Chapter 3
Norms

3.1 Banach Spaces

Since their creation by Banach in 1922, normed spaces have played a central role in functional analysis. Banach spaces are complete normed spaces. Completeness allows one to prove the convergence of a sequence or of a series without using the limit.

Definition 3.1.1. A norm on a real vector space X is a function

$$X \to \mathbb{R} : u \mapsto ||u||$$

such that

($\mathcal{N}_1$) for every $u \in X \setminus \{0\}$, $||u|| > 0$;
($\mathcal{N}_2$) for every $u \in X$ and for $\alpha \in \mathbb{R}$, $||\alpha u|| = |\alpha|\,||u||$;
($\mathcal{N}_3$) (Minkowski's inequality) for every $u, v \in X$,

$$||u + v|| \leq ||u|| + ||v||.$$

A (real) normed space is a (real) vector space together with a norm on that space.

Examples. 1. Let $(X, ||.||)$ be a normed space and let Y be a subspace of X. The space Y together with $||.||$ (restricted to Y) is a normed space.
2. Let $(X_1, ||.||_1), (X_2, ||.||_2)$ be normed spaces. The space $X_1 \times X_2$ together with

$$||(u_1, u_2)|| = \max(||u_1||_1, ||u_2||_2)$$

is a normed space.
3. We define the norm on the space $\mathbb{R}^N$ to be

$$|x|_\infty = \max\Big\{|x_1|, \ldots, |x_N|\Big\}.$$

Every normed space is a metric space.

M. Willem, *Functional Analysis: Fundamentals and Applications*, Cornerstones,
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Proposition 3.1.2. *Let X be a normed space. The function*

$$X \times X \to \mathbb{R} : (u, v) \mapsto \|u - v\|$$

is a distance on X. The following mappings are continuous:

$$\begin{aligned} &X \to \mathbb{R} : u \mapsto \|u\|, \\ &X \times X \to X : (u, v) \mapsto u + v, \\ &\mathbb{R} \times X \to X : (\alpha, u) \mapsto \alpha u. \end{aligned}$$

Proof. By $\mathcal{N}_1$ and $\mathcal{N}_2$,

$$d(u, v) = 0 \iff u = v, \quad d(u, v) = \| - (u - v)\| = \|v - u\| = d(v, u).$$

Finally, by Minkowski's inequality,

$$d(u, w) \leq d(u, v) + d(v, w).$$

Since by Minkowski's inequality,

$$\Big| \|u\| - \|v\| \Big| \leq \|u - v\|,$$

the norm is continuous on X. It is easy to verify the continuity of the sum and of the product by a scalar. □

Definition 3.1.3. Let X be a normed space and $(u_n) \subset X$. The series $\sum_{n=0}^{\infty} u_n$ converges, and its sum is $u \in X$ if the sequence $\sum_{n=0}^{k} u_n$ converges to u. We then write $\sum_{n=0}^{\infty} u_n = u$.

The series $\sum_{n=0}^{\infty} u_n$ converges normally if $\sum_{n=0}^{\infty} \|u_n\| < \infty$.

Definition 3.1.4. A Banach space is a complete normed space.

Proposition 3.1.5. *In a Banach space X, the following statements are equivalent:*

(a) $\sum_{n=0}^{\infty} u_n$ *converges;*

(b) $\lim\limits_{\substack{j \to \infty \\ j < k}} \sum_{n=j+1}^{k} u_n = 0.$

Proof. Define $S_k = \sum_{n=0}^{k} u_n$. Since X is complete, we have

$$(a) \Longleftrightarrow \lim_{\substack{j\to\infty \\ j<k}} \|S_k - S_j\| = 0 \Longleftrightarrow \lim_{\substack{j\to\infty \\ j<k}} \left\| \sum_{n=j+1}^{k} u_n \right\| = 0 \Longleftrightarrow b). \qquad \square$$

Proposition 3.1.6. *In a Banach space, every normally convergent series converges.*

Proof. Let $\sum_{n=0}^{\infty} u_n$ be a normally convergent series in the Banach space X. Minkowski's inequality implies that for $j < k$,

$$\left\| \sum_{n=j+1}^{k} u_n \right\| \le \sum_{n=j+1}^{k} \|u_n\|.$$

Since the series is normally convergent,

$$\lim_{\substack{j\to\infty \\ j<k}} \sum_{n=j+1}^{k} \|u_n\| = 0.$$

It suffices then to use the preceding proposition. $\square$

Examples. 1. The space of bounded continuous functions on the metric space X,

$$\mathcal{BC}(X) = \left\{ u \in C(X) : \sup_{x\in X} |u(x)| < \infty \right\},$$

together with the norm

$$\|u\|_\infty = \sup_{x\in X} |u(x)|,$$

is a Banach space. Convergence with respect to $\|.\|_\infty$ is uniform convergence.

2. Let μ be a positive measure on Ω. We denote by $L^1(\Omega, \mu)$ the quotient of $\mathcal{L}^1(\Omega, \mu)$ by the equivalence relation "equality almost everywhere." We define the norm

$$\|u\|_1 = \int_\Omega |u|\, d\mu.$$

Convergence with respect to $\|.\|_1$ is convergence in mean. We will prove in Sect. 4.2, on Lebesgue spaces, that $L^1(\Omega, \mu)$ is a Banach space.

3. Let dx be the Lebesgue measure on the open subset Ω of $\mathbb{R}^N$. We denote by $L^1(\Omega)$ the space $L^1(\Omega, dx)$. Convergence in mean is not implied by simple convergence, and almost everywhere convergence is not implied by convergence in mean.

If $m(\Omega) < \infty$, the comparison theorem implies that for every $u \in \mathcal{BC}(\Omega)$,

$$\|u\|_1 = \int_\Omega |u| dx \leq m(\Omega)\|u\|_\infty.$$

Hence $\mathcal{BC}(\Omega) \subset L^1(\Omega)$, and the canonical injection is continuous, since

$$\|u - v\|_1 \leq m(\Omega)\|u - v\|_\infty.$$

Proposition 3.1.7. *Let $u \in L^1(\Omega, \mu)$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable subset A of Ω satisfying $\mu(A) \leq \delta$, $\int_A |u| d\mu \leq \varepsilon$.*

Proof. Let $\varepsilon > 0$. Markov's inequality implies that for every $t > 0$ and for every measurable set A,

$$\int_A |u| d\mu \leq t\,\mu(A) + \int_{\{|u|>t\}} |u| d\mu \leq t\,\mu(A) + \|u\|_1/t.$$

We choose $t = 2\|u\|_1/\varepsilon$ and $\delta = \varepsilon/(2t)$. We obtain, when $\mu(A) \leq \delta$, that $\int_A |u| d\mu \leq \varepsilon$. □

Definition 3.1.8. A subset S of $L^1(\Omega, \mu)$ is uniformly integrable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable subset A of Ω satisfying $\mu(A) \leq \delta$, $\sup_{u \in S} \int_A |u| d\mu \leq \varepsilon$.

Theorem 3.1.9 (Vitali). *Let $\mu(\Omega) < \infty$ and let $(u_n) \subset L^1(\Omega, \mu)$ be a sequence almost everywhere converging to u. Then the following properties are equivalent:*

(a) $\{u_n : n \in \mathbb{N}\}$ *is uniformly integrable;*
(b) $\|u_n - u\|_1 \to 0, n \to \infty$.

Proof. Assume that (a) is satisfied and let $\varepsilon > 0$. For every n, we have

$$\int_\Omega |u_n - u| d\mu = \int_{|u_n - u| \leq \varepsilon} |u_n - u| d\mu + \int_{|u_n - u| > \varepsilon} |u_n - u| d\mu \qquad (*)$$

$$\leq \varepsilon\mu(\Omega) + \int_{|u_n - u| > \varepsilon} |u_n| d\mu + \int_{|u_n - u| > \varepsilon} |u| d\mu.$$

There exists, by assumption and Fatou's lemma, a $\delta > 0$ such that for every measurable subset A of Ω satisfying $\mu(A) \leq \delta$,

$$\sup_n \int_A |u_n| d\mu \leq \varepsilon, \int_A |u| d\mu \leq \varepsilon. \qquad (**)$$

By Lebesgue's dominated convergence theorem and the fact that $\mu(\Omega) < \infty$, there exists m such that for every $n \geq m$,

$$\mu\{|u_n - u| > \varepsilon\} \leq \delta.$$

It follows from $(*)$ and $(**)$ that for every $n \geq m$,

$$\int_\Omega |u_n - u| d\mu \leq (\mu(\Omega) + 2)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $||u_n - u||_1 \to 0$, $n \to \infty$.

Assume that (b) is satisfied. For every measurable subset A of Ω, we have

$$\int_A |u_n| d\mu \leq \int_A |u| d\mu + ||u_n - u||_1.$$

Let $\varepsilon > 0$. There exists m such that for every $n \geq m$, $||u_n - u||_1 \leq \varepsilon/2$ and there exists $\delta > 0$ such that for every measurable subset A of Ω, $\mu(A) \leq \delta$ implies that

$$\int_A |u| d\mu \leq \varepsilon/2, \int_A |u_1| d\mu \leq \varepsilon, \ldots \int_A |u_{m-1}| d\mu \leq \varepsilon.$$

Then for every n, $\int_A |u_n| d\mu \leq \varepsilon$ and $\{u_n : n \in \mathbb{N}\}$ is uniformly integrable. □

Theorem 3.1.10 (de la Vallée Poussin criterion). *Let $S \subset L^1(\Omega, \mu)$ be such that $c = \sup_{u \in S} ||u||_1 < +\infty$. The following properties are equivalent:*

(a) S is uniformly integrable;
(b) there exists an increasing convex function $F : [0, \infty[\to [0, \infty[$ such that

$$\lim_{t \to \infty} F(t)/t = +\infty \quad \textit{and} \quad M = \sup_{u \in S} \int_\Omega F(|u|) d\mu < \infty.$$

Proof. If S satisfies (b), then for every $\varepsilon > 0$, there exists $t > 0$ such that for every $s > t$, $F(s)/s > M/\varepsilon$. Hence for all $u \in S$, we have

$$\int_{\{|u| > t\}} |u| d\mu \leq \frac{\varepsilon}{M} \int_{\{|u| > t\}} F(|u|) d\mu \leq \varepsilon.$$

We choose $\delta = \varepsilon/t$. For every measurable subset A of Ω such that $\mu(A) \leq \delta$ and for every $u \in S$, we obtain

$$\int_A |u| d\mu \leq t\mu(A) + \int_{\{|u| > t\}} |u| d\mu \leq 2\varepsilon.$$

Markov's inequality implies that for every $u \in S$ and every $t > 0$,

$$\mu\{|u| > t\} \leq ||u||_1/t \leq c/t.$$

Assume that S satisfies (a). Then there exists a strictly increasing sequence of integers $a_n \geq 1$ such that $\sup_{u \in S} \int_{\{|u|>a_n\}} |u| d\mu \leq 2^{-n}$. We define $f(s) = 0$ on $[0, 1[$ and $f(s) = f(m)$ on $]m, m+1[$, where $f(m)$ is the number of integers n such that $a_n \leq m$. Let $F(t) = \int_0^t f(s)ds$. We choose $u \in S$, and we define $b_m = \mu\{|u| > m\}$. Since

$$\begin{aligned}\int_\Omega F(|u|)d\mu &\leq f(1)\mu\{1 < |u| \leq 2\} + (f(1) + f(2))\mu\{2 < |u| \leq 3\} + \cdots \\ &= \sum_{m=1}^{\infty} f(m)b_m,\end{aligned}$$

and

$$\sum_{m=a_n}^{\infty} b_m \leq \sum_{m=a_n}^{\infty} m\, \mu\{m < |u| \leq m+1\} \leq \int_{\{|u|>a_n\}} |u| d\mu \leq 2^{-n},$$

we find that $\sum_{m=1}^{\infty} f(m)b_m = \sum_{n=1}^{\infty} \sum_{m=a_n}^{\infty} b_m \leq 1$. □

3.2 Continuous Linear Mappings

On a le droit de faire la théorie générale des opérations sans définir l'opération que l'on considère, de même qu'on fait la théorie de l'addition sans définir la nature des termes à additionner.

Henri Poincaré

In general, linear mappings between normed spaces are not continuous.

Proposition 3.2.1. *Let X and Y be normed spaces and $A : X \to Y$ a linear mapping. The following properties are equivalent:*

(a) A is continuous;

(b) $c = \sup\limits_{\substack{u \in X \\ u \neq 0}} \dfrac{||Au||}{||u||} < \infty$.

Proof. If $c < \infty$, we obtain

$$||Au - Av|| = ||A(u - v)|| \leq c||u - v||.$$

Hence A is continuous.

If A is continuous, there exists $\delta > 0$ such that for every $u \in X$,

$$||u|| = ||u - 0|| \le \delta \Rightarrow ||Au|| = ||Au - A0|| \le 1.$$

Hence for every $u \in X \setminus \{0\}$,

$$||Au|| = \frac{||u||}{\delta} ||A\left(\frac{\delta}{||u||}u\right)|| \le \frac{||u||}{\delta}. \qquad \square$$

Proposition 3.2.2. *The function*

$$||A|| = \sup_{\substack{u \in X \\ u \neq 0}} \frac{||Au||}{||u||} = \sup_{\substack{u \in X \\ ||u|| = 1}} ||Au||$$

defines a norm on the space $\mathcal{L}(X, Y) = \{A : X \to Y : A$ *is linear and continuous*$\}$.

Proof. By the preceding proposition, if $A \in \mathcal{L}(X, Y)$, then $0 \le ||A|| < \infty$. If $A \neq 0$, it is clear that $||A|| > 0$. It follows from axiom $\mathcal{N}_2$ that

$$||\alpha A|| = \sup_{\substack{u \in X \\ ||u|| = 1}} ||\alpha Au|| = \sup_{\substack{u \in X \\ ||u|| = 1}} |\alpha| \, ||Au|| = |\alpha| \, ||A||.$$

It follows from Minkowski's inequality that

$$||A + B|| = \sup_{\substack{u \in X \\ ||u|| = 1}} ||Au + Bu|| \le \sup_{\substack{u \in X \\ ||u|| = 1}} (||Au|| + ||Bu||) \le ||A|| + ||B||. \qquad \square$$

Proposition 3.2.3 (Extension by density). *Let Z be a dense subspace of a normed space X, Y a Banach space, and $A \in \mathcal{L}(Z, Y)$. Then there exists a unique mapping $B \in \mathcal{L}(X, Y)$ such that $B\big|_Z = A$. Moreover, $||B|| = ||A||$.*

Proof. Let $u \in X$. There exists a sequence $(u_n) \subset Z$ such that $u_n \to u$. The sequence (Au_n) is a Cauchy sequence, since

$$||Au_j - Au_k|| \le ||A|| \, ||u_j - u_k|| \to 0, \quad j, k \to \infty$$

by Proposition 1.2.3. We denote by f its limit. Let $(v_n) \subset Z$ be such that $v_n \to u$. We have

$$||Av_n - Au_n|| \le ||A|| \, ||v_n - u_n|| \le ||A|| \, (||v_n - u|| + ||u - u_n||) \to 0, \quad n \to \infty.$$

Hence $Av_n \to f$, and we define $Bu = f$. By Proposition 3.1.2, B is linear. Since for every n,

$$||Au_n|| \le ||A|| \, ||u_n||,$$

we obtain by Proposition 3.1.2 that

$$||Bu|| \le ||A|| \, ||u||.$$

Hence B is continuous and $||B|| \le ||A||$. It is clear that $||A|| \le ||B||$. Hence $||A|| = ||B||$.
If $C \in \mathcal{L}(X, Y)$ is such that $C\big|_Z = A$, we obtain

$$Cu = \lim_{n\to\infty} Cu_n = \lim_{n\to\infty} Au_n = \lim_{n\to\infty} Bu_n = Bu. \qquad \square$$

Proposition 3.2.4. *Let X and Y be normed spaces, and let $(A_n) \subset \mathcal{L}(X, Y)$ and $A \in \mathcal{L}(X, Y)$ be such that $||A_n - A|| \to 0$. Then (A_n) converges simply to A.*

Proof. For every $u \in X$, we have

$$||A_n u - Au|| = ||(A_n - A)u|| \le ||A_n - A|| \, ||u||. \qquad \square$$

Proposition 3.2.5. *Let Z be a dense subset of a normed space X, let Y be a Banach space, and let $(A_n) \subset \mathcal{L}(X, Y)$ be such that*

(a) $c = \sup_n ||A_n|| < \infty$;
(b) for every $v \in Z$, $(A_n v)$ converges.

Then A_n converges simply to $A \in \mathcal{L}(X, Y)$, and

$$||A|| \le \varliminf_{n\to\infty} ||A_n||.$$

Proof. Let $u \in X$ and $\varepsilon > 0$. By density, there exists $v \in B(u, \varepsilon) \cap Z$. Since $(A_n v)$ converges, Proposition 1.2.3 implies the existence of n such that

$$j, k \ge n \Rightarrow ||A_j v - A_k v|| \le \varepsilon.$$

Hence for $j, k \ge n$, we have

$$\begin{aligned} ||A_j u - A_k u|| &\le ||A_j u - A_j v|| + ||A_j v - A_k v|| + ||A_k v - A_k u|| \\ &\le 2c \, ||u - v|| + \varepsilon \\ &= (2c + 1)\varepsilon. \end{aligned}$$

The sequence $(A_n u)$ is a Cauchy sequence, since $\varepsilon > 0$ is arbitrary. Hence $(A_n u)$ converges to a limit Au in the complete space Y. It follows from Proposition 3.1.2 that A is linear and that

$$||Au|| = \lim_{n\to\infty} ||A_n u|| \le \varliminf_{n\to\infty} ||A_n|| \, ||u||.$$

But then A is continuous and $||A|| \le \varliminf_{n\to\infty} ||A_n||$. $\square$

Theorem 3.2.6 (Banach–Steinhaus theorem). *Let X be a Banach space, Y a normed space, and let $(A_n) \subset \mathcal{L}(X, Y)$ be such that for every $u \in X$,*

$$\sup_n \|A_n u\| < \infty.$$

Then

$$\sup_n \|A_n\| < \infty.$$

First Proof. Theorem 1.3.13 applied to the sequence $F_n : u \mapsto \|A_n u\|$ implies the existence of a ball $B(v, r)$ such that

$$c = \sup_n \sup_{u \in B(v,r)} \|A_n u\| < \infty.$$

It is clear that for every $y, z \in Y$,

$$\|y\| \leq \max\{\|z + y\|, \|z - y\|\}. \tag{$*$}$$

Hence for every n and for every $w \in B(0, r)$, $\|A_n w\| \leq c$, so that

$$\sup_n \|A_n\| \leq c/r.$$

Second Proof. Assume to obtain a contradiction that $\sup_n \|A_n\| = +\infty$. By considering a subsequence, we assume that $n\, 3^n \leq \|A_n\|$. Let us define inductively a sequence (u_n). We choose $u_0 = 0$. There exists v_n such that $\|v_n\| = 3^{-n}$ and $\frac{3}{4} 3^{-n} \|A_n\| \leq \|A_n v_n\|$. By $(*)$, replacing if necessary v_n by $-v_n$, we obtain

$$\frac{3}{4} 3^{-n} \|A_n\| \leq \|A_n v_n\| \leq \|A_n(u_{n-1} + v_n)\|.$$

We define $u_n = u_{n-1} + v_n$, so that $\|u_n - u_{n-1}\| = 3^{-n}$. It follows that for every $k \geq n$,

$$\|u_k - u_n\| \leq 3^{-n}/2.$$

Hence (u_n) is a Cauchy sequence that converges to u in the complete space X. Moreover,

$$\|u - u_n\| \leq 3^{-n}/2.$$

We conclude that

$$\begin{aligned} \|A_n u\| &\geq \|A_n u_n\| - \|A_n(u_n - u)\| \\ &\geq \|A_n\| \left[\frac{3}{4} 3^{-n} - \|u_n - u\| \right] \\ &\geq n\, 3^n \left[\frac{3}{4} 3^{-n} - \frac{1}{2} 3^{-n} \right] = n/4. \end{aligned}$$

□

Corollary 3.2.7. *Let X be a Banach space, Y a normed space, and $(A_n) \subset \mathcal{L}(X,Y)$ a sequence converging simply to A. Then (A_n) is bounded, $A \in \mathcal{L}(X,Y)$, and*

$$||A|| \leq \varliminf_{n\to\infty} ||A_n||.$$

Proof. For every $u \in X$, the sequence $(A_n u)$ is convergent, hence bounded, by Proposition 1.2.3. The Banach–Steinhaus theorem implies that $\sup_n ||A_n|| < \infty$. It follows from Proposition 3.1.2 that A is linear and

$$||Au|| = \lim_{n\to\infty} ||A_n u|| \leq \varliminf_{n\to\infty} ||A_n||\, ||u||,$$

so that A is continuous and $||A|| \leq \varliminf_{n\to\infty} ||A_n||$. □

The preceding corollary explains why every natural linear mapping defined on a Banach space is continuous.

Example (Convergence of functionals). We define the linear continuous functionals f_n on $L^1(]0,1[)$ to be

$$\langle f_n, u\rangle = \int_0^1 u(x) x^n \, dx.$$

Since for every $u \in L^1(]0,1[)$ such that $||u||_1 = 1$, we have

$$|\langle f_n, u\rangle| < \int_0^1 |u(x)| dx = 1,$$

it is clear that

$$||f_n|| = \sup_{\substack{u \in L^1 \\ ||u||_1 = 1}} |\langle f_n, u\rangle| \leq 1.$$

Choosing $v_k(x) = (k+1)x^k$, we obtain

$$\lim_{k\to\infty} \langle f_n, v_k\rangle = \lim_{k\to\infty} \frac{k+1}{k+n+1} = 1.$$

It follows that $||f_n|| = 1$, and for every $u \in L^1(]0,1[)$ such that $||u||_1 = 1$,

$$|\langle f_n, u\rangle| < ||f_n||.$$

Lebesgue's dominated convergence theorem implies that (f_n) converges simply to $f = 0$. Observe that

$$||f|| < \varliminf_{n\to\infty} ||f_n||.$$

3.3 Hilbert Spaces

Hilbert spaces are Banach spaces with a norm derived from a scalar product.

Definition 3.3.1. A scalar product on the (real) vector space X is a function

$$X \times X \to \mathbb{R} : (u, v) \mapsto (u|v)$$

such that

($\mathcal{S}_1$) for every $u \in X \setminus \{0\}$, $(u|u) > 0$;
($\mathcal{S}_2$) for every $u, v, w \in X$ and for every $\alpha, \beta \in \mathbb{R}$, $(\alpha u + \beta v|w) = \alpha(u|w) + \beta(v|w)$;
($\mathcal{S}_3$) for every $u, v \in X$, $(u|v) = (v|u)$.

We define $||u|| = \sqrt{(u|u)}$. A (real) pre-Hilbert space is a (real) vector space together with a scalar product on that space.

Proposition 3.3.2. *Let $u, v, w \in X$ and let $\alpha, \beta \in \mathbb{R}$. Then*

(a) $(u|\alpha v + \beta w) = \alpha(u|v) + \beta(u|w)$;
(b) $||\alpha u|| = |\alpha|\, ||u||$.

Proposition 3.3.3. *Let X be a pre-Hilbert space and let $u, v \in X$. Then*

(a) (parallelogram identity) $||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$;
(b) (polarization identity) $(u|v) = \frac{1}{4}||u + v||^2 - \frac{1}{4}||u - v||^2$;
(c) (Pythagorean identity) $(u|v) = 0 \iff ||u + v||^2 = ||u||^2 + ||v||^2$.

Proof. Observe that

$$||u + v||^2 = ||u||^2 + 2(u|v) + ||v||^2, \qquad (*)$$

$$||u - v||^2 = ||u||^2 - 2(u|v) + ||v||^2. \qquad (**)$$

By adding and subtracting, we obtain parallelogram and polarization identities. The Pythagorean identity is clear. □

Proposition 3.3.4. *Let X be a pre-Hilbert space and let $u, v \in X$. Then*

(a) (Cauchy–Schwarz inequality) $|(u|v)| \leq ||u||\, ||v||$;
(b) (Minkowski's inequality) $||u + v|| \leq ||u|| + ||v||$.

Proof. It follows from $(*)$ and $(**)$ that for $||u|| = ||v|| = 1$,

$$|(u|v)| \leq \frac{1}{2}\left(||u||^2 + ||v||^2\right) = 1.$$

Hence for $u \neq 0 \neq v$, we obtain

$$\frac{|(u|v)|}{||u||\, ||v||} = \left|\left(\frac{u}{||u||}\Big|\frac{v}{||v||}\right)\right| \leq 1.$$

By (∗) and the Cauchy–Schwarz inequality, we have

$$\|u+v\|^2 \le \|u\|^2 + 2\|u\|\,\|v\| + \|v\|^2 = \big(\|u\| + \|v\|\big)^2. \qquad \square$$

Corollary 3.3.5. *(a) The function* $\|u\| = \sqrt{(u|u)}$ *defines a norm on the pre-Hilbert space* X.
(b) The function

$$X \times X \to \mathbb{R} : (u, v) \mapsto (u|v)$$

is continuous.

Definition 3.3.6. A family $(e_j)_{j\in J}$ in a pre-Hilbert space X is orthonormal if

$$\begin{aligned}(e_j|e_k) &= 1, \qquad j = k,\\ &= 0, \qquad j \ne k.\end{aligned}$$

Proposition 3.3.7 (Bessel's inequality). *Let* (e_n) *be an orthonormal sequence in a pre-Hilbert space* X *and let* $u \in X$. *Then*

$$\sum_{n=0}^{\infty} \big|(u|e_n)\big|^2 \le \|u\|^2.$$

Proof. It follows from the Pythagorean identity that

$$\begin{aligned}\|u\|^2 &= \left\| u - \sum_{n=0}^{k} (u|e_n)e_n + \sum_{n=0}^{k} (u|e_n)e_n \right\|^2\\ &= \left\| u - \sum_{n=0}^{k} (u|e_n)e_n \right\|^2 + \sum_{n=0}^{k} \big|(u|e_n)\big|^2\\ &\ge \sum_{n=0}^{k} \big|(u|e_n)\big|^2.\end{aligned}$$

$\square$

Proposition 3.3.8. *Let* $(e_0, \ldots, e_k)$ *be a finite orthonormal sequence in a pre-Hilbert space* X, $u \in X$, *and* $x_0, \ldots, x_k \in \mathbb{R}$. *Then*

$$\left\| u - \sum_{n=0}^{k} (u \mid e_n)e_n \right\| \le \left\| u - \sum_{n=0}^{k} x_n e_n \right\|.$$

Proof. It follows from the Pythagorean identity that

$$\left\|u - \sum_{n=0}^{k} x_n e_n\right\|^2 = \left\|u - \sum_{n=0}^{k}(u \mid e_n)e_n + \sum_{n=0}^{k}((u \mid e_n) - x_n)e_n\right\|^2$$

$$= \left\|u - \sum_{n=0}^{k}(u \mid e_n)e_n\right\|^2 + \sum_{n=0}^{k} \left|(u \mid e_n) - x_n\right|^2.$$

□

Definition 3.3.9. A Hilbert basis of a pre-Hilbert space X is an orthonormal sequence generating a dense subspace of X.

Proposition 3.3.10. *Let (e_n) be a Hilbert basis of a pre-Hilbert space X and let $u \in X$. Then*

(a) $u = \sum_{n=0}^{\infty}(u \mid e_n)e_n$;

(b) (Parseval's identity) $||u||^2 = \sum_{n=0}^{\infty}|(u \mid e_n)|^2$.

Proof. Let $\varepsilon > 0$. By definition, there exists a sequence $x_0, \dots, x_j \in \mathbb{R}$ such that

$$||u - \sum_{n=0}^{j} x_n e_n|| < \varepsilon.$$

It follows from the preceding proposition that for $k \geq j$,

$$||u - \sum_{n=0}^{k}(u \mid e_n)e_n|| < \varepsilon.$$

Hence $u = \sum_{n=0}^{\infty}(u \mid e_n)e_n$, and by Proposition 3.1.2,

$$\left\|\lim_{k\to\infty} \sum_{n=0}^{k}(u \mid e_n)e_n\right\|^2 = \lim_{k\to\infty}\left\|\sum_{n=0}^{k}(u \mid e_n)e_n\right\|^2 = \lim_{k\to\infty} \sum_{n=0}^{k} \left|(u \mid e_n)\right|^2 = \sum_{n=0}^{\infty} \left|(u \mid e_n)\right|^2.$$

□

We characterize pre-Hilbert spaces having a Hilbert basis.

Proposition 3.3.11. *Assume the existence of a sequence (f_j) generating a dense subset of the normed space X. Then X is separable.*

Proof. By assumption, the space of (finite) linear combinations of (f_j) is dense in X. Hence the space of (finite) linear combinations with rational coefficients of (f_j) is dense in X. Since this space is countable, X is separable. □

Proposition 3.3.12. *Let X be an infinite-dimensional pre-Hilbert space. The following properties are equivalent:*

(a) X is separable;
(b) X has a Hilbert basis.

Proof. By the preceding proposition, (b) implies (a).

If X is separable, it contains a sequence (f_j) generating a dense subspace. We may assume that (f_j) is free. Since the dimension of X is infinite, the sequence (f_j) is infinite. We define by induction the sequences (g_n) and (e_n):

$$e_0 = f_0/||f_0||,$$

$$g_n = f_n - \sum_{j=0}^{n-1}(f_n|e_j)e_j, e_n = g_n/||g_n||, \quad n \geq 1.$$

The sequence (e_n) generated from (f_n) by the Gram–Schmidt orthonormalization process is a Hilbert basis of X. □

Definition 3.3.13. A Hilbert space is a complete pre-Hilbert space.

Theorem 3.3.14 (Riesz–Fischer). *Let (e_n) be an orthonormal sequence in the Hilbert space X. The sequence $\sum_{n=0}^{\infty} c_n e_n$ converges if and only if $\sum_{n=0}^{\infty} c_n^2 < \infty$. Then*

$$\left\|\sum_{n=0}^{\infty} c_n e_n\right\|^2 = \sum_{n=0}^{\infty} c_n^2.$$

Proof. Define $S_k = \sum_{n=0}^{k} c_n e_n$. The Pythagorean identity implies that for $j < k$,

$$||S_k - S_j||^2 = \left\|\sum_{n=j+1}^{k} c_n e_n\right\|^2 = \sum_{n=j+1}^{k} c_n^2.$$

Hence

$$\lim_{\substack{j \to \infty \\ j < k}} ||S_k - S_j||^2 = 0 \iff \lim_{\substack{j \to \infty \\ j < k}} \sum_{n=j+1}^{k} c_n^2 = 0 \iff \sum_{n=0}^{\infty} c_n^2 < \infty.$$

Since X is complete, (S_k) converges if and only if $\sum_{n=0}^{\infty} c_n^2 < \infty$. Then $\sum_{n=0}^{\infty} c_n e_n = \lim_{k\to\infty} S_k$, and by Proposition 3.1.2,

$$\|\lim_{k\to\infty} S_k\|^2 = \lim_{k\to\infty} \|S_k\|^2 = \lim_{k\to\infty} \sum_{n=0}^{k} c_n^2 = \sum_{n=0}^{\infty} c_n^2. \qquad \square$$

Examples. 1. Let μ be a positive measure on Ω. We denote by $L^2(\Omega, \mu)$ the quotient of

$$\mathcal{L}^2(\Omega, \mu) = \left\{ u \in \mathcal{M}(\Omega, \mu) : \int_\Omega |u|^2 d\mu < \infty \right\}$$

by the equivalence relation "equality almost everywhere." If $u, v \in L^2(\Omega, \mu)$, then $u + v \in L^2(\Omega, \mu)$. Indeed, almost everywhere on Ω, we have

$$|u(x) + v(x)|^2 \le 2(|u(x)|^2 + |v(x)|^2).$$

We define the scalar product

$$(u|v) = \int_\Omega uv \, d\mu$$

on the space $L^2(\Omega, \mu)$.

The scalar product is well defined, since almost everywhere on Ω,

$$|u(x)\, v(x)| \le \frac{1}{2}(|u(x)|^2 + |v(x)|^2).$$

By definition,

$$\|u\|_2 = \left(\int_\Omega |u|^2 d\mu \right)^{1/2}.$$

Convergence with respect to $\|.\|_2$ is convergence in quadratic mean. We will prove in Sect. 4.2, on Lebesgue spaces, that $L^2(\Omega, \mu)$ is a Hilbert space. If $\mu(\Omega) < \infty$, it follows from the Cauchy–Schwarz inequality that for every $u \in L^2(\Omega, \mu)$,

$$\|u\|_1 = \int_\Omega |u| \, d\mu \le \mu(\Omega)^{1/2} \|u\|_2.$$

Hence $L^2(\Omega, \mu) \subset L^1(\Omega, \mu)$, and the canonical injection is continuous.

2. Let dx be the Lebesgue measure on the open subset Ω of $\mathbb{R}^N$. We denote by $L^2(\Omega)$ the space $L^2(\Omega, dx)$. Observe that

$$\frac{1}{x} \in L^2(]1, \infty[) \setminus L^1(]1, \infty[) \text{ and } \frac{1}{\sqrt{x}} \in L^1(]0, 1[) \setminus L^2(]0, 1[).$$

If $m(\Omega) < \infty$, the comparison theorem implies that for every $u \in \mathcal{BC}(\Omega)$,

$$\|u\|_2^2 = \int_\Omega u^2 dx \le m(\Omega)\|u\|_\infty^2.$$

Hence $\mathcal{BC}(\Omega) \subset L^2(\Omega)$, and the canonical injection is continuous.

Theorem 3.3.15 (Vitali 1921, Dalzell 1945). *Let (e_n) be an orthonormal sequence in $L^2(]a,b[)$. The following properties are equivalent:*

(a) (e_n) is a Hilbert basis;

(b) for every $a \le t \le b$, $\sum_{n=1}^{\infty} \left(\int_a^t e_n(x)dx \right)^2 = t - a$;

(c) $\sum_{n=1}^{\infty} \int_a^b \left(\int_a^t e_n(x)dx \right)^2 dt = \frac{(b-a)^2}{2}$.

Proof. Property (b) follows from (a) and Parseval's identity applied to $\chi_{[a,t]}$. Property (c) follows from (b) and Levi's theorem. The converse is left to the reader. □

Example. The sequence $e_n(x) = \sqrt{\frac{2}{\pi}} \sin n\, x$ is orthonormal in $L^2(]0,\pi[)$. Since

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \int_0^\pi \left(\int_0^t \sin n\, x\, dx \right)^2 dt = 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and since by a classical identity due to Euler,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

the sequence (e_n) is a Hilbert basis of $L^2(]0,\pi[)$.

3.4 Spectral Theory

Spectral theory allows one to diagonalize symmetric compact operators.

Definition 3.4.1. Let X be a real vector space and let $A : X \to X$ be a linear mapping. The eigenvectors corresponding to the eigenvalue $\lambda \in \mathbb{R}$ are the nonzero solutions of

$$Au = \lambda u.$$

The multiplicity of λ is the dimension of the space of solutions. The eigenvalue λ is simple if its multiplicity is equal to 1. The rank of A is the dimension of the range of A.

Definition 3.4.2. Let X be a pre-Hilbert space. A symmetric operator is a linear mapping $A : X \to X$ such that for every $u, v \in X$, $(Au|v) = (u|Av)$.

Proposition 3.4.3. *Let X be a pre-Hilbert space and $A : X \to X$ a symmetric continuous operator. Then*

$$||A|| = \sup_{\substack{u \in X \\ ||u|| = 1}} |(Au|u)|.$$

Proof. It is clear that

$$a = \sup_{\substack{u \in X \\ ||u|| = 1}} |(Au|u)| \leq b = \sup_{\substack{u, v \in X \\ ||u|| = ||v|| = 1}} |(Au|v)| = ||A||.$$

If $||u|| = ||v|| = 1$, it follows from the parallelogram identity that

$$\begin{aligned} |(Au|v)| &= \frac{1}{4}|(A(u+v)|u+v) - (A(u-v)|u-v)| \\ &\leq \frac{a}{4}[||u+v||^2 + ||u-v||^2] \\ &= \frac{a}{4}[2||u||^2 + 2||v||^2] = a. \end{aligned}$$

Hence $b = a$. □

Corollary 3.4.4. *Under the assumptions of the preceding proposition, there exists a sequence $(u_n) \subset X$ such that*

$$||u_n|| = 1, ||Au_n - \lambda_1 u_n|| \to 0, |\lambda_1| = ||A||.$$

Proof. Consider a maximizing sequence (u_n):

$$||u_n|| = 1, |(Au_n|u_n)| \to \sup_{\substack{u \in X \\ ||u|| = 1}} |(Au|u)| = ||A||.$$

By passing if necessary to a subsequence, we can assume that $(Au_n|u_n) \to \lambda_1$, $|\lambda_1| = ||A||$. Hence

$$\begin{aligned} 0 \leq ||Au_n - \lambda_1 u_n||^2 &= ||Au_n||^2 - 2\lambda_1(Au_n|u_n) + \lambda_1^2||u_n||^2 \\ &\leq 2\lambda_1^2 - 2\lambda_1(Au_n|u_n) \to 0, \quad n \to \infty. \end{aligned}$$ □

Definition 3.4.5. Let X and Y be normed spaces. A mapping $A : X \to Y$ is compact if the set $\{Au : u \in X, ||u|| \leq 1\}$ is precompact in Y.

By Proposition 3.2.1, every linear compact mapping is continuous.

Theorem 3.4.6. *Let X be a Hilbert space and let $A : X \to X$ be a symmetric compact operator. Then there exists an eigenvalue λ_1 of A such that $|\lambda_1| = \|A\|$.*

Proof. We can assume that $A \neq 0$. The preceding corollary implies the existence of a sequence $(u_n) \subset X$ such that

$$\|u_n\| = 1, \|Au_n - \lambda_1 u_n\| \to 0, |\lambda_1| = \|A\|.$$

Passing if necessary to a subsequence, we can assume that $Au_n \to v$. Hence $u_n \to u = \lambda_1^{-1} v$, $\|u\| = 1$, and $Au = \lambda_1 u$. □

Theorem 3.4.7 (Poincaré's principle). *Let X be a Hilbert space and $A : X \to X$ a symmetric compact operator with infinite rank. Let there be given the eigenvectors $(e_1, \ldots, e_{n-1})$ and the corresponding eigenvalues $(\lambda_1, \ldots, \lambda_{n-1})$. Then there exists an eigenvalue λ_n of A such that*

$$|\lambda_n| = \max\{|(Au|u)| : u \in X, \|u\| = 1, (u|e_1) = \ldots = (u|e_{n-1}) = 0\}$$

and $\lambda_n \to 0$, $n \to \infty$.

Proof. The closed subspace of X

$$X_n = \{u \in X : (u|e_1) = \ldots = (u|e_{n-1}) = 0\}$$

is invariant by A. Indeed, if $u \in X_n$ and $1 \leq j \leq n-1$, then

$$(Au|e_j) = (u|Ae_j) = \lambda_j (u|e_j) = 0.$$

Hence $A_n = A\Big|_{X_n}$ is a nonzero symmetric compact operator, and there exist an eigenvalue λ_n of A_n such that $|\lambda_n| = \|A_n\|$ and a corresponding eigenvector $e_n \in X_n$ such that $\|e_n\| = 1$. By construction, the sequence (e_n) is orthonormal, and the sequence $(|\lambda_n|)$ is decreasing. Hence $|\lambda_n| \to d$, $n \to \infty$, and for $j \neq k$,

$$\|Ae_j - Ae_k\|^2 = \lambda_j^2 + \lambda_k^2 \to 2d^2, \quad j, k \to \infty.$$

Since A is compact, $d = 0$. □

Theorem 3.4.8. *Under the assumptions of the preceding theorem, for every $u \in X$, the series $\sum_{n=1}^{\infty} (u|e_n)e_n$ converges and $u - \sum_{n=1}^{\infty} (u|e_n)e_n$ belongs to the kernel of A:*

$$Au = \sum_{n=1}^{\infty} \lambda_n (u|e_n) e_n. \tag{$*$}$$

Proof. For every $k \geq 1$, $u - \sum_{n=1}^{k}(u|e_n)e_n \in X_{k+1}$. It follows from Proposition 3.3.8. that

$$\left\| Au - \sum_{n=1}^{k} \lambda_n (u|e_n)e_n \right\| \leq \|A_{k+1}\| \left\| u - \sum_{n=1}^{k}(u|e_n)e_n \right\| \leq \|A_{k+1}\| \, \|u\| \to 0, \quad k \to \infty.$$

Bessel's inequality implies that $\sum_{n=1}^{\infty}|(u|e_n)|^2 \leq \|u\|^2$. We deduce from the Riesz–Fischer theorem that $\sum_{n=1}^{\infty}(u|e_n)e_n$ converges to $v \in X$. Since A is continuous,

$$Av = \sum_{n=1}^{\infty} \lambda_n (u|e_n)e_n = Au$$

and $A(u - v) = 0$. □

Formula (∗) is the diagonalization of symmetric compact operators.

3.5 Comments

The de la Vallée Poussin criterion was proved in the beautiful paper [17].

The first proof of the Banach–Steinhaus theorem in Sect. 3.2 is due to Favard [22], and the second proof to Royden [66].

3.6 Exercises for Chap. 3

1. Prove that $\mathcal{BC}(\Omega) \cap L^1(\Omega) \subset L^2(\Omega)$.
2. Define a sequence $(u_n) \subset \mathcal{BC}(]0,1[)$ such that $\|u_n\|_1 \to 0$, $\|u_n\|_2 = 1$, and $\|u_n\|_\infty \to \infty$.
3. Define a sequence $(u_n) \subset \mathcal{BC}(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\|u_n\|_1 \to \infty$, $\|u_n\|_2 = 1$ and $\|u_n\|_\infty \to 0$.
4. Define a sequence $(u_n) \subset \mathcal{BC}(]0,1[)$ converging simply to u such that $\|u_n\|_\infty = \|u\|_\infty = \|u_n - u\|_\infty = 1$.
5. Define a sequence $(u_n) \subset L^1(]0,1[)$ such that $\|u_n\|_1 \to 0$ and for every $0 < x < 1$, $\overline{\lim_{n\to\infty}}\, u_n(x) = 1$. *Hint*: Use characteristic functions of intervals.
6. On the space $C([0,1])$ with the norm $\|u\|_1 = \int_0^1 |u(x)|dx$, is the linear functional

$$f : C([0,1]) \to \mathbb{R} : u \mapsto u(1/2)$$

continuous?

7. Let X be a normed space such that every normally convergent series converges. Prove that X is a Banach space.
8. A linear functional defined on a normed space is continuous if and only if its kernel is closed. If this is not the case, the kernel is dense.
9. Is it possible to derive the norm on $L^1(]0,1[)$ (respectively $\mathcal{BC}(]0,1[)$) from a scalar product?
10. Prove *Lagrange's identity* in pre-Hilbert spaces:

$$\big\| \|v\|u - \|u\|v \big\|^2 = 2\|u\|^2\|v\|^2 - 2\|u\|\,\|v\|(u|v).$$

11. Let X be a pre-Hilbert space and $u, v \in X \setminus \{0\}$. Then

$$\left\| \frac{u}{\|u\|^2} - \frac{v}{\|v\|^2} \right\| = \frac{\|u-v\|}{\|u\|\,\|v\|}.$$

Let $f, g, h \in X$. Prove *Ptolemy's inequality*:

$$\|f\|\,\|g-h\| \le \|h\|\,\|f-g\| + \|g\|\,\|h-f\|.$$

12. (The Jordan–von Neumann theorem.) Assume that the parallelogram identity is valid in the normed space X. Then it is possible to derive the norm from a scalar product. Define

$$(u|v) = \frac{1}{2}(\|u+v\|^2 - \|u-v\|^2).$$

Verify that

$$(f+g|h) + (f-g|h) = 2(f|h),$$

$$(u|h) + (v|h) = 2\left(\frac{u+v}{2}|h\right) = (u+v|h).$$

13. Let f be a linear functional on $L^2(]0,1[)$ such that $u \ge 0 \Rightarrow \langle f, u\rangle \ge 0$. Prove, by contradiction, that f is continuous with respect to the norm $\|.\|_2$. Prove that f is not necessarily continuous with respect to the norm $\|.\|_1$.
14. Prove that every symmetric operator defined on a Hilbert space is continuous. *Hint*: If this were not the case, there would exist a sequence (u_n) such that $\|u_n\| = 1$ and $\|Au_n\| \to \infty$. Then use the Banach–Steinhaus theorem to obtain a contradiction.
15. In a Banach space an algebraic basis is either finite or uncountable. *Hint*: Use Baire's theorem.

16. Assume that $\mu(\Omega) < \infty$. Let $(u_n) \subset L^1(\Omega, \mu)$ be such that

(a) $\sup_n \int_\Omega |u_n| \ell n(1 + |u_n|) d\mu < +\infty$;

(b) (u_n) converges almost everywhere to u.

Then $u_n \to u$ in $L^1(\Omega, \mu)$.