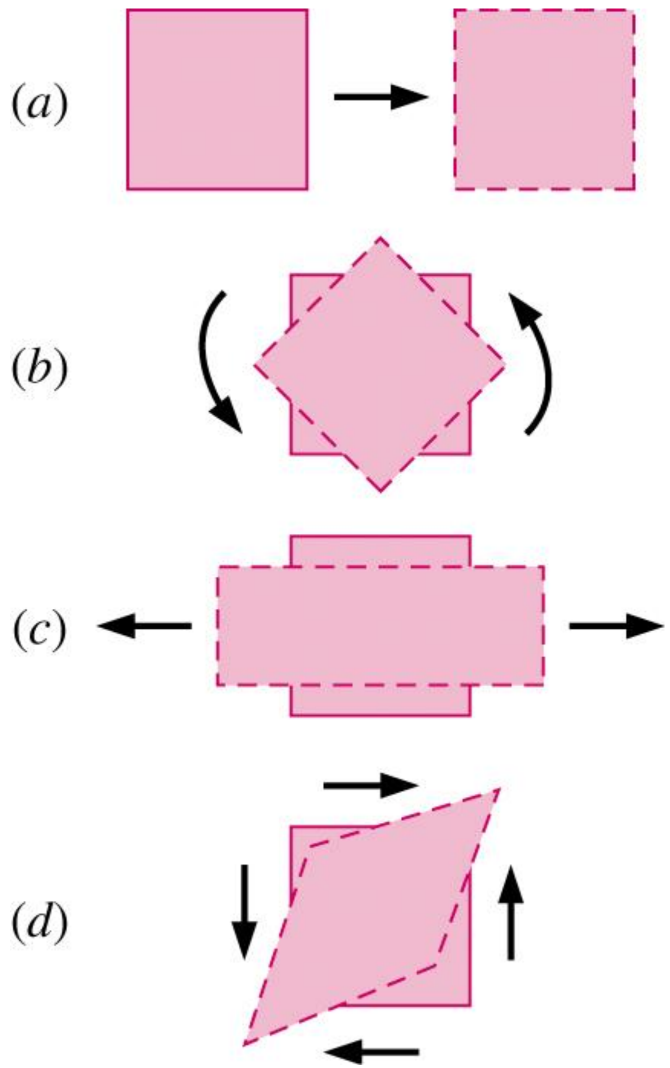
Kinematic Description



- In fluid mechanics, an element may undergo four fundamental types of motion.
  a) Translation
  b) Rotation
  c) Linear strain
  d) Shear strain

- Because fluids are in constant motion and deformation, they are better described in terms of rates
  a) velocity: rate of translation
  b) angular velocity: rate of rotation
  c) linear strain rate: rate of linear strain
  d) shear strain rate: rate of shear strain

Rate of Translation and Rotation

- To be useful, these rates must be expressed in terms of velocity and derivatives of velocity
- The **rate of translation vector** is described as the velocity vector. In Cartesian coordinates:

$$\vec{V} = \underbrace{u\vec{i}}_{u_x} + \underbrace{v\vec{j}}_{u_y} + \underbrace{w\vec{k}}_{u_z}$$

- **Rate of rotation** at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point. The rate of rotation vector in Cartesian coordinates:

$$\vec{V} = \vec{\omega} \times \vec{r}$$

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$= \boxed{\frac{1}{2}\nabla \times \vec{V}}$$

Tabela com o operador del em coordenadas cartesianas, cilíndricas e esféricas

Operação	Coordenadas cartesianas (x, y, z)	Coordenadas cilíndricas (ρ, φ, z)	Coordenadas esféricas (r, θ, φ), onde φ é o polar e θ é o ângulo azimutal [α]
campo vetorial $\mathbf{A}$	$A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$	$A_\rho\hat{\boldsymbol{\rho}} + A_\varphi\hat{\boldsymbol{\varphi}} + A_z\hat{\mathbf{z}}$	$A_r\hat{\mathbf{r}} + A_\theta\hat{\boldsymbol{\theta}} + A_\varphi\hat{\boldsymbol{\varphi}}$
Gradiente ∇f	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}}$
Divergência $\nabla \cdot \mathbf{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$
Rotacional $\nabla \times \mathbf{A}$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}}$ $+\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}}$ $+\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$	$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}}$ $+\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}}$ $+\frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}}$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\varphi\sin\theta) - \frac{\partial A_\theta}{\partial \varphi}\right)\hat{\mathbf{r}}$ $+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}}$ $+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\boldsymbol{\varphi}}$
Operador de Laplace $\nabla^2 f \equiv \Delta f$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$
Vetor de Laplace $\nabla^2\mathbf{A} \equiv \Delta\mathbf{A}$	$\nabla^2 A_x\hat{\mathbf{x}} + \nabla^2 A_y\hat{\mathbf{y}} + \nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\rho}}$ $+\left(\nabla^2 A_\varphi - \frac{A_\varphi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}}$ $+\nabla^2 A_z\hat{\mathbf{z}}$	$\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(A_\theta\sin\theta)}{\partial \theta} - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\mathbf{r}}$ $+\left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\theta}}$ }} $+\left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial \varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}}$

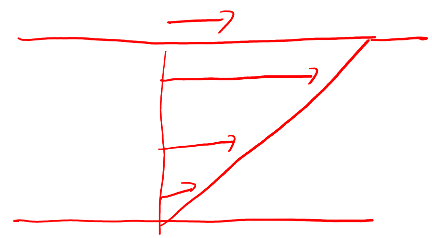
https://pt.wikipedia.org/wiki/Del_em_coordenadas_cil%C3%ADndricas_e_esf%C3%A9ricas

Linear Strain Rate

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

- **Linear Strain Rate** is defined as the rate of increase in length per unit length.
- In Cartesian coordinates

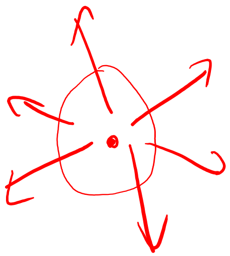
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$



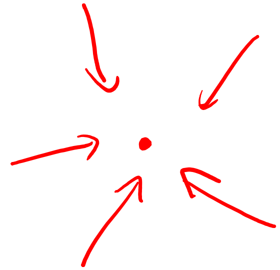
- Volumetric strain rate in Cartesian coordinates

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V}$$

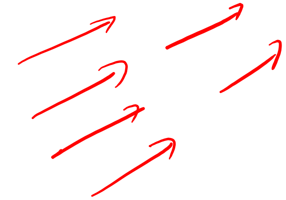
- Since the volume of a fluid element is constant for an incompressible flow, the volumetric strain rate must be zero.



$\nabla \cdot \vec{v} > 0$



$\nabla \cdot \vec{v} < 0$



$\nabla \cdot \vec{v} = 0$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ (Analogia: Lei de Gauss)

Shear Strain Rate

- **Shear Strain Rate** at a point is defined as *half of the rate of decrease of the angle between two initially perpendicular lines that intersect at a point*.

- Shear strain rate can be expressed in Cartesian coordinates as:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Shear Strain Rate

We can combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain-rate tensor.**

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

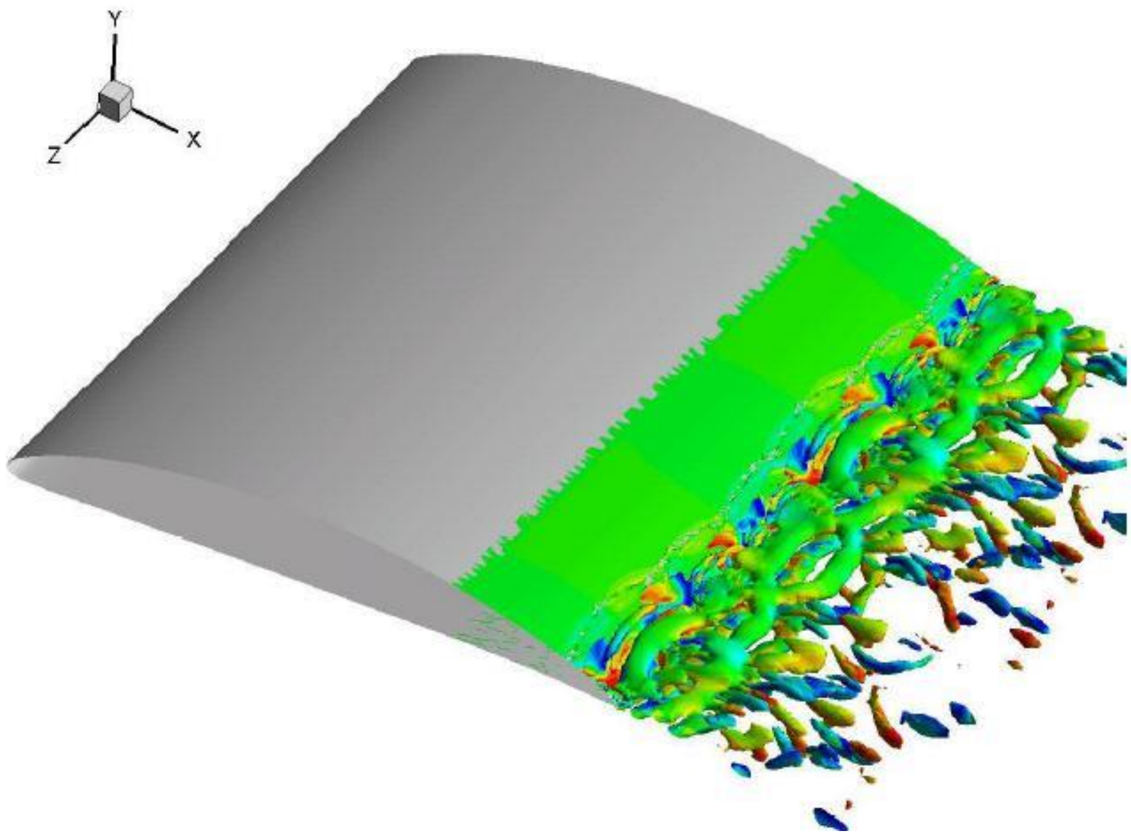
Shear Strain Rate

- Purpose of our discussion of fluid element kinematics:
  - Better appreciation of the inherent complexity of fluid dynamics
  - Mathematical sophistication required to fully describe fluid motion

- Strain-rate tensor is important for numerous reasons. For example,
  - Develop relationships between fluid stress and strain rate.
  - Feature extraction and flow visualization in CFD simulations.

Shear Strain Rate

Example: Visualization of trailing-edge turbulent eddies
for a hydrofoil with a beveled trailing edge



Feature extraction method is based upon eigen-analysis of the strain-rate tensor.

Vorticity and Rotationality

- The **vorticity vector** is defined as the curl of the velocity vector

$$\vec{\zeta} = \vec{\nabla} \times \vec{V}$$

- Vorticity is equal to twice the angular velocity of a fluid particle.

$$\boxed{\vec{\zeta} = 2\vec{\omega}}$$

Cartesian coordinates

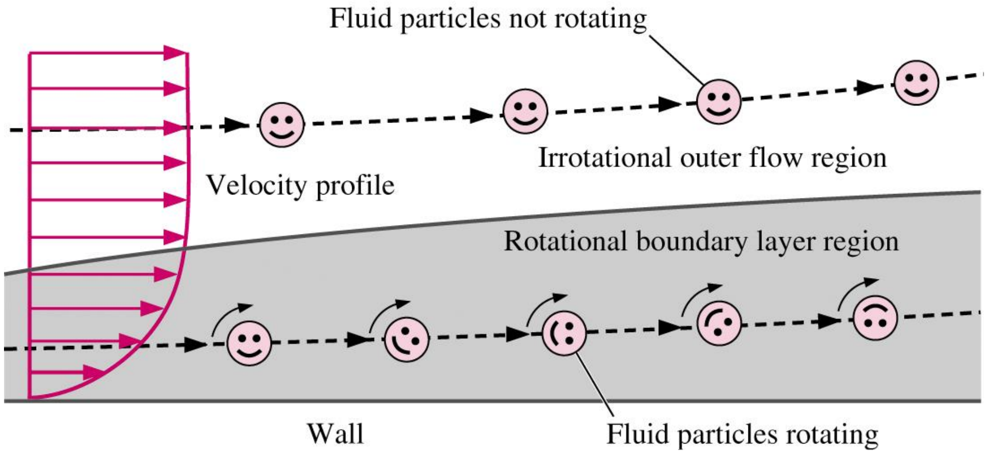
$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}$$

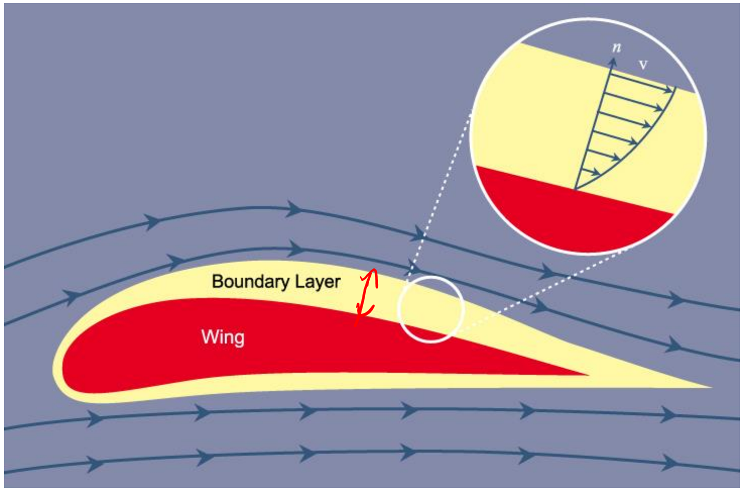
Cylindrical coordinates

$$\vec{\zeta} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \vec{e}_\theta + \left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \vec{e}_z$$

- In regions where $\zeta = 0$, the flow is called **irrotational.**
- Elsewhere, the flow is called **rotational.**

Vorticity and Rotationality



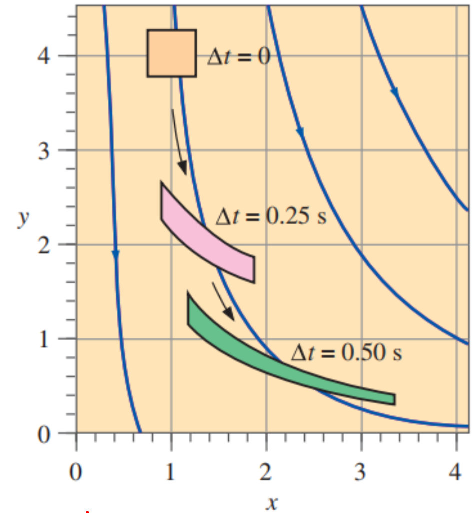


EXAMPLE 4–8 **Determination of Rotationality in a Two-Dimensional Flow**

Consider the following steady, incompressible, two-dimensional velocity field:

$$\vec{V} = (u, v) = x^2\vec{i} + (-2xy - 1)\vec{j} \tag{1}$$

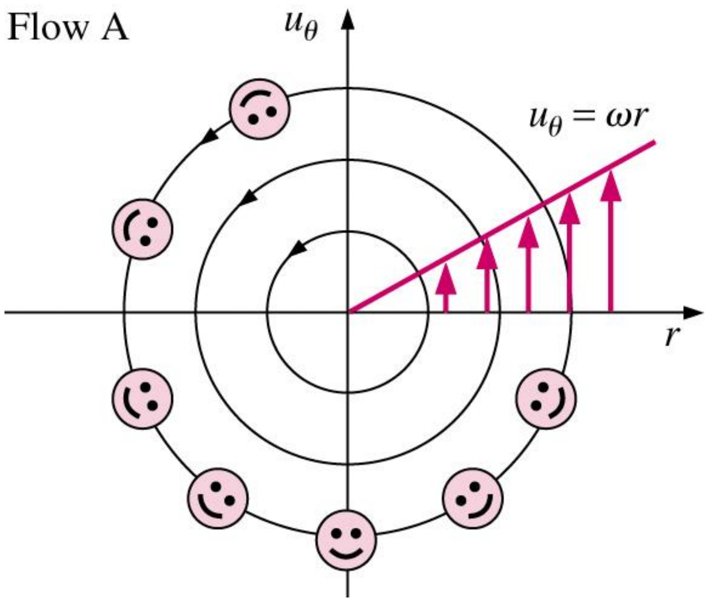
Is this flow rotational or irrotational? Sketch some streamlines in the first quadrant and discuss.

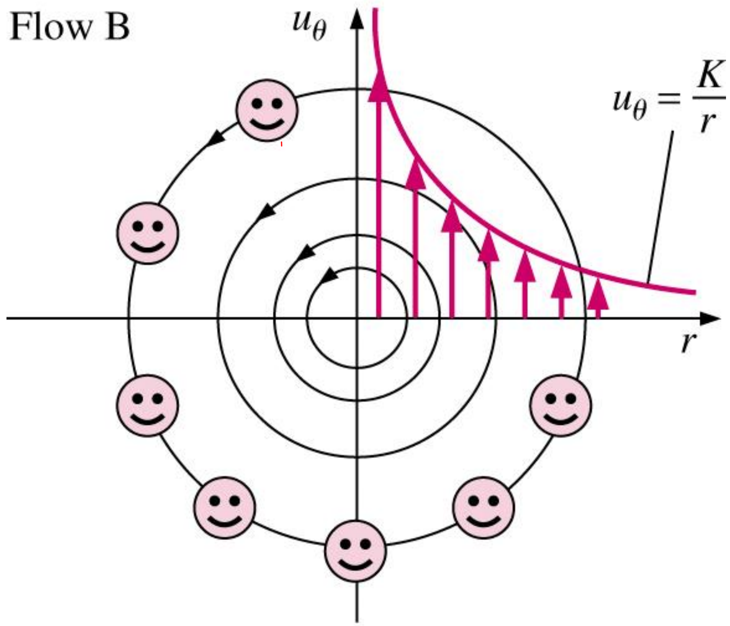


$$\vec{\zeta} = \nabla \times \vec{V} = \hat{z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \hat{z}\left(-2y - \frac{\partial}{\partial y}(x^2)\right) = -2y\,\hat{z}$$

Comparison of Two Circular Flows

Special case: consider two flows with circular streamlines





(b)

$$u_r = 0, u_\theta = \omega r$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(\omega r^2)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$

$$u_r = 0, u_\theta = \frac{K}{r}$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial(K)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$