u| \in \mathcal{K}(\mathbb{R}^N)$;
(b) for every $u, v \in \mathcal{K}(\mathbb{R}^N)$ and every $\alpha, \beta \in \mathbb{R}$,

$$\int_{\mathbb{R}^N} \alpha u + \beta v\,dx = \alpha \int_{\mathbb{R}^N} u\,dx + \beta \int_{\mathbb{R}^N} v\,dx;$$

(c) for every $u \in \mathcal{K}(\mathbb{R}^N)$ such that $u \geq 0$, $\int_{\mathbb{R}^N} u\,dx \geq 0$;
(d) for every sequence $(u_n) \subset \mathcal{K}(\mathbb{R}^N)$ such that $u_n \downarrow 0$, $\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n\,dx = 0$.

Proof. Properties (a)–(c) are clear. Property (d) follows from Dini's theorem. By definition, there is $R > 1$ such that

$$\operatorname{spt} u_0 \subset K = \{x \in \mathbb{R}^N : |x|_\infty \leq R - 1\}.$$

By Dini's theorem, (u_n) converges uniformly to 0 on K. Hence

$$0 \leq \int_{\mathbb{R}^N} u_n dx \leq (2R)^N \max_{x\in K} u_n(x) \to 0, \quad n \to \infty. \qquad \square$$

The above properties define an elementary integral. They suffice for constructing the *Lebesgue integral*.

The (concrete) Lebesgue integral is the smallest extension of the Cauchy integral satisfying the property of *monotone convergence*,

(e) if (u_n) is an increasing sequence of integrable functions such that

$$\sup_n \int_{\mathbb{R}^N} u_n dx < \infty,$$

then $u(x) = \lim_{n\to\infty} u_n(x)$ is integrable and

$$\int_{\mathbb{R}^N} u\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} u_n\, dx,$$

and *linearity*,
(f) if u and v are integrable functions and if α and β are real numbers, then

$$\int_{\mathbb{R}^N} \alpha u + \beta v\, dx = \alpha \int u\, dx + \beta \int v\, dx.$$

Let us sketch the construction of the (concrete) Lebesgue integral.

By definition, the function u belongs to $\mathcal{L}^+(\mathbb{R}^N, dx)$ if there exists an increasing sequence (u_n) of functions of $\mathcal{K}(\mathbb{R}^N)$ such that $u_n \uparrow u$ and $\sup_n \int_{\mathbb{R}^N} u_n\, dx < \infty$. The integral, defined by the formula

$$\int_{\mathbb{R}^N} u\, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} u_n dx,$$

satisfies property (e). We shall prove that the integral depends only on u.

Let $f, g \in \mathcal{L}^+(\mathbb{R}^N, dx)$. The difference $f(x) - g(x)$ is well defined except if $f(x) = g(x) = +\infty$. A subset S of $\mathbb{R}^N$ is *negligible* if there exists $h \in \mathcal{L}^+(\mathbb{R}^N, dx)$ such that for every $x \in S$, $h(x) = +\infty$.

By definition a function u belongs to $\mathcal{L}^1(\mathbb{R}^N, dx)$ if there exists $f, g \in \mathcal{L}^+(\mathbb{R}^N, dx)$ such that $u = f - g$ except on a negligible subset of $\mathbb{R}^N$. The integral defined by

$$\int_{\mathbb{R}^N} u\, dx = \int_{\mathbb{R}^N} f\, dx - \int_{\mathbb{R}^N} g\, dx$$

satisfies properties (e) and (f). Again we shall prove that the integral depends only on u.

The Lebesgue integral will be constructed in an abstract framework, the *elementary integral*, generalizing the Cauchy integral.

Example (Limit of integrals). It is not always permitted to permute limit and integral. Let us define, on $[0, 1]$, $u_n(x) = 2nx(1 - x^2)^{n-1}$. Since for every $x \in]0, 1[$,

$$\lim_{n\to\infty} \frac{u_{n+1}(x)}{u_n(x)} = (1 - x^2) < 1,$$

u_n converges simply to 0 on $[0, 1]$. But

$$0 = \int_0^1 \lim_{n\to\infty} u_n(x)dx < \lim_{n\to\infty} \int_0^1 u_n(x)dx = 1.$$

2.2 The Lebesgue Integral

Les inégalités peuvent s'intégrer.

Paul Lévy

Elementary integrals were defined by Daniell in 1918.

Definition 2.2.1. An elementary integral on the set Ω is defined by a vector space $\mathcal{L} = \mathcal{L}(\Omega, \mu)$ of functions from Ω to $\mathbb{R}$ and by a functional

$$\mu : \mathcal{L} \to \mathbb{R} : u \mapsto \int_\Omega u \, d\mu$$

such that

($\mathcal{J}_1$) for every $u \in \mathcal{L}$, $|u| \in \mathcal{L}$;
($\mathcal{J}_2$) for every $u, v \in \mathcal{L}$ and every $\alpha, \beta \in \mathbb{R}$,

$$\int_\Omega \alpha u + \beta v \, d\mu = \alpha \int_\Omega u \, d\mu + \beta \int_\Omega v \, d\mu;$$

($\mathcal{J}_3$) for every $u \in \mathcal{L}$ such that $u \geq 0$, $\int_\Omega u \, d\mu \geq 0$;

($\mathcal{J}_4$) for every sequence $(u_n) \subset \mathcal{L}$ such that $u_n \downarrow 0$, $\lim_{n\to\infty} \int_\Omega u_n \, d\mu = 0$.

Proposition 2.2.2. *Let $u, v \in \mathcal{L}$. Then $u^+, u^-, \max(u, v), \min(u, v) \in \mathcal{L}$.*

Proof. Let us recall that $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$,

$$\max(u, v) = \frac{1}{2}(u + v) + \frac{1}{2}|u - v|, \quad \min(u, v) = \frac{1}{2}(u + v) - \frac{1}{2}|u - v|. \qquad \square$$

Proposition 2.2.3. *Let $u, v \in \mathcal{L}$ be such that $u \leq v$. Then $\int_\Omega u \, d\mu \leq \int_\Omega v \, d\mu$.*

Proof. We deduce from $(\mathcal{J}_2)$ and $(\mathcal{J}_3)$ that

$$0 \leq \int_\Omega v - u \, d\mu = \int_\Omega v \, d\mu - \int_\Omega u \, d\mu. \qquad \square$$

Definition 2.2.4. A fundamental sequence is an increasing sequence $(u_n) \subset \mathcal{L}$ such that

$$\lim_{n\to\infty} \int_\Omega u_n d\mu = \sup_n \int_\Omega u_n d\mu < \infty.$$

Definition 2.2.5. A subset S of Ω is negligible (with respect to μ) if there is a fundamental sequence (u_n) such that for every $x \in S$, $\lim_{n\to\infty} u_n(x) = +\infty$. A property is true almost everywhere if the set of points of Ω where it is false is negligible.

Let us justify the definition of a negligible set.

Proposition 2.2.6. *Let (u_n) be a decreasing sequence of functions of $\mathcal{L}$ such that everywhere $u_n \geq 0$ and almost everywhere, $\lim_{n\to\infty} u_n(x) = 0$. Then $\lim_{n\to\infty} \int_\Omega u_n d\mu = 0$.*

Proof. Let $\varepsilon > 0$. By assumption, there is a fundamental sequence (v_n) such that if $\lim_{n\to\infty} u_n(x) > 0$, then $\lim_{n\to\infty} v_n(x) = +\infty$. We replace v_n by v_n^+, and we multiply by a strictly positive constant such that

$$v_n \geq 0, \qquad \int_\Omega v_n d\mu \leq \varepsilon.$$

We define $w_n = (u_n - v_n)^+$. Then $w_n \downarrow 0$, and we deduce from axiom $(\mathcal{J}_4)$ that

$$\begin{aligned} 0 \leq \lim \int_\Omega u_n d\mu \leq \lim \int_\Omega w_n + v_n d\mu &= \lim \int_\Omega w_n d\mu + \lim \int_\Omega v_n d\mu \\ &= \lim \int_\Omega v_n d\mu \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. $\square$

Proposition 2.2.7. *Let (u_n) and (v_n) be fundamental sequences such that almost everywhere,*

$$u(x) = \lim_{n\to\infty} u_n(x) \leq \lim_{n\to\infty} v_n(x) = v(x).$$

Then

$$\lim_{n\to\infty} \int_\Omega u_n d\mu \leq \lim_{n\to\infty} \int_\Omega v_n d\mu.$$

Proof. We choose k and we define $w_n = (u_k - v_n)^+$. Then $(w_n) \subset \mathcal{L}$ is a decreasing sequence of positive functions such that almost everywhere,

$$\lim w_n(x) = (u_k(x) - v(x))^+ \leq (u(x) - v(x))^+ = 0.$$

We deduce from the preceding proposition that

$$\int_\Omega u_k d\mu \leq \lim \int_\Omega w_n + v_n \, d\mu = \lim \int_\Omega w_n d\mu + \lim \int_\Omega v_n d\mu = \lim \int_\Omega v_n d\mu.$$

Since k is arbitrary, the proof is complete. □

Definition 2.2.8. A function $u : \Omega \to]-\infty, +\infty]$ belongs to $\mathcal{L}^+ = \mathcal{L}^+(\Omega, \mu)$ if there exists a fundamental sequence (u_n) such that $u_n \uparrow u$. The integral (with respect to μ) of u is defined by

$$\int_\Omega u \, d\mu = \lim_{n\to\infty} \int_\Omega u_n d\mu.$$

By the preceding proposition, the integral of u is well defined.

Proposition 2.2.9. *Let $u, v \in \mathcal{L}^+$ and $\alpha, \beta \geq 0$. Then*

(a) $\max(u, v), \min(u, v), u^+ \in \mathcal{L}^+$*;*
(b) $\alpha u + \beta v \in \mathcal{L}^+$ *and* $\int_\Omega \alpha u + \beta v \, d\mu = \alpha \int_\Omega u \, d\mu + \beta \int_\Omega v \, d\mu$*;*
(c) if $u \leq v$ *almost everywhere, then* $\int_\Omega u \, d\mu \leq \int_\Omega v \, d\mu$.

Proof. Proposition 2.2.7 is equivalent to (c). □

Proposition 2.2.10 (Monotone convergence in $\mathcal{L}^+$). *Let $(u_n) \subset \mathcal{L}^+$ be everywhere (or almost everywhere) increasing and such that*

$$c = \sup_n \int_\Omega u_n d\mu < \infty.$$

Then (u_n) converges everywhere (or almost everywhere) to $u \in \mathcal{L}^+$ and

$$\int_\Omega u \, d\mu = \lim_{n\to\infty} \int_\Omega u_n d\mu.$$

Proof. We consider almost everywhere convergence. For every k, there is a fundamental sequence $(u_{k,n})$ such that $u_{k,n} \uparrow u_k$.

The sequence $v_n = \max(u_{1,n}, \dots, u_{n,n})$ is increasing, and almost everywhere,

$$v_n \leq \max(u_1, \dots, u_n) = u_n.$$

Since

$$\int_\Omega v_n d\mu \leq \int_\Omega u_n d\mu \leq c,$$

the sequence $(v_n) \subset \mathcal{L}$ is fundamental. By definition, $v_n \uparrow u$, $u \in \mathcal{L}^+$, and

$$\int_\Omega u\,d\mu = \lim_{n\to\infty} \int_\Omega v_n d\mu.$$

For $k \le n$, we have almost everywhere that

$$u_{k,n} \le v_n \le u_n.$$

Hence we obtain, almost everywhere, that $u_k \le u \le \lim\limits_{n\to\infty} u_n$ and

$$\int_\Omega u_k d\mu \le \int_\Omega u\,d\mu \le \lim_{n\to\infty} \int_\Omega u_n d\mu.$$

It is easy to conclude the proof. □

Corollary 2.2.11. *Every countable union of negligible sets is negligible.*

Proof. Let (S_k) be a sequence of negligible sets. For every k, there exists $v_k \in \mathcal{L}^+$ such that for every $x \in S_k$, $v_k(x) = +\infty$. We replace v_k by v_k^+, and we multiply by a strictly positive constant such that

$$v_k \ge 0, \qquad \int_\Omega v_k d\mu \le \frac{1}{2^k}.$$

The sequence $u_n = \sum\limits_{k=1}^{n} v_k$ is increasing and

$$\int_\Omega u_n d\mu \le \sum_{k=1}^{n} \frac{1}{2^k} \le 1.$$

Hence $u_n \uparrow u$ and $u \in \mathcal{L}^+$. Since for every $x \in \bigcup\limits_{k=1}^{\infty} S_k$, $u(x) = +\infty$, the set $\bigcup\limits_{k=1}^{\infty} S_k$ is negligible. □

By definition, functions of $\mathcal{L}^+$ are finite almost everywhere. Hence the difference of two functions of $\mathcal{L}^+$ is well defined almost everywhere. Assume that $f, g, v, w \in \mathcal{L}^+$ and that $f - g = v - w$ almost everywhere. Then $f + w = v + g$ almost everywhere and

$$\int_\Omega f\,d\mu + \int_\Omega w\,dv\mu = \int_\Omega f + w\,d\mu = \int_\Omega v + g\,d\mu = \int_\Omega v\,d\mu + \int_\Omega g\,d\mu,$$

so that

$$\int_\Omega f\,d\mu - \int_\Omega g\,d\mu = \int_\Omega v\,d\mu - \int_\Omega w\,d\mu.$$

Definition 2.2.12. A real function u almost everywhere defined on Ω belongs to $\mathcal{L}^1 = \mathcal{L}^1(\Omega, \mu)$ if there exist $f, g \in \mathcal{L}^+$ such that $u = f - g$ almost everywhere. The integral (with respect to μ) of u is defined by

$$\int_\Omega u\, d\mu = \int_\Omega f\, d\mu - \int_\Omega g\, d\mu.$$

By the preceding computation, the integral is well defined.

Proposition 2.2.13. *(a) If $u \in \mathcal{L}^1$, then $|u| \in \mathcal{L}^1$.*
(b) If $u, v \in \mathcal{L}^1$ and if $\alpha, \beta \in \mathbb{R}$, then $\alpha u + \beta v \in \mathcal{L}^1$ and
$$\int_\Omega \alpha u + \beta v\, d\mu = \alpha \int_\Omega u\, d\mu + \beta \int_\Omega v\, d\mu.$$
(c) If $u \in \mathcal{L}^1$ and if $u \geq 0$ almost everywhere, then $\int_\Omega u\, d\mu \geq 0$.

Proof. Observe that
$$|f - g| = \max(f, g) - \min(f, g).$$
□

Lemma 2.2.14. *Let $u \in \mathcal{L}^1$ and $\varepsilon > 0$. Then there exist $v, w \in \mathcal{L}^+$ such that $u = v - w$ almost everywhere, $w \geq 0$, and $\int_\Omega w\, d\mu \leq \varepsilon$.*

Proof. By definition, there exist $f, g \in \mathcal{L}^+$ such that $u = f - g$ almost everywhere. Let (g_n) be a fundamental sequence such that $g_n \uparrow g$. Since

$$\int_\Omega g\, d\mu = \lim_{n\to\infty} \int_\Omega g_n d\mu,$$

there exists n such that $\int_\Omega g - g_n\, d\mu \leq \varepsilon$. We choose $w = g - g_n \geq 0$ and $v = f - g_n$.
□

We extend the property of monotone convergence to $\mathcal{L}^1$.

Theorem 2.2.15 (Levi's monotone convergence theorem). *Let $(u_n) \subset \mathcal{L}^1$ be an almost everywhere increasing sequence such that*

$$c = \sup_n \int_\Omega u_n d\mu < \infty.$$

Then $\lim_{n\to\infty} u_n \in \mathcal{L}^1$ and

$$\int_\Omega \lim_{n\to\infty} u_n d\mu = \lim_{n\to\infty} \int_\Omega u_n d\mu.$$

Proof. After replacing u_n by $u_n - u_0$, we can assume that $u_0 = 0$. By the preceding lemma, for every $k \geq 1$, there exist $v_k, w_k \in \mathcal{L}^+$ such that $w_k \geq 0$, $\int_\Omega w_k d\mu \leq 1/2^k$, and, almost everywhere,
$$u_k - u_{k-1} = v_k - w_k.$$

Since (u_k) is almost everywhere increasing, $v_k \geq 0$ almost everywhere.

We define

$$f_n = \sum_{k=1}^{n} v_k, \quad g_n = \sum_{k=1}^{n} w_k.$$

The sequences (f_n) and (g_n) are almost everywhere increasing, and

$$\int_\Omega g_n d\mu = \sum_{k=1}^{n} \int_\Omega w_k d\mu \leq \sum_{k=1}^{n} \frac{1}{2^k} \leq 1, \quad \int_\Omega f_n d\mu = \int_\Omega u_n + g_n d\mu \leq c + 1.$$

Proposition 2.2.10 implies that almost everywhere,

$$\lim_{n\to\infty} f_n = f \in \mathcal{L}^+, \lim_{n\to\infty} g_n = g \in \mathcal{L}^+$$

and

$$\int_\Omega f\, d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu, \int_\Omega g\, d\mu = \lim_{n\to\infty} \int_\Omega g\, d\mu.$$

We deduce from Corollary 2.2.11 that almost everywhere,

$$f - g = \lim_{n\to\infty} (f_n - g_n) = \lim_{n\to\infty} u_n.$$

Hence $\lim\limits_{n\to\infty} u_n \in \mathcal{L}^1$ and

$$\int_\Omega \lim_{n\to\infty} u_n d\mu = \int_\Omega f\, d\mu - \int_\Omega g\, d\mu = \lim_{n\to\infty} \int_\Omega f_n - g_n d\mu = \lim_{n\to\infty} \int_\Omega u_n d\mu. \qquad \square$$

Theorem 2.2.16 (Fatou's lemma). *Let $(u_n) \subset \mathcal{L}^1$ and $f \in \mathcal{L}^1$ be such that*

(a) $\sup\limits_n \int_\Omega u_n d\mu < \infty$;

(b) for every n, $f \leq u_n$ almost everywhere.

Then $\underline{\lim}_{n\to\infty} u_n \in \mathcal{L}^1$ and

$$\int_\Omega \underline{\lim}_{n\to\infty} u_n d\mu \leq \underline{\lim}_{n\to\infty} \int_\Omega u_n d\mu.$$

Proof. We choose k, and we define, for $m \geq k$,

$$u_{k,m} = \min(u_k, \ldots, u_m).$$

The sequence $(u_{k,m})$ decreases to $v_k = \inf\limits_{n\geq k} u_n$, and

$$\int_\Omega f\,d\mu \le \int_\Omega u_{k,m} d\mu.$$

The preceding theorem, applied to $(-u_{k,m})$, implies that $v_k \in \mathcal{L}^1$ and

$$\int_\Omega v_k d\mu = \lim_{m\to\infty} \int_\Omega u_{k,m} d\mu \le \lim_{m\to\infty} \min_{k\le n\le m} \int_\Omega u_n d\mu = \inf_{n\ge k} \int_\Omega u_n d\mu.$$

The sequence (v_k) increases to $\varliminf_{n\to\infty} u_n$ and

$$\int_\Omega v_k d\mu \le \sup_n \int_\Omega u_n d\mu < \infty.$$

It follows from the preceding theorem that $\varliminf_{n\to\infty} u_n \in \mathcal{L}^1$ and

$$\int_\Omega \varliminf_{n\to\infty} u_n d\mu = \lim_{k\to\infty} \int_\Omega v_k d\mu \le \lim_{k\to\infty} \inf_{n\ge k} \int_\Omega u_n d\mu = \varliminf_{n\to\infty} \int_\Omega u_n d\mu. \qquad \square$$

Theorem 2.2.17 (Lebesgue's dominated convergence theorem). *Let $(u_n) \subset \mathcal{L}^1$ and $f \in \mathcal{L}^1$ be such that*

(a) u_n converges almost everywhere;
(b) for every n, $|u_n| \le f$ almost everywhere.

Then $\lim_{n\to\infty} u_n \in \mathcal{L}^1$ and

$$\int_\Omega \lim_{n\to\infty} u_n d\mu = \lim_{n\to\infty} \int_\Omega u_n d\mu.$$

Proof. Fatou's lemma implies that $u = \lim_{n\to\infty} u_n \in \mathcal{L}^1$ and

$$2\int_\Omega f\,d\mu \le \varliminf_{n\to\infty} \int 2f - |u_n - u| d\mu = 2\int_\Omega f\,d\mu - \varlimsup_{n\to\infty} \int_\Omega |u_n - u| d\mu.$$

Hence

$$\lim_{n\to\infty} \left|\int_\Omega u_n - u\,d\mu\right| \le \lim_{n\to\infty} \int_\Omega |u_n - u| d\mu = 0. \qquad \square$$

Theorem 2.2.18 (Comparison theorem). *Let $(u_n) \subset \mathcal{L}^1$ and $f \in \mathcal{L}^1$ be such that*

(a) u_n converges almost everywhere to u;
(b) $|u| \le f$ almost everywhere.

Then $u \in \mathcal{L}^1$.

Proof. We define
$$v_n = \max(\min(u_n, f), -f).$$
The sequence $(v_n) \subset \mathcal{L}^1$ is such that

(a) v_n converges almost everywhere to u;
(b) for every n, $|v_n| \leq f$ almost everywhere.

The preceding theorem implies that $u = \lim_{n\to\infty} v_n \in \mathcal{L}^1$. □

Definition 2.2.19. A real function u defined almost everywhere on Ω is measurable (with respect to μ) if there exists a sequence $(u_n) \subset \mathcal{L}$ such that $u_n \to u$ almost everywhere. We denote the space of measurable functions (with respect to μ) on Ω by $\mathcal{M} = \mathcal{M}(\Omega, \mu)$.

Proposition 2.2.20. *(a)* $\mathcal{L} \subset \mathcal{L}^+ \subset \mathcal{L}^1 \subset \mathcal{M}$.
(b) If $u \in \mathcal{M}$, then $|u| \in \mathcal{M}$.
(c) If $u, v \in \mathcal{M}$ and if $\alpha, \beta \in \mathbb{R}$, then $\alpha u + \beta v \in \mathcal{M}$.
(d) If $u \in \mathcal{M}$ and if, almost everywhere, $|u| \leq f \in \mathcal{L}^1$, then $u \in \mathcal{L}^1$.

Proof. Property (d) follows from the comparison theorem. □

Notation. Let $u \in \mathcal{M}$ be such that $u \geq 0$ and $u \notin \mathcal{L}^1$. We write $\int_\Omega u\, d\mu = +\infty$. Hence the integral of a measurable nonnegative function always exists.

Measurability is preserved by almost everywhere convergence.

Lemma 2.2.21. *Let $(u_n) \subset \mathcal{L}^+$ be an almost everywhere increasing sequence converging to an almost everywhere finite function u. Then $u \in \mathcal{M}$.*

Proof. For every k, there exists a fundamental sequence $(u_{k,n})$ such that $u_{k,n} \uparrow u_k$. The increasing sequence $v_n = \max(u_{1,n}, \ldots, u_{n,n})$ converges to v, and almost everywhere,
$$v_n \leq \max(u_1, \ldots, u_n) = u_n.$$
For $k \leq n$, we have, almost everywhere, $u_{k,n} \leq v_n \leq u_n$. Hence almost everywhere, $u_k \leq v \leq u$. It is now easy to conclude the proof. □

Lemma 2.2.22. *Let $(u_n) \subset \mathcal{L}^1$ be an increasing sequence converging to an almost everywhere finite function u. Then $u \in \mathcal{M}$.*

Proof. By Lemma 2.2.14, for every $n \geq 1$ there exist $v_n, w_n \in \mathcal{L}^+$ such that almost everywhere,
$$0 \leq u_n - u_{n-1} = v_n - w_n, w_n \geq 0, \int_\Omega w_n d\mu \leq 1/2^n.$$
Proposition 2.2.10 and the preceding lemma imply that

$$\sum_{n=1}^{\infty} w_n = w \in \mathcal{L}^+, \quad \sum_{n=1}^{\infty} v_n = v \in \mathcal{M}.$$

Since almost everywhere, $u = v - w + u_0$, $u \in \mathcal{M}$. □

Lemma 2.2.23. *Let $(u_n) \subset \mathcal{M}$ be an increasing sequence converging to an almost everywhere finite function u. Then $u \in \mathcal{M}$.*

Proof. Replacing u_n by $u_n - u_0$, we can assume that $u_n \geq 0$. For every k, there exists a sequence $(u_{k,m}) \subset \mathcal{L}$ converging almost everywhere to u_k. We can assume that $u_{k,m} \geq 0$. By Levi's theorem,

$$v_{k,n} = \inf_{m \geq n} u_{k,m} \in \mathcal{L}^1.$$

For every k, $(v_{k,n})$ is increasing and converges almost everywhere to u_k. We define

$$v_n = \max(v_{1,n}, \ldots, v_{n,n}) \in \mathcal{L}^1.$$

The sequence (v_n) is increasing and converges almost everywhere to u. By the preceding lemma, $u \in \mathcal{M}$. □

Theorem 2.2.24. *Let $(u_n) \subset \mathcal{M}$ be a sequence converging almost everywhere to a finite limit. Then $u \in \mathcal{M}$.*

Proof. By the preceding lemma,

$$v_k = \sup_{n \geq k} u_n \in \mathcal{M} \text{ and } \lim u_n = -\sup_k(-v_k) \in \mathcal{M}.$$ □

The class of measurable functions is the smallest class containing $\mathcal{L}$ that is closed under almost everywhere convergence.

Definition 2.2.25. A subset A of Ω is measurable (with respect to μ) if the characteristic function of A is measurable. The measure of A is defined by $\mu(A) = \int_\Omega \chi_A d\mu$.

Proposition 2.2.26. *Let A and B be measurable sets and let (A_n) be a sequence of measurable sets. Then $A \setminus B$, $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$ are measurable, and*

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

If, moreover, for every n, $A_n \subset A_{n+1}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

If, moreover, $\mu(A_1) < \infty$, and for every n, $A_{n+1} \subset A_n$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Proof. Observe that

$$\chi_{A\cup B} + \chi_{A\cap B} = \max(\chi_A, \chi_B) + \min(\chi_A, \chi_B) = \chi_A + \chi_B,$$

$$\chi_{A\setminus B} = \chi_A - \min(\chi_A, \chi_B),$$

$$\chi_{\bigcup_{n=1}^{\infty} A_n} = \lim_{n\to\infty} \max(\chi_{A_1}, \ldots, \chi_{A_n}),$$

$$\chi_{\bigcap_{n=1}^{\infty} A_n} = \lim_{n\to\infty} \min(\chi_{A_1}, \ldots, \chi_{A_n}).$$

The proposition follows then from the preceding theorem and Levi's theorem. □

Proposition 2.2.27. *A subset of Ω is negligible if and only if it is measurable and its measure is equal to* 0.

Proof. Let $A \subset \Omega$ be a negligible set. Since $\chi_A = 0$ almost everywhere, we have by definition that $\chi_A \in \mathcal{L}^1$ and $\mu(A) = \int_\Omega \chi_A d\mu = 0$.

Let A be a measurable set such that $\mu(A) = 0$. For every n, $\int_\Omega n\chi_A d\mu = 0$. By Levi's theorem, $u = \lim_{n\to\infty} n\chi_A \in \mathcal{L}^1$. Since u is finite almost everywhere and $u(x) = +\infty$ on A, the set A is negligible. □

The hypothesis in the following definition will be used to prove that the set $\{u > t\}$ is measurable when the function $u \geq 0$ is measurable.

Definition 2.2.28. A positive measure on Ω is an elementary integral $\mu : \mathcal{L} \to \mathbb{R}$ on Ω such that

($\mathcal{J}_5$) for every $u \in \mathcal{L}$, $\min(u, 1) \in \mathcal{L}$.

Proposition 2.2.29. *Let μ be a positive measure on Ω, $u \in \mathcal{M}$, and $t \geq 0$. Then* $\min(u, t) \in \mathcal{M}$.

Proof. If $t = 0$, $\min(u, 0) = u^+ \in \mathcal{M}$. Let $t > 0$. There is a sequence $(u_n) \subset \mathcal{L}$ converging to u almost everywhere. Then $v_n = t\min(t^{-1}u_n, 1) \in \mathcal{L}$ and $v_n \to \min(u, t)$ almost everywhere. □

Theorem 2.2.30. *Let μ be a positive measure on Ω and let $u : \Omega \to [0, +\infty]$ be almost everywhere finite. The following properties are equivalent:*

(a) u is measurable;
(b) for every $t \geq 0$, $\{u > t\} = \{x \in \Omega : u(x) > t\}$ is measurable.

Proof. Assume that u is measurable. For every $t \geq 0$ and $n \geq 1$, the preceding proposition implies that

$$u_n = n[\min(u, t + 1/n) - \min(u, t)]$$

is measurable. It follows from Theorem 2.2.24 that

$$\chi_{\{u>t\}} = \lim_{n\to\infty} u_n \in \mathcal{M}.$$

Hence $\{u > t\}$ is measurable.

Assume that u satisfies (b). Let us define, for $n \geq 1$, the function

$$u_n = \frac{1}{2^n} \sum_{k=1}^{\infty} \chi_{\{u>k/2^n\}}. \tag{$*$}$$

For every $x \in \Omega$, $u(x) - 1/2^n \leq u_n(x) \leq u(x)$. Hence (u_n) is simply convergent to u. Theorem 2.2.24 implies that $(u_n) \subset \mathcal{M}$ and $u \in \mathcal{M}$. □

Corollary 2.2.31. *Let $u, v \in \mathcal{M}$. Then $uv \in \mathcal{M}$.*

Proof. If f is measurable, then for every $t \geq 0$, the set

$$\{f^2 > t\} = \{|f| > t\}$$

is measurable. Hence f^2 is measurable. We conclude that

$$uv = \frac{1}{4}[(u + v)^2 - (u - v)^2] \in \mathcal{M}.$$ □

Definition 2.2.32. A function $u : \Omega \to [0, +\infty]$ is admissible (with respect to the positive measure μ) if u is measurable and if for every $t > 0$,

$$\mu_u(t) = \mu(\{u > t\}) = \mu(\{x \in \Omega : u(x) > t\}) < +\infty.$$

The function μ_u is the distribution function of u.

Corollary 2.2.33 (Markov inequality). *Let $u \in \mathcal{L}^1$, $u \geq 0$. Then u is admissible, and for every $t > 0$,*

$$\mu_u(t) \leq t^{-1} \int_\Omega u \, d\mu.$$

Proof. Observe that for every $t > 0$, $v = t\chi_{\{u>t\}} \leq u$. By the comparison theorem, $v \in \mathcal{L}^1$ and $\int_\Omega v \, d\mu \leq \int_\Omega u \, d\mu$. □

Corollary 2.2.34 (Cavalieri's principle). *Let $u \in \mathcal{L}^1$, $u \geq 0$. Then*

$$\int_\Omega u\, d\mu = \int_0^\infty \mu_u(t)dt.$$

Proof. The sequence (u_n) defined by $(*)$ is increasing and converges simply to u. The function $\mu_u\ :]0,+\infty[\to [0,+\infty[$ is nonincreasing. We deduce from Levi's theorem that

$$\int_\Omega u\, d\mu = \lim_{n\to\infty} \int_\Omega u_n d\mu = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^\infty \mu_u\left(\frac{k}{2^n}\right) = \int_0^\infty \mu_u(t)dt. \qquad \square$$

Definition 2.2.35. Let Ω be an open set of $\mathbb{R}^N$. The Lebesgue measure on Ω is the positive measure defined by the Cauchy integral

$$\mathcal{K}(\Omega) \to \mathbb{R} : u \mapsto \int_\Omega u\, dx.$$

The Lebesgue measure of a measurable subset A of Ω is defined by

$$m(A) = \int_\Omega \chi_A dx.$$

Topology is not used in the abstract theory of the *Lebesgue integral*. In contrast, the concrete theory of the *Lebesgue measure* depends on the topology of $\mathbb{R}^N$.

Theorem 2.2.36. *We consider the Lebesgue measure on $\mathbb{R}^N$.*

(a) Every open set is measurable, and every closed set is measurable.
(b) For every measurable set A of $\mathbb{R}^N$, there exist a sequence (G_k) of open sets of $\mathbb{R}^N$ and a negligible set S of $\mathbb{R}^N$ such that $A \cup S = \bigcap_{k=1}^\infty G_k$.
(c) For every measurable set A of $\mathbb{R}^N$, there exist a sequence (F_k) of closed sets of $\mathbb{R}^N$ and a negligible set T of $\mathbb{R}^N$ such that $A = \bigcup_{k=1}^\infty F_k \cup T$.

Proof. (a) Let G be an open bounded set and define

$$u_n(x) = \min\{1, n\, d(x, \mathbb{R}^N \setminus G)\}. \qquad (*)$$

Since $(u_n) \subset \mathcal{K}(\mathbb{R}^N)$ and $u_n \to \chi_G$, the set G is measurable. For every open set G, $G_n = G \cap B(0,n)$ is measurable. Hence $G = \bigcup_{n=1}^\infty G_n$ is measurable. Taking the complement, every closed set is measurable.

(b) Let A be a measurable set of $\mathbb{R}^N$. By definition, there exist a sequence $(u_n) \subset \mathcal{K}(\mathbb{R}^N)$ and a negligible set R of $\mathbb{R}^N$ such that $u_n \to \chi_A$ on $\mathbb{R}^N \setminus R$. There is also $f \in \mathcal{L}^+$ such that $R \subset S = \{f = +\infty\}$. By Proposition 1.3.10, f is l.s.c. Proposition 1.3.12 implies that for every $t \in \mathbb{R}$, $\{f > t\}$ is open. Let us define the open sets

$$U_n = \{u_n > 1/2\} \cup \{f > n\} \quad \text{and} \quad G_k = \bigcup_{n=k}^{\infty} U_n.$$

It is clear that for every k, $A \cup S \subset G_k$ and $A \cup S = \bigcap_{k=1}^{\infty} G_k$. Since S is negligible by definition, the proof is complete.

(c) Taking the complement, there exist a sequence (F_k) of closed sets of $\mathbb{R}^N$ and a negligible set S of $\mathbb{R}^N$ such that

$$A \cap (\mathbb{R}^N \setminus S) = \bigcup_{k=1}^{\infty} F_k.$$

It suffices then to define $T = A \cap S$. □

Corollary 2.2.37. *Let $a < b$. Then*

$$m(]a,b[) = m([a,b]) = b - a.$$

In particular, $m(\{a\}) = 0$, and every countable set is negligible.

Proof. Let (u_n) be the sequence defined by $(*)$. Proposition 2.2.10 implies that

$$m(]a,b[) = \int_{\mathbb{R}} \chi_{]a,b[} dx = \lim_{n\to\infty} \int_{\mathbb{R}} u_n dx = b - a.$$

Since $[a,b] = \bigcap_{n=1}^{\infty}]a - 1/n, b + 1/n[$, it follows from Proposition 2.2.26 that

$$m([a,b]) = \lim_{n\to\infty} b - a + 2/n = b - a.$$ □

Example. Let $\lambda > -1$. For every $n \geq 2$, the function

$$u_n(x) = x^\lambda \chi_{]1/n,1[}(x)$$

is integrable by the comparison theorem. It follows from Levi's monotone convergence theorem that

$$\int_0^1 x^\lambda dx = 1/(\lambda + 1).$$

Let $\lambda < -1$. For every $n \geq 2$, the function

$$v_n(x) = x^\lambda \chi_{]1,n[}(x)$$

is integrable. It follows that

$$\int_1^\infty x^\lambda dx = 1/|\lambda + 1|.$$

Example (Cantor sets). Let $0 < \varepsilon \leq 1$ and $(\ell_n) \subset \;]0, 1[$ be such that

$$\varepsilon = \sum_{n=0}^{\infty} 2^n \ell_n.$$

From the interval $C_0 = [0, 1]$, remove the open middle interval $J_{0,1}$ of length ℓ_0. Remove from the two remaining closed intervals the middle open intervals $J_{1,1}$ and $J_{1,2}$ of length ℓ_1. In general, remove from the 2^n remaining closed intervals the middle open intervals $J_{n,1}, \dots, J_{n,2^n}$ of length ℓ_n. Define

$$C_{n+1} = C_n \setminus \bigcup_{k=1}^{2^n} J_{n,k}, \quad C = \bigcap_{n=1}^{\infty} C_n.$$

The set C is the *Cantor set* (corresponding to (ℓ_n)). Let us describe the fascinating properties of the Cantor set.

The set C is closed. Indeed, each C_n is closed.

The interior of C is empty. Indeed, each C_n consists of 2^n closed intervals of equal length, so that ϕ is the only open subset in C.

The Lebesgue measure of C is equal to $1 - \varepsilon$. By induction, we have for every n that

$$m(C_{n+1}) = 1 - \sum_{j=0}^{n} 2^j \ell_j.$$

Proposition 2.2.26 implies that

$$m(C) = 1 - \sum_{j=0}^{\infty} 2^j \ell_j = 1 - \varepsilon.$$

The set C is not countable. Let $(x_n) \subset C$. Denote by $[a_1, b_1]$ the interval of C_1 not containing x_1. Denote by $[a_2, b_2]$ the first interval of $C_2 \cap [a_1, b_1]$ not containing x_2. In general, let $[a_n, b_n]$ denote the first interval of $C_n \cap [a_{n-1}, b_{n-1}]$ not containing x_n. Define $x = \sup_n a_n = \lim_{n\to\infty} a_n$. For every n, we have

$$[a_n, b_n] \subset C_n, x_n \notin [a_n, b_n], x \in [a_n, b_n].$$

Hence $x \in C$, and for every n, $x_n \neq x$.

For $\varepsilon = 1$, C is not countable and negligible.

Finally, the characteristic function of C is u.s.c., integrable, and discontinuous at every point of C.

The first Cantor sets were defined by Smith in 1875, by Volterra in 1881, and by Cantor in 1883.

2.3 Multiple Integrals

Fubini's theorem reduces the computation of a double integral to the computation of two simple integrals.

Definition 2.3.1. Define on $\mathbb{R}$, $f(t) = (1-|t|)^+$. The family $f_{j,k}(x) = \prod_{n=1}^{N} f(2^j x_n - k_n)$, $j \in \mathbb{N}$, $k \in \mathbb{Z}^N$, is such that $f_{j,k} \in \mathcal{K}(\mathbb{R}^N)$,

$$\text{spt } f_{j,k} = B_\infty[k/2^j, 1/2^j], \sum_{k \in \mathbb{Z}^N} f_{j,k} = 1, f_{j,k} \geq 0.$$

Proposition 2.3.2. *Let Ω be an open set in $\mathbb{R}^N$ and let $u \in \mathcal{K}(\Omega)$. Then the sequence*

$$u_j = \sum_{k \in \mathbb{Z}^N} u(k/2^j) f_{j,k}$$

converges uniformly to u on Ω.

Proof. Let $\varepsilon > 0$. By uniform continuity, there exists m such that $\omega_u(1/2^m) \leq \varepsilon$. Hence for $j \geq m$,

$$|u(x) - u_j(x)| = |\sum_{k \in \mathbb{Z}^N} (u(x) - u(k/2^j)) f_{j,k}(x)| \leq \varepsilon \sum_{k \in \mathbb{Z}^N} f_{j,k}(x) = \varepsilon. \qquad \square$$

Proposition 2.3.3. *Let $u \in \mathcal{K}(\mathbb{R}^N)$. Then*

(a) for every $x_N \in \mathbb{R}$, $u(., x_N) \in \mathcal{K}(\mathbb{R}^{N-1})$;

(b) $\int_{\mathbb{R}^{N-1}} u(x', .)dx' \in \mathcal{K}(\mathbb{R})$;

(c) $\int_{\mathbb{R}^N} u(x)dx = \int_{\mathbb{R}} dx_N \int_{\mathbb{R}^{N-1}} u(x', x_N)dx'$.

Proof. Every restriction of a continuous function is continuous.

Let us define $v(x_N) = \int_{\mathbb{R}^{N-1}} u(x', x_N)dx'$. Lebesgue's dominated convergence theorem implies that v is continuous on $\mathbb{R}$. Since the support of u is a compact subset of $\mathbb{R}^N$, the support of v is a compact subset of $\mathbb{R}$.

We have, for every $j \in \mathbb{N}$ and every $k \in \mathbb{Z}$, by definition of the integral that

$$\int_{\mathbb{R}^N} f_{j,k}(x)dx = \int_{\mathbb{R}} dx_N \int_{\mathbb{R}^{N-1}} f_{j,k}(x', x_N)dx'.$$

Hence for every $j \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} u_j(x)dx = \int_{\mathbb{R}} dx_N \int_{\mathbb{R}^{N-1}} u_j(x', x_N)dx'.$$

There is $R > 1$ such that

$$\text{spt } u \subset \{x \in \mathbb{R}^N : |x|_\infty \leq R - 1\}.$$

For every $j \in \mathbb{N}$, by the definition of the integral

$$\left| \int_{\mathbb{R}^N} u(x) - u_j(x)dx \right| \leq (2R)^N \max_{x \in \mathbb{R}^N} \left| u(x) - u_j(x) \right|,$$

we obtain

$$\left| \int_{\mathbb{R}} dx_N \int_{\mathbb{R}^{N-1}} u(x', x_N) - u_j(x', x_N)dx' \right| \leq (2R)^N \max_{x \in \mathbb{R}^N} \left| u(x) - u_j(x) \right|.$$

It is easy to conclude the proof using the preceding proposition. □

Definition 2.3.4. The elementary integral μ on $\Omega = \Omega_1 \times \Omega_2$ is the product of the elementary integrals μ_1 on Ω_1 and μ_2 on Ω_2 if for every $u \in \mathcal{L}(\Omega, \mu)$,

(a) $u(., x_2) \in \mathcal{L}(\Omega_1, \mu_1)$ for every $x_2 \in \Omega_2$;

(b) $\int_{\Omega_1} u(x_1, .)d\mu_1 \in \mathcal{L}(\Omega_2, \mu_2)$;

(c) $\int_{\Omega} u(x_1, x_2)d\mu = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} u(x_1, x_2)d\mu_1$.

We assume that μ is the product of μ_1 and μ_2.

Lemma 2.3.5. *Let $u \in \mathcal{L}^+(\Omega, \mu)$. Then*

(a) for almost every $x_2 \in \Omega_2$, $u(., x_2) \in \mathcal{L}^+(\Omega_1, \mu_1)$;

(b) $\int_{\Omega_1} u(x_1, .)d\mu_1 \in \mathcal{L}^+(\Omega_2, \mu_2)$;

(c) $\int_{\Omega} u(x_1, x_2)d\mu = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} u(x_1, x_2)d\mu_1$.

Proof. Let $(u_n) \subset \mathcal{L}(\Omega, \mu)$ be a fundamental sequence such that $u_n \uparrow u$. By definition,

$$v_n = \int_\Omega u_n(x_1, .)d\mu_1 \in \mathcal{L}(\Omega_2, \mu_2),$$

and (v_n) is a fundamental sequence. But then $v_n \uparrow v$, $v \in \mathcal{L}^+(\Omega_2, \mu_2)$, and

$$\int_{\Omega_2} v(x_2)d\mu_2 = \lim_{n\to\infty} \int_{\Omega_2} v_n(x_2)d\mu_2.$$

For almost every $x_2 \in \Omega_2$, $v(x_2) \in \mathbb{R}$. In this case, $(u_n(., x_2)) \subset \mathcal{L}(\Omega_1, \mu_1)$ is a fundamental sequence and $u_n(., x_2) \uparrow u(., x_2)$. Hence $u(., x_2) \in \mathcal{L}^+(\Omega_1, \mu_1)$ and

$$\int_{\Omega_1} u(x_1, x_2)d\mu_1 = \lim_{n\to\infty} \int_{\Omega_1} u_n(x_1, x_2)d\mu_1 = \lim_{n\to\infty} v_n(x_2) = v(x_2).$$

It follows that $\int_{\Omega_1} u(x_1, .)d\mu_1 \in \mathcal{L}^+(\Omega_2, \mu_2)$ and

$$\begin{aligned}
\int_\Omega u(x_1, x_2)d\mu &= \lim_{n\to\infty} \int_\Omega u_n(x_1, x_2)d\mu \\
&= \lim_{n\to\infty} \int_{\Omega_2} d\mu_2 \int_{\Omega_1} u_n(x_1, x_2)d\mu_1 \\
&= \lim_{n\to\infty} \int_{\Omega_2} v_n(x_2)d\mu_2 \\
&= \int_{\Omega_2} v(x_2)d\mu_2 = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} u(x_1, x_2)d\mu_1.
\end{aligned}$$

□

Lemma 2.3.6. *Let $S \subset \Omega$ be negligible with respect to μ. Then for almost every $x_2 \in \Omega_2$,*

$$S_{x_2} = \{x_1 \in \Omega_1 : (x_1, x_2) \in S\}$$

is negligible with respect to μ_1.

Proof. By assumption, there is $u \in \mathcal{L}^+(\Omega, \mu)$ such that

$$S \subset \{(x_1, x_2) \in \Omega : u(x_1, x_2) = +\infty\}.$$

The preceding lemma implies that for almost every $x_2 \in \Omega_2$,

$$S_{x_2} \subset \{x_1 \in \Omega_1 : u(x_1, x_2) = +\infty\}$$

is negligible with respect to μ_1. □

Theorem 2.3.7 (Fubini). *Let $u \in \mathcal{L}^1(\Omega, \mu)$. Then*

(a) for almost every $x_2 \in \Omega_2$, $u(., x_2) \in \mathcal{L}^1(\Omega_1, \mu_1)$;
(b) $\int_{\Omega_1} u(x_1, .)d\mu_1 \in \mathcal{L}^1(\Omega_2, \mu_2)$;
(c) $\int_{\Omega} u(x_1, x_2)d\mu = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} u(x_1, x_2)d\mu_1$.

Proof. By assumption, there is $f, g \in \mathcal{L}^+(\Omega, \mu)$ such that $u = f - g$ almost everywhere on Ω. By the preceding lemma, for almost every $x_2 \in \Omega_2$,

$$u(x_1, x_2) = f(x_1, x_2) - g(x_1, x_2)$$

almost everywhere on Ω_1. The conclusion follows from Lemma 2.3.5. □

The following result provides a way to prove that a function on a product space is integrable.

Theorem 2.3.8 (Tonelli). *Let $u : \Omega \to [0, +\infty[$ be such that*

(a) for every $n \in \mathbb{N}$, $\min(n, u) \in L^1(\Omega, \mu)$;
(b) $c = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} u(x_1, x_2)d\mu_1 < +\infty$.

Then $u \in \mathcal{L}^1(\Omega, \mu)$.

Proof. Let us define $u_n = \min(n, u)$. Fubini's theorem implies that

$$\int_{\Omega} u_n(x_1, x_2)d\mu = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} u_n(x_1, x_2)d\mu_1 \leq c.$$

The conclusion follows from Levi's dominated convergence theorem. □

2.4 Change of Variables

Let Ω be an open set of $\mathbb{R}^N$ and let dx be the Lebesgue measure on Ω. We define

$$\mathcal{L}^+(\Omega) = \mathcal{L}^+(\Omega, dx), \mathcal{L}^1(\Omega) = \mathcal{L}^1(\Omega, dx).$$

Definition 2.4.1. Let Ω and ω be open. A diffeomorphism is a continuously differentiable map $f : \Omega \to \omega$ such that for every $x \in \Omega$,

$$J_f(x) = \det\ f'(x) \neq 0.$$

We assume that $f : \Omega \to \omega$ is a diffeomorphism. The next theorem is proved in Sect. 9.1.

Theorem 2.4.2. *Let $u \in \mathcal{K}(\omega)$. Then $u(f)|J_f| \in \mathcal{K}(\Omega)$ and*

$$\int_\Omega u(f(x))|J_f(x)|dx = \int_\omega u(y)dy. \qquad (*)$$

Lemma 2.4.3. *Let $u \in \mathcal{L}^+(\omega)$. Then $u(f)|J_f| \in \mathcal{L}^+(\Omega)$, and $(*)$ is valid.*

Proof. Let $(u_n) \subset \mathcal{K}(\omega)$ be a fundamental sequence such that $u_n \uparrow u$. By the preceding theorem, $v_n = u_n(f)|J_f| \in \mathcal{K}(\Omega)$ and (v_n) is a fundamental sequence. It follows that

$$\int_\Omega u(f(x))|J_f(x)|dx = \lim_{n\to\infty} \int_\Omega u_n(f(x))|J_f(x)|dx = \lim_{n\to\infty} \int_\omega u_n(y)dy = \int_\omega u(y)dy.$$

□

Lemma 2.4.4. *Let $S \subset \omega$ be a negligible set. Then $f^{-1}(S)$ is a negligible set.*

Proof. By assumption, there is $u \in \mathcal{L}^+(\omega)$ such that

$$S \subset \{y \in \omega : u(y) = +\infty\}.$$

The preceding lemma implies that the set

$$f^{-1}(S) \subset \{x \in \Omega : u(f(x)) = +\infty\}$$

is negligible. □

Theorem 2.4.5. *Let $u \in \mathcal{L}^1(\omega)$. Then $u(f)|J_f| \in \mathcal{L}^1(\Omega)$, and $(*)$ is valid.*

Proof. By assumption, there exist $v, w \in \mathcal{L}^+(\omega)$ such that $u = v - w$ almost everywhere on ω. It follows from the preceding lemma that

$$u(f)|J_f| = v(f)|J_f| - w(f)|J_f|$$

almost everywhere on Ω. It is easy to conclude the proof using Lemma 2.4.3. □

Let

$$B_N = \{x \in \mathbb{R}^N : |x| < 1\}$$

be the *unit ball* in $\mathbb{R}^N$, and let $V_N = m(B_N)$ be its volume. By the preceding theorem, for every $r > 0$,

$$m(B(0,r)) = \int_{|y|<r} dy = r^N \int_{|x|<1} dx = r^N V_N.$$

We now define *polar coordinates*. Let $N \geq 2$ and $\mathbb{R}^N_* = \mathbb{R}^N \setminus \{0\}$. Let

$$\mathbb{S}^{N-1} = \{\sigma \in \mathbb{R}^N : |\sigma| = 1\}$$

be the *unit sphere* in $\mathbb{R}^N$. The *polar change of variables* is the homeomorphism

$$]0, \infty[\times\mathbb{S}^{N-1} \longrightarrow \mathbb{R}^N_* : (r, \sigma) \longmapsto r\sigma.$$

Definition 2.4.6. The surface measure on $\mathbb{S}^{N-1}$ is defined on $\mathcal{C}(\mathbb{S}^{N-1})$ by

$$\int_{\mathbb{S}^{N-1}} f(\sigma)d\sigma = N \int_{B_N} f\left(\frac{x}{|x|}\right) dx.$$

Observe that the function $f(x/|x|)$ is bounded and continuous on $B_N \setminus \{0\}$.

Since $\mathbb{S}^{N-1}$ is compact, Dini's theorem implies that the surface measure is a positive measure.

Lemma 2.4.7. *Let $u \in \mathcal{K}(\mathbb{R}^N)$. Then*

(a) for every $r > 0$, the function $\sigma \mapsto u(r\sigma)$ belongs to $\mathcal{C}(\mathbb{S}^{N-1})$;

(b) $\displaystyle\frac{d}{dr}\int_{|x|<r} u(x)dx = r^{N-1}\int_{\mathbb{S}^{N-1}} u(r\sigma)d\sigma;$

(c) $\displaystyle\int_{\mathbb{R}^N} u(x)dx = \int_0^\infty r^{N-1}dr\int_{\mathbb{S}^{N-1}} u(r\sigma)d\sigma.$

Proof. (a) The restriction of a continuous function is a continuous function.

(b) Let $w(r) = \int_{|x|<r} u(x)dx$ and $v(r) = \int_{\mathbb{S}^{N-1}} u(r\sigma)d\sigma$, $r > 0$. By definition, we have

$$v(r) = N \int_{B_N} u\left(\frac{r}{|x|}x\right) dx.$$

Choose $r > 0$ and $\varepsilon > 0$. By definition of the modulus of continuity, we have

$$\left| w(r+\varepsilon) - w(r) - \int_{r<|x|<r+\varepsilon} u(rx/|x|)dx \right| = \left| \int_{r<|x|<r+\varepsilon} u(x) - u(rx/|x|)dx \right|$$

$$\leq \omega_u(\varepsilon) V_N [(r+\varepsilon)^N - r^N].$$

The preceding theorem implies that

$$\int_{r<|x|<r+\varepsilon} u(rx/|x|)dx = \int_{|x|<r+\varepsilon} u(rx/|x|)dx - \int_{|x|<r} u(rx/|x|)dx = \frac{(r+\varepsilon)^N - r^N}{N} v(r).$$

Hence we find that

$$\left| w(r+\varepsilon) - w(r) - \frac{(r+\varepsilon)^N - r^N}{N} v(r) \right| \leq \omega_u(\varepsilon) V_N [(r+\varepsilon)^N - r^N],$$

so that

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \left| \frac{w(r+\varepsilon) - w(r)}{\varepsilon} - r^{N-1} v(r) \right| = 0.$$

The right derivative of w is equal to $r^{N-1}v$. Similarly, the left derivative of w is equal to $r^{N-1}v$.

(c) The fundamental theorem of calculus implies that for $0 < a < b$,

$$\int_{a<|x|<b} u(x)dx = w(b) - w(a) = \int_a^b v(r) r^{N-1} dr = \int_a^b r^{N-1} dr \int_{\mathbb{S}^{N-1}} u(r\sigma) d\sigma.$$

Taking the limit as $a \to 0$ and $b \to +\infty$, we obtain (c). □

Theorem 2.4.8. *Let $u \in \mathcal{L}^1(\mathbb{R}^N)$. Then*

(a) for almost every $r > 0$, the function $\sigma \to u(r\sigma)$ belongs to $\mathcal{L}^1(\mathbb{S}^{N-1}, d\sigma)$;

(b) the function $r \to \int_{\mathbb{S}^{N-1}} u(r\sigma)d\sigma$ belongs to $\mathcal{L}^1(]0, \infty[, r^{N-1}dr)$;

(c) $\int_{\mathbb{R}^N} u(x)dx = \int_0^\infty r^{N-1} dr \int_{\mathbb{S}^{N-1}} u(r\sigma)d\sigma.$

Proof. By the preceding theorem, the Lebesgue measure on $\mathbb{R}^N$ is the product of the surface measure on $\mathbb{S}^{N-1}$ and the measure $r^{N-1}dr$ on $]0, \infty[$. It suffices then to use Fubini's theorem. □

Theorem 2.4.9. *The volume V_N is given by the formulas*

$$V_1 = 2, V_2 = \pi \quad and \quad V_N = \frac{2\pi}{N} V_{N-2}.$$

Proof. Let $N \geq 3$. Fubini's theorem and Theorems 2.4.5 and 2.4.8 imply that

$$\begin{aligned}
V_N &= \int_{|x|<1} dx \\
&= \int_{x_3^2+\ldots+x_N^2<1} dx_3 \ldots dx_N \int_{x_1^2+x_2^2<1-(x_3^2+\ldots+x_N^2)} dx_1 dx_2 \\
&= \pi \int_{x_3^2+\ldots+x_N^2<1} 1 - (x_3^2 + \ldots + x_N^2) dx_3 \ldots dx_N \\
&= \pi (N-2) V_{N-2} \int_0^1 (1-r^2) r^{N-3} dr = \frac{2\pi}{N} V_{N-2}.
\end{aligned}$$

□

2.5 Comments

The construction of the Lebesgue integral in Chap. 2 follows the article [65] by Roselli and the author. Our source was an outline by Riesz on p. 133 of [62]. However, the space $\mathcal{L}^+$ defined by Riesz is much larger, since it consists of all functions u that are almost everywhere equal to the limit of an almost everywhere increasing sequence (u_n) of elementary functions such that

$$\sup_n \int_\Omega u_n \, d\mu < \infty.$$

Using our definition, it is almost obvious that in the case of the concrete Lebesgue integral:

- Every integrable function is almost everywhere equal to the difference of two lower semicontinuous functions.
- The Lebesgue integral is the smallest extension of the Cauchy integral satisfying the properties of monotone convergence and linearity.

Our approach was used in *Analyse Réelle et Complexe* by Golse et al. [30].

Lemma 2.4.7 is due to Baker [4]. The book by Saks [67] is still an excellent reference on integration theory.

The history of integration theory is described in [39, 57]. See also [31] on the life and the work of Émile Borel.

An informal version of the Lebesgue dominated convergence theorem appears (p. 121) in *Théorie du Potentiel Newtonien*, by Henri Poincaré (1899).

2.6 Exercises for Chap. 2

1. (Independence of $\mathcal{J}_4$.) The functional defined on

$$\mathcal{L} = \left\{ u : \mathbb{N} \to \mathbb{R} : \lim_{k\to\infty} u(k) \text{ exists} \right\}$$

by $\langle f, u \rangle = \lim_{k\to\infty} u(k)$ satisfies $(\mathcal{J}_{1-2-3})$ but not $\mathcal{J}_4$.

2. (Independence of $\mathcal{J}_5$.) The elementary integral defined on

$$\mathcal{L} = \{ u : [0,1] \to \mathbb{R} : x \mapsto ax : a \in \mathbb{R} \}$$

by

$$\int u \, d\mu = u(1)$$

is not a positive measure.

3. (Counting measure.) Let Ω be a set. The elementary integral defined on

$$\mathcal{L} = \{u : \Omega \to \mathbb{R} : \{u(x) \neq 0\} \text{ is finite}\}$$

by

$$\int_\Omega u \, d\mu = \sum_{u(x)\neq 0} u(x),$$

satisfies

$$\mathcal{L}^1(\mathbb{N}, \mu) = \left\{u : \mathbb{N} \to \mathbb{R} : \sum_{n=0}^{\infty} |u(n)| < \infty\right\}$$

and

$$\int_{\mathbb{N}} u \, d\mu = \sum_{n=0}^{\infty} u(n).$$

Prove also that when $\Omega = \mathbb{R}$, the set $\mathbb{R}$ is not measurable.

4. (Axiomatic definition of the Cauchy integral.) Let us recall that $\tau_y u(x) = u(x - y)$. Let $f : \mathcal{K}(\mathbb{R}^N) \to \mathbb{R}$ be a linear functional such that

(a) for every $u \in \mathcal{K}(\mathbb{R}^N)$, $u \geq 0 \Rightarrow \langle f, u\rangle \geq 0$;
(b) for every $y \in \mathbb{R}^N$ and for every $u \in \mathcal{K}(\mathbb{R}^N)$, $\langle f, \tau_y u\rangle = \langle f, u\rangle$.

Then there exists $c \geq 0$ such that for every $u \in \mathcal{K}(\mathbb{R}^N)$, $\langle f, u\rangle = c\int_{\mathbb{R}^N} u \, dx$. *Hint*: Use Proposition 2.3.2.

5. Let μ be an elementary integral on Ω. Then the following statements are equivalent:

(a) $u \in \mathcal{L}^1(\Omega, \mu)$.
(b) There exists a decreasing sequence $(u_n) \subset \mathcal{L}^+(\Omega, \mu)$ such that almost everywhere, $u = \lim_{n\to\infty} u_n$ and $\inf \int_\Omega u_n d\mu > -\infty$.

6. Let $\Omega = B(0, 1) \subset \mathbb{R}^N$. Then

$$\lambda + N > 0 \iff |x|^\lambda \in \mathcal{L}^1(\Omega), \lambda + N < 0 \iff |x|^\lambda \in \mathcal{L}^1(\mathbb{R}^N \setminus \overline{\Omega}).$$

7. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be such that for every $y \in \mathbb{R}$, $u(., y)$ is continuous and for every $x \in \mathbb{R}$, $u(x, .)$ is continuous. Then u is Lebesgue measurable. *Hint*: Prove the existence of a sequence of continuous functions converging simply to u on $\mathbb{R}^2$.

8. Construct a sequence (ω_k) of open dense subsets of $\mathbb{R}$ such that $m\left(\bigcap_{k=0}^{\infty} \omega_k\right) = 0$.
Hint: Let (q_n) be an enumeration of $\mathbb{Q}$ and let $I_{n,k}$ be the open interval with center q_n and length $1/2^{n+k}$. Define $\omega_k = \bigcup_{n=0}^{\infty} I_{n,k}$.

9. Prove, using Baire's theorem, that the set of nowhere differentiable functions is dense in $X = C([0,1])$ with the distance $d(u,v) = \max_{0 \le x \le 1} |u(x) - v(x)|$.

   Hint: Let Y be the set of functions in X that are differentiable at at least one point and define, for $n \ge 1$,

$$F_n = \{u \in X : \text{there exists } 0 \le x \le 1 \text{ such that,} \\ \text{for all } 0 \le y \le 1, |u(x) - u(y)| \le n|x - y|\}.$$

   Since $Y \subset \bigcup_{n=1}^{\infty} F_n$, it suffices to prove that $\bigcap_{n=1}^{\infty} G_n$ is dense in X, where $G_n = X \backslash F_n$. By Baire's theorem, it suffices to prove that every G_n is open and dense.

   It is clear that

$$G_n = \{u \in X : \text{for all } 0 \le x \le 1, \text{ there exists } 0 \le y \le 1 \\ \text{such that } n|x - y| < |u(x) - u(y)|\}.$$

   Let $u \in G_n$. The function

$$f(x) = \max\{|u(x) - u(y)| - n(x - y)| : 0 \le y \le 1\},$$

   is such that

$$\inf_{0 \le x \le 1} f(x) = \min_{0 \le x \le 1} f(x) > 0.$$

   It follows that G_n is open.

   We use the functions $f_{j,k}$ of Definition 2.3.1. Let $u \in X$ and $\varepsilon > 0$. Define

$$u_j(x) = \sum_{0 \le k \le 2^j} u(k/2^j) f_{j,k}(x),$$

$$g_m(x) = \varepsilon\, d(2^m x, \mathbb{N}).$$

   Then for j and m large enough,

$$d(u, u_j) < \varepsilon, \quad u_j + g_m \in G_n.$$

   It follows that G_n is dense.

10. (Iterated integrals, Baker 1990.) Let $K = [0,1]^N$ and let μ be an elementary integral on Ω. Assume that $f \in \mathcal{L}^1(\Omega, \mu)$ and

$$F : K \times \Omega \to \mathbb{R} : (x, y) \mapsto F(x, y)$$

   are such that

(a) For almost all $y \in \Omega$, $F(., y)$ is continuous;
(b) For all $x \in K$, $F(x, .)$ is μ-measurable;
(c) $|F(x, y)| \leq f(y)$.

Then:

(a) The function $G(x) = \int_\Omega F(x, y)d\mu$ is continuous on K.
(b) The function $H(y) = \int_K F(x, y)dx$ is μ-measurable on Ω.
(c) $\int_K G(x)dx = \int_\Omega H(y)d\mu$.

Hint: Define on Ω

$$H_j(y) = 2^{-jN} \sum_{\substack{k\in\mathbb{N}^N \\ |k|_\infty<2^j}} F(k/2^j, y)$$

and observe that

$$\lim_{j\to\infty} H_j(y) = H(y), \quad \lim_{j\to\infty} \int_\Omega H_j(y)d\mu = \int_\Omega H(y)d\mu.$$

11. (Proof of Euler's identity by M. Ivan, 2008).

(a) $$\int_{-1}^{1} dy \int_{-1}^{1} \frac{dx}{1+2xy+y^2} = \int_{-1}^{1} \frac{\log\frac{1+y}{1-y}}{y}\, dy = 2\sum_{n=0}^{\infty}\int_{-1}^{1}\frac{y^{2n}}{2n+1}dy$$
$$= 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

(b) $$\int_{-1}^{1} dx \int_{-1}^{1} \frac{dy}{1+2xy+y^2} = \int_{-1}^{1} \frac{\pi}{2\sqrt{1-x^2}}\, dx = \frac{\pi^2}{2}.$$

(c) The formula $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$ is equivalent to the formula $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

12. Let $u \in C^1(\mathbb{R}^N) \bigcap \mathcal{K}(\mathbb{R}^N)$. Then

$$u(x) = \frac{1}{NV_N}\int_{\mathbb{R}^N}\frac{\nabla u(x-y)\cdot y}{|y|^N}dy.$$

Hint: For every $\sigma \in \mathbb{S}^{N-1}$,

$$u(x) = \int_0^\infty \nabla u(x - r\sigma)\cdot\sigma dr.$$

13. The *Newton potential* of the ball $B_R = B(0, R) \subset \mathbb{R}^3$ is defined, for $|y| > R$, by

$$\varphi(y) = \int_{B_R} \frac{dx}{|y - x|}.$$

Since B_R is invariant by rotation, we may assume that $y = (0, 0, a)$, where $a = |y|$. It follows that

$$\begin{aligned}\varphi(y) &= \int_{B_R} \frac{dx}{\sqrt{x_1^2 + x_2^2 + (x_3 - a)^2}} \\ &= 2\pi \int_{-R}^{R} dx_3 \int_0^{\sqrt{R^2 - x_3^2}} \frac{r}{\sqrt{r^2 + (x_3 - a)^2}} dr \\ &= \pi \int_{-R}^{R} \left(\sqrt{R^2 + a^2 - 2ax_3} - a + x_3 \right) dx_3 \\ &= \frac{4}{3}\pi \frac{R^3}{a} = \frac{4}{3}\pi \frac{R^3}{|y|}.\end{aligned}$$

14. The *Newton potential* of the sphere $\mathbb{S}^2$ is defined, for $|y| \neq 1$, by

$$\psi(y) = \int_{\mathbb{S}^2} \frac{d\sigma}{|y - \sigma|}.$$

For $|y| > R$, we have that

$$\frac{4}{3}\pi \frac{R^3}{|y|} = \int_0^R r^2 f(r, y) dr,$$

where

$$f(r, y) = \int_{\mathbb{S}^2} \frac{d\sigma}{|y - r\sigma|}.$$

It follows that

$$4\pi \frac{R^2}{|y|} = R^2 f(R, y).$$

In particular, for $|y| > 1$,

$$\psi(y) = f(1, y) = \frac{4\pi}{|y|}.$$