Ciências ULisboa

Euler Equation and Bernoulli's Theorem

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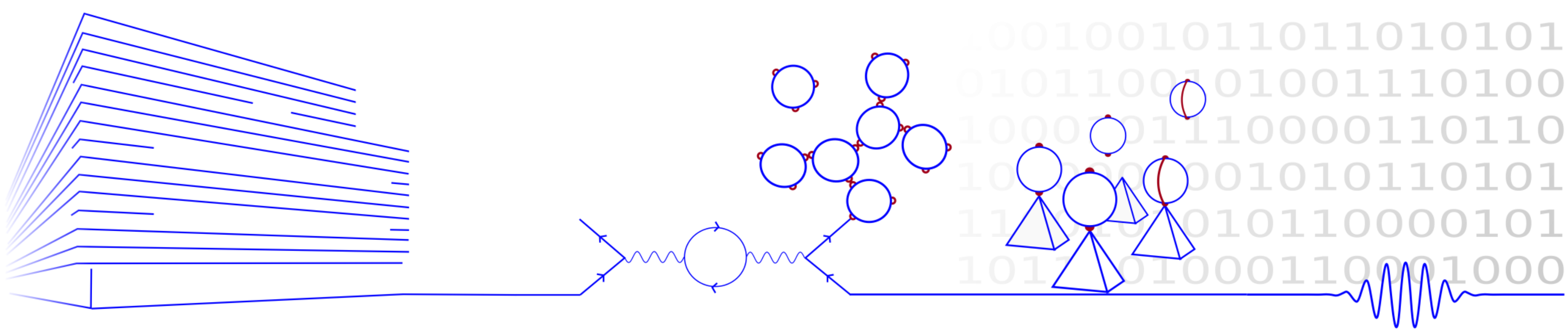

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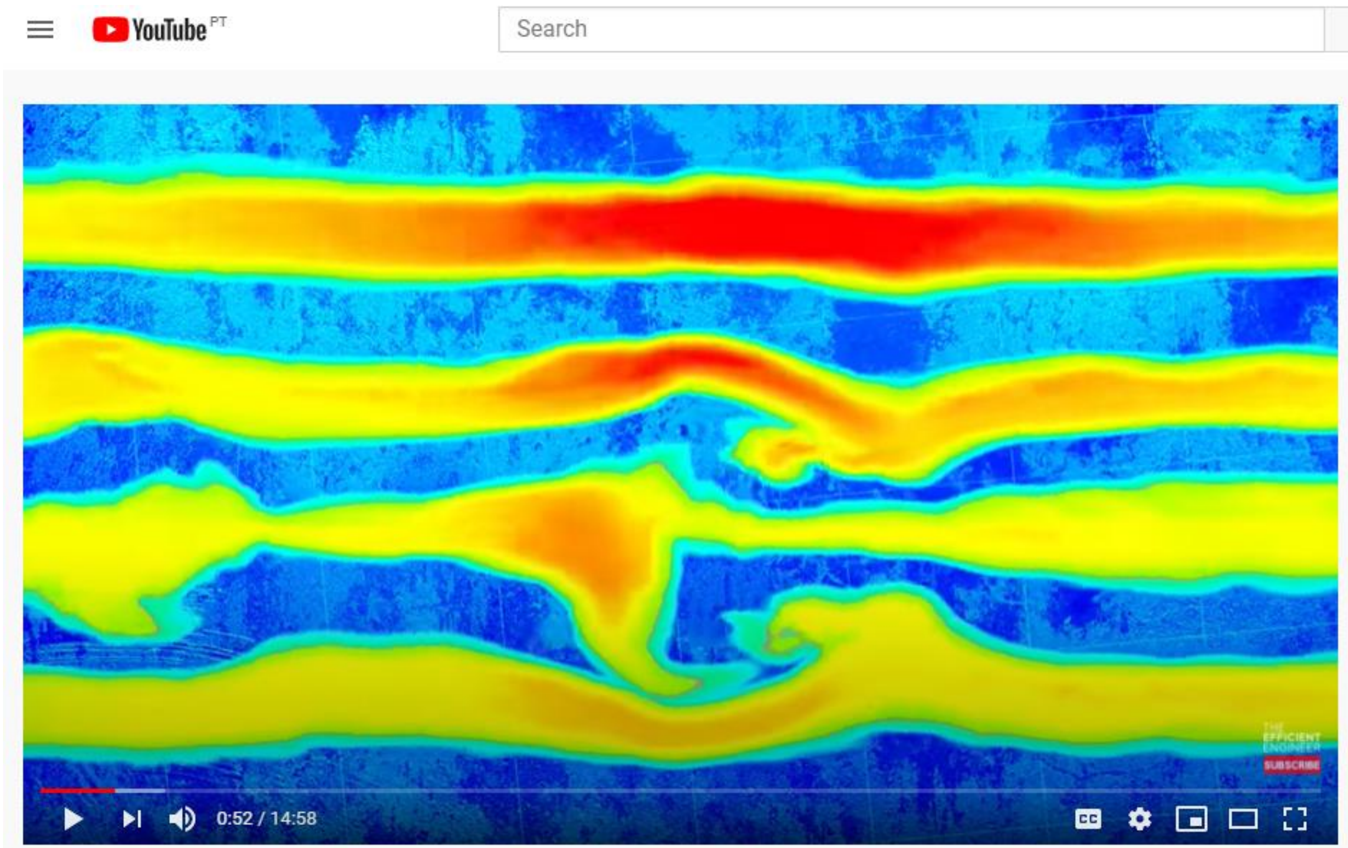
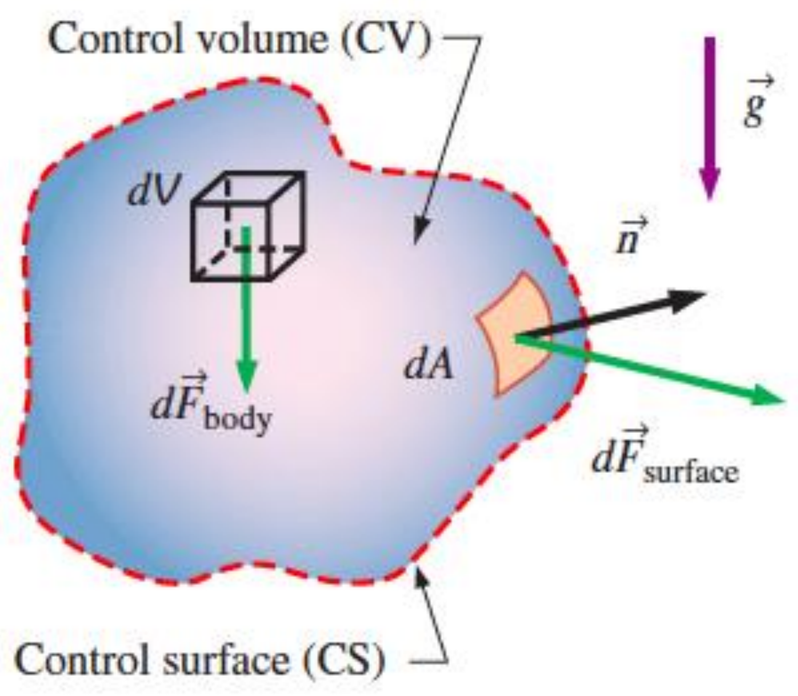


Understanding Laminar and Turbulent Flow

Overview

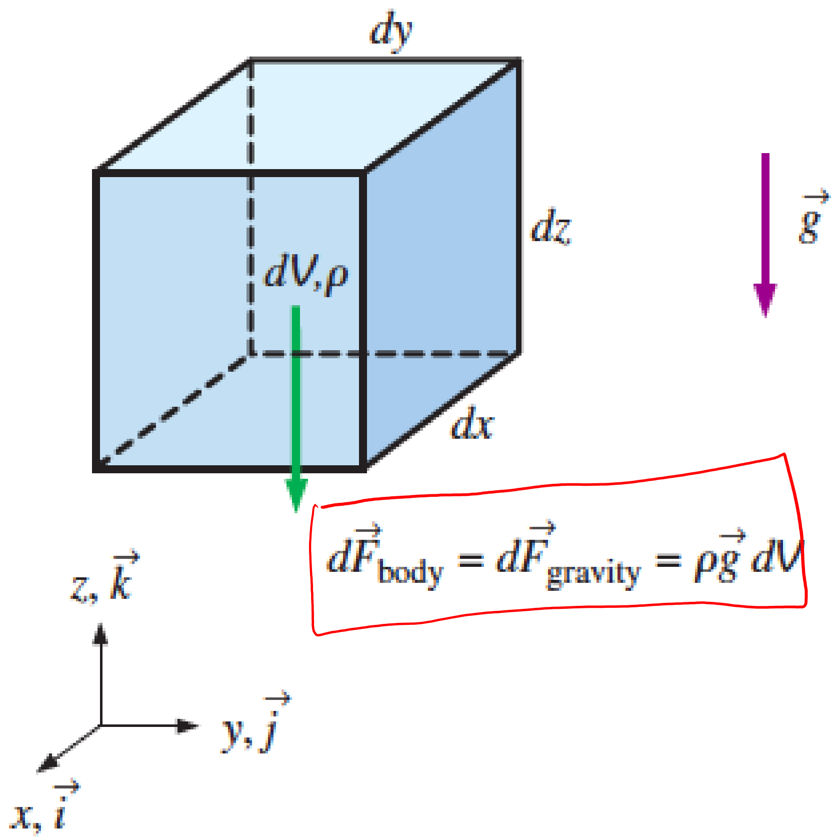
- References: chap. 5 Çengel, chap. 2 Faber and chap. 4 Acheson.
- Fluid dynamics deals with the relation between the motion of fluids considering the forces and moments which create the motion.
- Forces acting on fluid elements are of two types: body forces acting on the center of mass of the fluid element and surface forces acting through the surfaces.
- For ideal fluids (zero viscosity and compressibility) the surface forces reduce to an isotropic pressure (Pascal's Theorem) and the governing dynamical equation was derived by Euler.
- Euler's equation is a PDE that can be integrated along the streamlines and the integral is known as Bernoulli's equation (Bernoulli's first Theorem).

Forces on a control volume



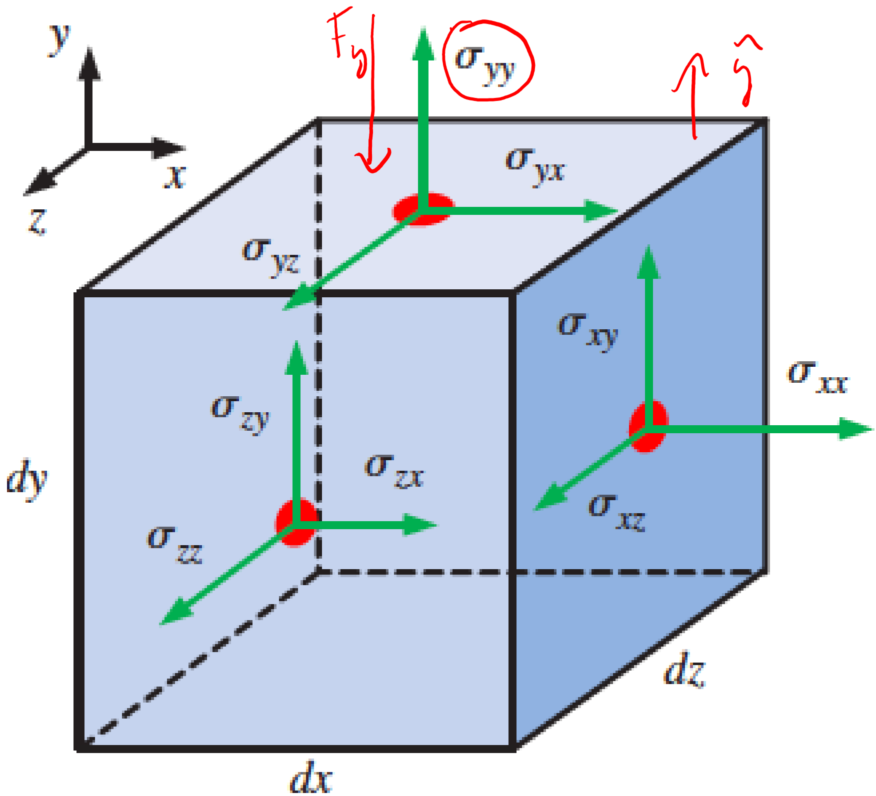
The forces acting on a control volume consist of body forces that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and surface forces that act on the control surface (such as pressure and viscous forces and reaction forces at points of contact).

Body forces: 1 vector or rank 1 tensor



$$\boxed{d\vec{F}_{\text{body}} = d\vec{F}_{\text{gravity}} = \rho \vec{g}\, dV}$$

Surface forces: 2 vectors or rank 2 tensor



σ_{ij}

σ_{ij} is the force per unit area in the direction of j through the plane (perpendicular to) i

Stress tensor in cartesian coordinates

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The diagonal components of the stress tensor are called normal stresses; they are composed of pressure (which always acts inwardly normal) and viscous stresses.
Viscous stresses are discussed in more detail later. The off-diagonal components, are called shear stresses; since pressure can act only normal to a surface, shear stresses are composed entirely of viscous stresses.

$$\sigma_{21} = -P \underbrace{\cancel{\delta_{21}}}_{0} + \dots$$

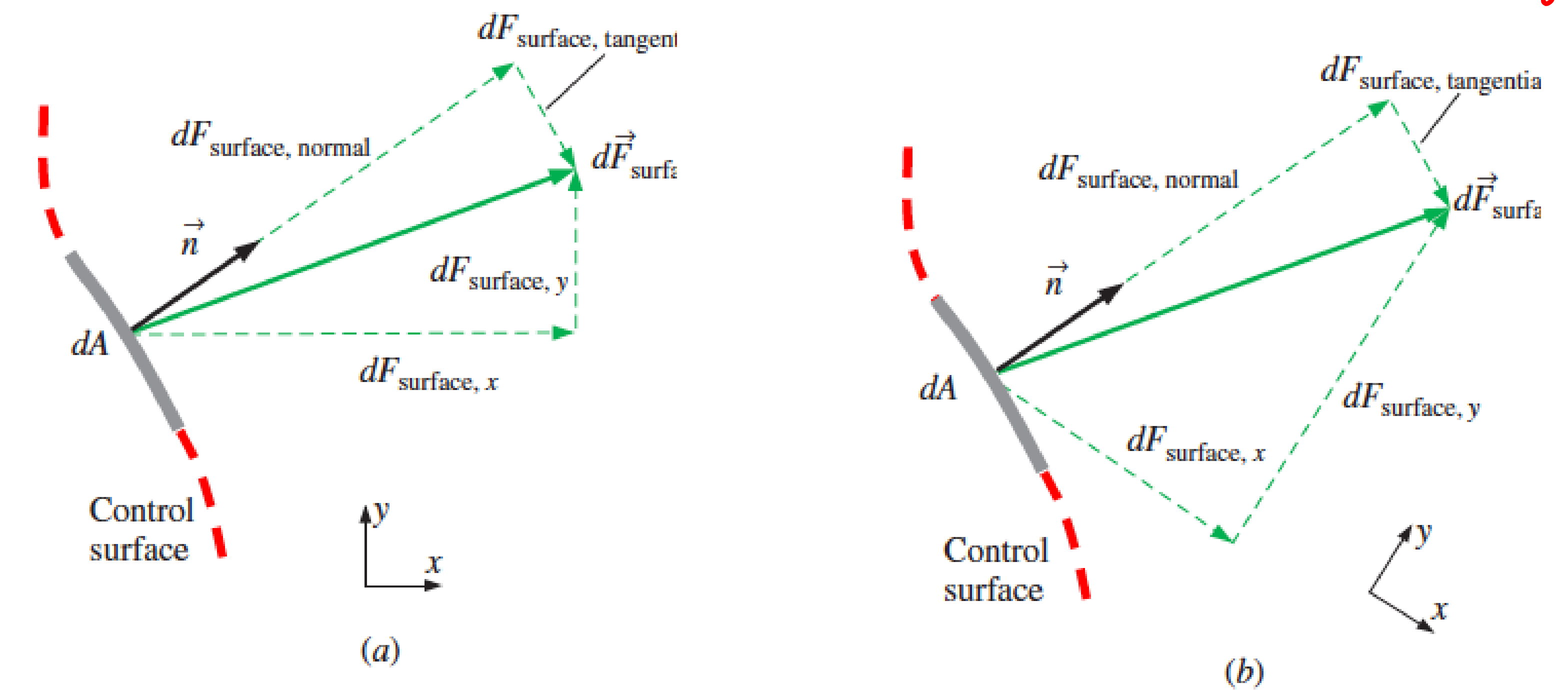
viscosidade

$$\sigma_{ij} = -P\delta_{ij} + \eta \underbrace{\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)}_{= 2\varepsilon_{ij}}$$

$$\delta_{ij} = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

Surface forces

$$A_i B_i = A_x B_x + A_y B_y + A_z B_z$$



$$dF_i = \sigma_{ij} n_j \, dA$$

Surface force acting on a differential surface element: $d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n} \, dA$

Total force

Body forces act on each volumetric portion of the control volume. The body force acts on a differential element of fluid of volume dV within the control volume, and we must perform a volume integral to account for the net body force on the entire control volume.

Total body force acting on control volume: $$\sum \vec{F}_{\text{body}} = \int_{\text{CV}} \rho \vec{g}\, dV = m_{\text{CV}} \vec{g}$$

Surface forces act on each portion of the control surface. A differential surface element of area dA and unit outward normal $\vec{n}$ is shown, along with the surface force acting on it. We must perform an area integral to obtain the net surface force acting on the entire control surface.

Total surface force acting on control surface: $$\sum \vec{F}_{\text{surface}} = \int_{\text{CS}} \sigma_{ij} \cdot \vec{n}\, dA$$

The linear momentum equation

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{sys}} \rho \vec{V}\, dV$$

- Therefore, Newton's second law can be stated as the sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system.

- Applying the Reynolds transport theorem we find

$$\frac{d(m\vec{V})_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho \vec{V}\, dV + \int_{\text{CS}} \rho \vec{V}\,(\vec{V}_r \cdot \vec{n})\, dA$$

$b \rightarrow \vec{V}$

$B = b \cdot m$

Newton's law for a control volume

General: $$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

which is stated in words as

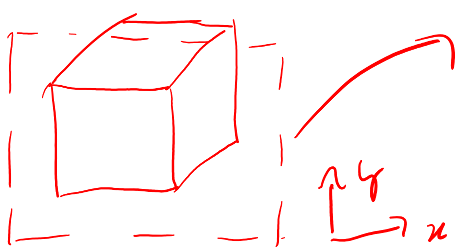
$$\begin{pmatrix} \text{The sum of all} \\ \text{external forces} \\ \text{acting on a CV} \end{pmatrix} = \begin{pmatrix} \text{The time rate of change} \\ \text{of the linear momentum} \\ \text{of the contents of the CV} \end{pmatrix} + \begin{pmatrix} \text{The net flow rate of} \\ \text{linear momentum out of the} \\ \text{control surface by mass flow} \end{pmatrix}$$

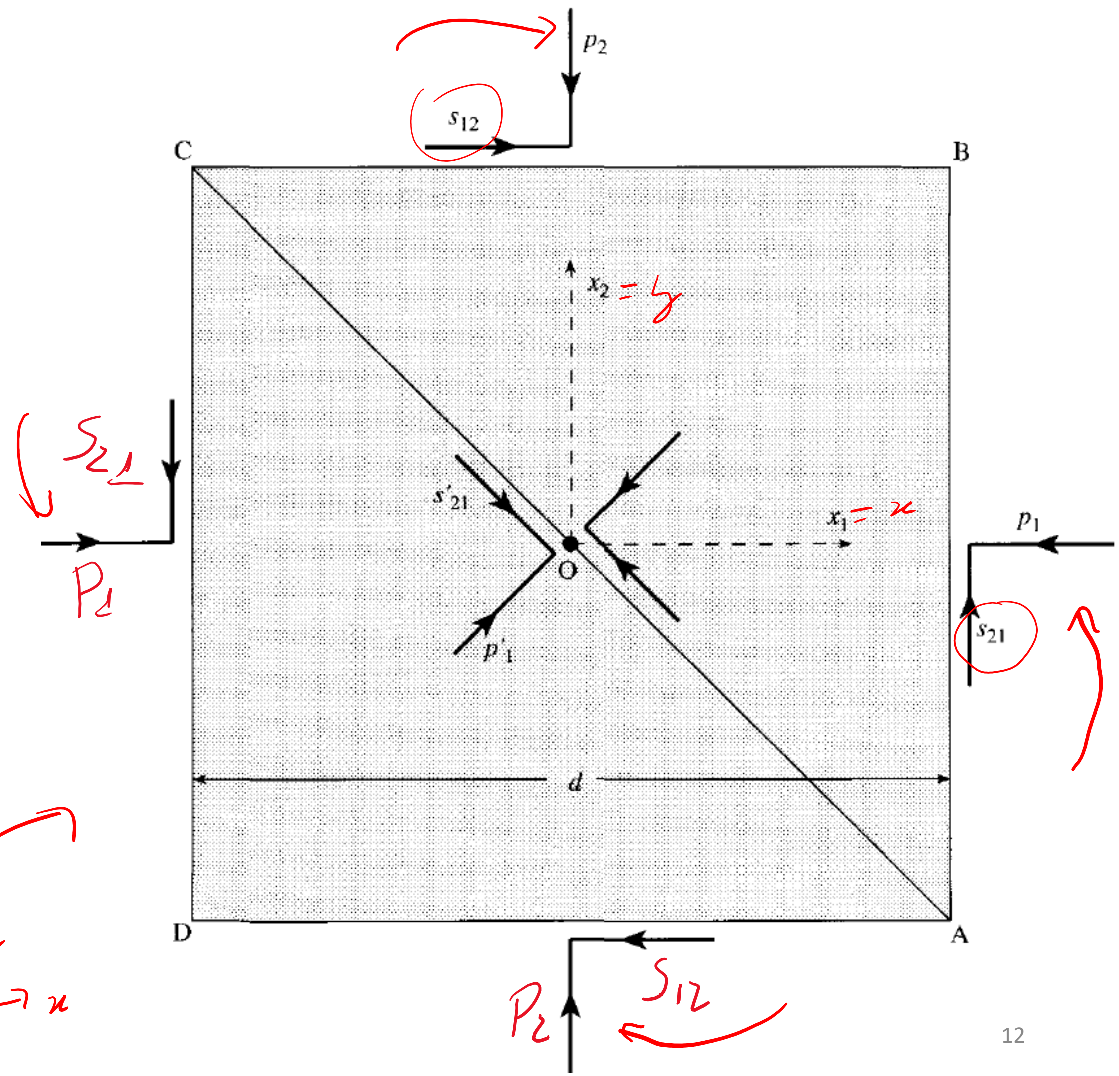
Fixed CV: $$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V} \cdot \vec{n})\, dA$$

The stress tensor is symmetric

(Sec 1.3, Faber)

- Choose axes paralel to the directions of a cubic fluid element (dashed lines)
- Consider a cubic fluid element of side *d*
- Consider one face of the cube, *ABCD.* The plane that passes through *AB* has a normal in the *x (1)* direction.



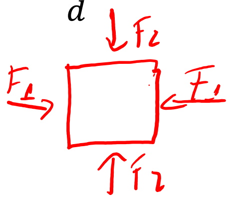


Balance of forces and torques

$$P = \frac{F}{A} = \frac{F}{d^2}$$

- Forces

Normal forces are: $\overbrace{d^2\, p_1}^{F_1}$, $\overbrace{d^2\, p_2}^{F_2}$ and $\overbrace{d^2\, p_3}^{F_3}$ with three other forces acting on the opposite faces. The forces act on an element of fluid of volume d^3. As $d \to 0$ the acceleration diverges as $\frac{1}{d}$ unless the forces on opposite faces of the cube balance.



$$\underbrace{\vec{F}}_{\sim d^2} = \underbrace{m}_{d^3} \cdot \vec{a} \Rightarrow \boxed{a \sim \frac{1}{d}} \quad \} \; m = \rho\, d^3$$

- Torques

$$\tau \sim d^3$$

Tangencial forces: $d^2\, s_{21}$ and $d^2\, s_{12}$ produce a torque $d^3\,(s_{21} - s_{12})$ which produces an angular acceleration that diverges as $\frac{1}{d^2}$ (the moment of inertia of the element of fluid scales with the 5th power of *d*) unless the torque vanishes. So $\boxed{s_{21} = s_{12} = s_3}$ (Faber's notation, 3 is the axis of rotation).

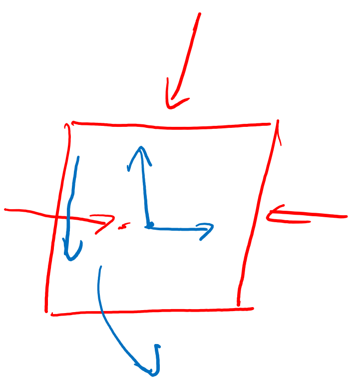
$$\tau = I \cdot \dot{\theta} \;, \quad I = \frac{m d^2}{6} \sim d^5 \Rightarrow \boxed{\dot{\theta} \sim \frac{1}{d^2}}$$

$$m = \rho \cdot d^3$$

Average pressure

- The average pressure is an invariant, i.e. it is the same for any rotation.

$$p = \frac{1}{3}(p_1 + p_2 + p_3)$$



$$\sigma_{ij} = \underbrace{\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}}_{9\ \text{comp}} \rightarrow 6\ \text{comp.}$$

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

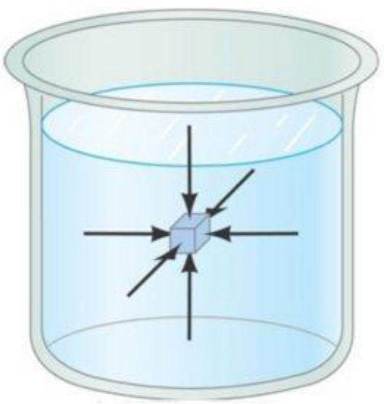
Fluids at mechanical equilibrium

$\Rightarrow \vec{v} = 0$

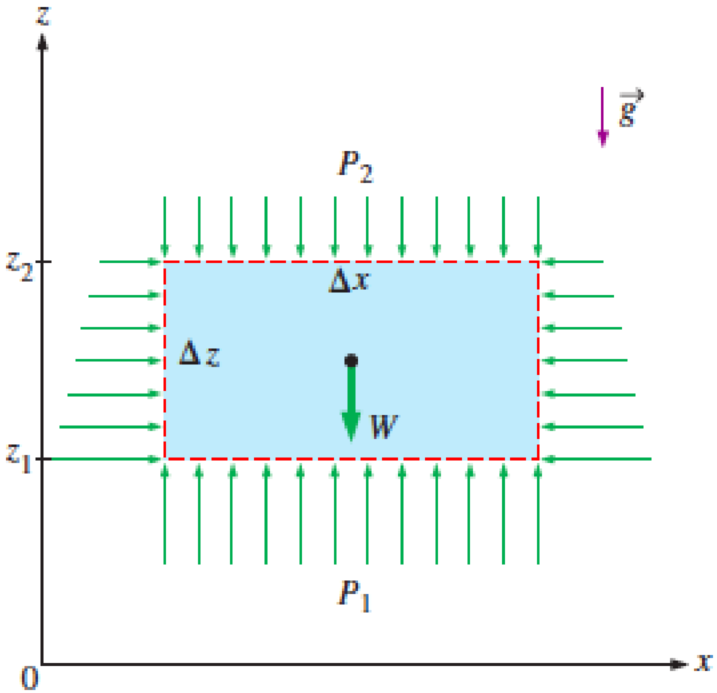
- At equilibrium the shear stresses vanish (otherwise there would be flow). This means that $p_1 = p_2 = p_3 = p$ in any frame of reference, i.e. the pressure is a scalar.

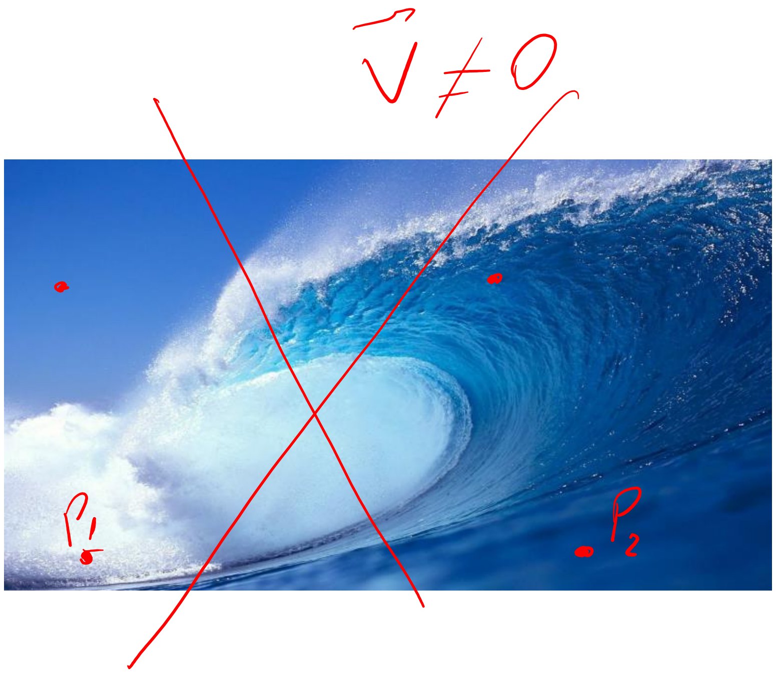
Pascal's principle

At mechanical equilibrium the pressure is a scalar field, $p(\vec{r})$.



Variation of pressure with depth

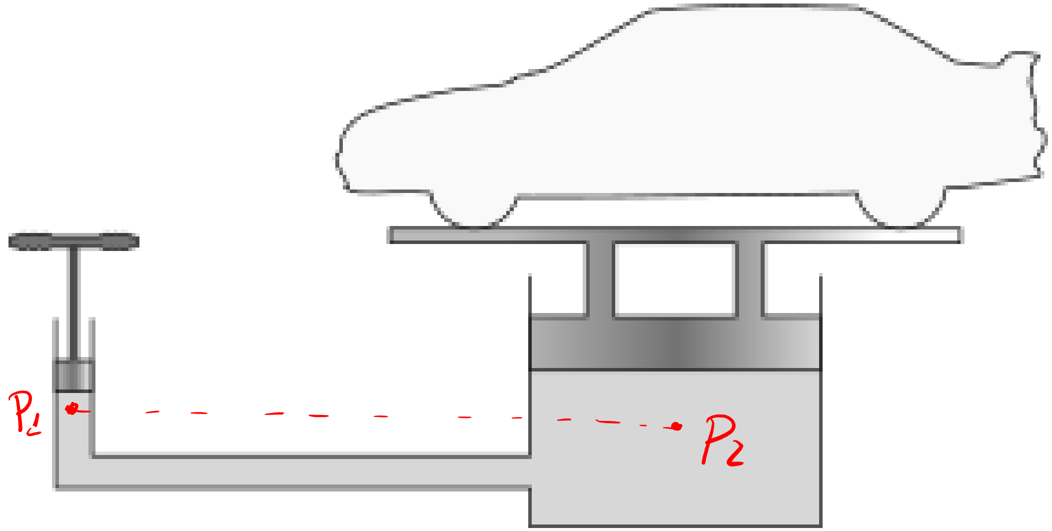




$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g \, dz$$

$$P = P_{atm} + \rho g h \quad \text{or} \quad P_{gage} = \rho g h$$

Hydraulic press



$$P_1 = P_2$$

$$\boxed{\frac{F_1}{A_1} = \frac{F_2}{A_2}}$$