Universo Primitivo
2022-2023 (1º Semestre)

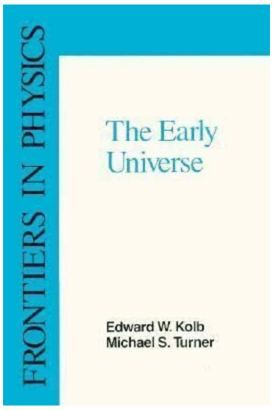
Mestrado em Física - Astronomia

Chapter 4

4 Decoupling

- Decoupling from local equilibrium;
  - Electroweak and week Interaction rates;
  - Particle distributions after decoupling;
  - Decoupling and Freeze-Out
- Neutrino decoupling;
- Electron-positron Annihilation;
- Cosmic Neutrino Background;
- Beyond thermal equilibrium: Boltzmann Equation

References



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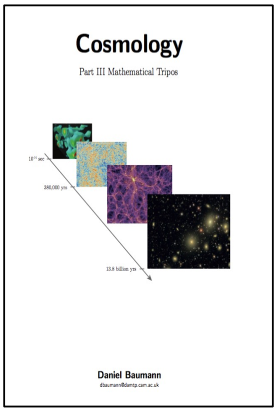

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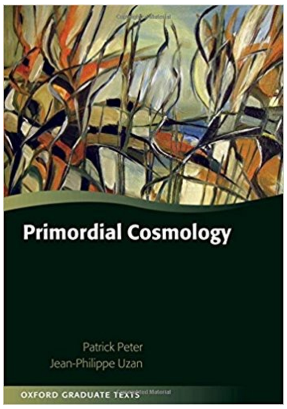

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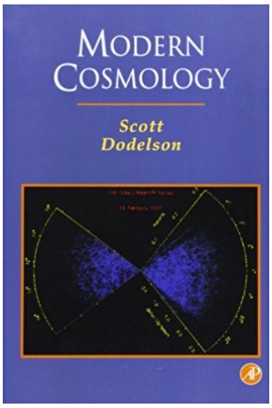

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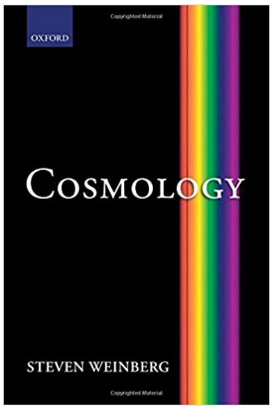


Ch. 3

Decoupling from equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

Thermal equilibrium of a fluid species can be established if the interaction rate, $\Gamma(t)$, is larger than the expansion rate, $H(t) = \dot{a}/a$:

$$\Gamma(t) \gg H(t)$$

The timescale for particle interactions, $t_c = 1/\Gamma$, is therefore much shorter than the characteristic timescale of expansion, $t_H = 1/H$:

$$t_c \equiv \frac{1}{\Gamma} \ll t_H \equiv \frac{1}{H}$$

The interaction rate is the number of interaction events of the species per unit of time. It is given by:

$$\boxed{\Gamma \equiv n\sigma v}$$

where $\boldsymbol{n}$ is the **number density of target particles**, $\boldsymbol{\sigma}$, is the **interaction cross section** and, $\boldsymbol{v}$, is the **relative speed** between particles.
The SI unit of Γ ``one over second'': $[\Gamma] = [n][\sigma][v] = s^{-1}$.

Decoupling from equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

For example, in the interaction process: $1 + 2 \leftrightarrow 3 + 4$ one has:

- $\Gamma_1 = n_2 \sigma v_{12}$ is the iteration rate of the particle species 1
- $\Gamma_2 = n_1 \sigma v_{21}$ is the iteration rate of the particle species 2 ($v_{21} = v_{12}$)
- $\Gamma_3 = n_4 \sigma v_{34}$ is the iteration rate of the particle species 3
- $\Gamma_4 = n_3 \sigma v_{43}$ is the iteration rate of the particle species 4 ($v_{43} = v_{34}$)

Reverting the equilibrium condition ($\Gamma \gg H$), one should expect that **a given particle specie has conditions to decouple** from the thermal bath when:

$$\frac{\Gamma}{H} \lesssim 1$$

For a relativistic fluid the expansion rate of the universe reads (SI):

$$H^2 = \frac{8\pi G}{3}\rho_r = \frac{\hbar c}{3M_{\rm pl}^2}\rho_r = \frac{\hbar c}{3M_{\rm pl}^2}\frac{\pi^2}{30}g_* T^4$$

where, $M_{\rm pl}$ is the **Planck Mass**:

$$\boxed{M_{\rm pl} \equiv \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18}\,\mathrm{GeV}}$$

Decoupling from thermal equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

From this, one obtains (in natural units):

$$H = \sqrt{\frac{\hbar c}{3M_{\rm pl}^2}\frac{\pi^2}{30}g_* T^4} = \pi\left(\frac{g_*}{90}\right)^{1/2}\frac{T^2}{M_{\rm pl}}$$

Let us now estimate the interaction rate, $\Gamma = n\sigma v$, for the **fluid of relativistic particles**:

- Since particles are relativistic: $v \sim c = 1$
- The number density in equilibrium is: $n_i \sim \frac{\zeta(3)}{\pi^2} g_i T^3 \propto T^3$
- The interaction cross section will depend on the type of interaction and mediators. For interactions mediated by bosons of mass m_X:

$$\sigma_X = \begin{cases} \alpha_X^2/T^2, & T \gg m_X \quad \text{(photon, massless bosons)} \\ \alpha_X^2 T^2/m_X^4, & T \ll m_X \quad \text{(massive bosons)} \end{cases}$$

where α_X is the generalized structure constant with the gauge boson X

Decoupling from thermal equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

So the equilibrium condition ratio, Γ/H, becomes:

$$\frac{\Gamma}{H} \simeq \frac{n\sigma}{H} = \frac{\frac{\zeta(3)}{\pi^2} g_i T^3}{\pi\left(\frac{g_*}{90}\right)^{1/2} T^2/M_{\rm pl}} \times \begin{cases} \alpha_X^2/T^2 & T \gg m_X \ \text{(photon, massless bosons)} \\ \alpha_X^2 T^2/m_X^4 & T \ll m_X \ \text{(massive bosons)} \end{cases}$$

Implications:

1. **At high temperature** ($T \gtrsim 100$ GeV – the electroweak symmetry breaking $m_{W^\pm,Z,H} \sim 100\text{GeV}$) all interactions are mediated by massless Gauge bosons (γ, W, Z, g). For the electroweak interaction, $\alpha^2 = 0.01$, this gives:

$$\boxed{\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{\rm pl}}{T} \sim \frac{10^{16}\,\text{GeV}}{T}}$$

So thermal equilibrium is possible via electroweak interactions for

$$100\ \text{GeV} \lesssim T \lesssim 10^{16}\ \text{GeV}$$

Above the 10^{16} GeV (the **Grand Unification Theory**) scale the Universe is not able to acquire an equilibrium state via the electroweak interaction. Actually it may have never been in thermal equilibrium!

Decoupling from thermal equilibrium

Equilibrium condition, interaction timescale, and interaction rate:

So the equilibrium condition ratio, Γ/H, becomes:

$$\frac{\Gamma}{H} \simeq \frac{n\sigma}{H} = \frac{\frac{\zeta(3)}{\pi^2} g_i T^3}{\pi\left(\frac{g_*}{90}\right)^{1/2} T^2/M_{\rm pl}} \times \begin{cases} \alpha_X^2/T^2 & T \gg m_X \ \text{(photon, massless bosons)} \\ \alpha_X^2 T^2/m_X^4 & T \ll m_X \ \text{(massive bosons)} \end{cases}$$

2. **At lower temperature** ($T \lesssim 100$ GeV – below the electroweak symmetry breaking) the weak interaction between relativistic particles is mediated by massive bosons, m_X. One has:

$$\boxed{\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{\rm pl} T^3}{M_W^4} \sim \left(\frac{T}{1\ \text{MeV}}\right)^3}$$

Which **drops below unity** $\boldsymbol{T \lesssim 1}$ **MeV.**

So relativistic particles interacting via weak force (e.g. neutrinos) are able to remain in equilibrium with the fluid in the temperature range:

$$1\ \text{MeV} \lesssim T \lesssim 100\ \text{GeV}$$

Below this temperature they should decouple from the fluid.

Thermal history of the Universe:

Key events in the thermal history of the universe

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

Electroweak interacting species may attain thermal equilibrium up to the GUT 10^{16}GeV scale

Particle species interacting via the weak force have conditions to attain thermal equilibrium

Decoupling from thermal equilibrium

Particle distributions after decoupling

Let us now study what happens to the phase space distribution of a given particle species, $f(x, p, t > t_D)$, after that species decouples from the fluid at time t_D.

The **number of particles** within the volume element $dV d^3p$ around the point (x, p) of the phase space is:

$$dN = f(x, p, t) dV d^3p$$

If no particles are created or destroyed after decoupling, the left hand side of this equation remains constant. On the right hand side, we know that the volume element element scales with a^3. For the momentum we have:

$$E^2 = m^2 + p^2$$

- For massless particles (e.g. radiation) momentum scales as energy;
- For massive (non-relativistic) particles we also have that momentum scales as energy (m is the rest mass of the particle species).

So, for massless or massive particles:

$$\boxed{p \propto E = hc/\lambda \propto a^{-1}}$$

Decoupling from thermal equilibrium

Particle distributions after decoupling

Let us now study what happens to the phase space distribution of a given particle species, $f(x, p, t > t_D)$, after that species decouples from the fluid at time t_D.

The **number of particles** within the volume element $dV d^3p$ around the point (x, p) of the phase space is:

$$dN = f(x, p, t) dV d^3p$$

Is constant if no particles are created/destroyed after decoupling

a^3 $\quad a^{-3}$

The shape of the distribution function should not change after decoupling… However f depends on x and p, so it's argument scales with time as a^{-1}. We can then relate the **momentum at decoupling**, p_D , with the **momentum after decoupling**, p, as:

$$\boxed{\frac{x_D}{x} = \frac{a_D}{a} \Leftrightarrow x_D = x\frac{a_D}{a}} \qquad \boxed{\frac{p_D}{p} = \frac{a_D^{-1}}{a^{-1}} \Leftrightarrow p_D = p\frac{a}{a_D}}$$

where a_D is the scale factor at decoupling $a = a(t_D)$.

Decoupling from thermal equilibrium

Particle distributions after decoupling

So the **distribution function at decoupling**, t_D, can be written as

$$f(x_D, p_D, t_D) = f(x\frac{a_D}{a}, p\frac{a}{a_D}, t_D)$$

The **right-hand side** of the equation is the **distribution function after decoupling**. Dropping x (because f is independent of position) one has:

$$\boxed{f(x, p, t \geq t_D) = f(p(t)\frac{a(t)}{a_D}, t \geq t_D)}$$

So, depending on the relativistic state of particles, one has two possibilities:

- **The species decouples while is relativistic** (e.g. massless neutrinos)

$$f(p, t \geq t_D) = \frac{g}{2\pi^3}\frac{1}{\exp(\frac{pa/a_D}{T_D}) \pm 1} = \frac{g}{2\pi^3}\frac{1}{\exp(\frac{p}{T}) \pm 1}$$

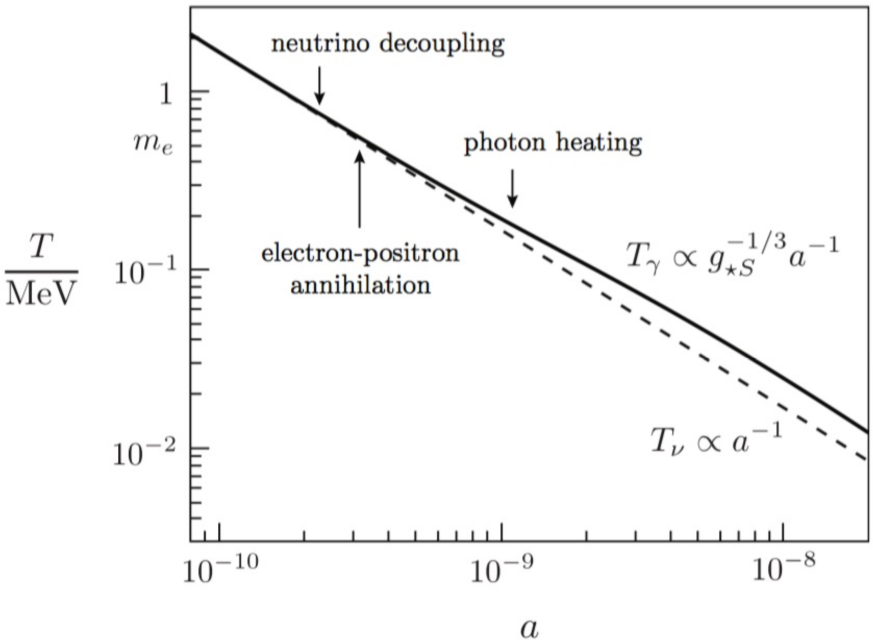
So the functional form of f remains the same if T scales as:

$$\boxed{T(t) = T_D\frac{a_D}{a(t)} = \frac{T_D}{(1+z_D)}(1+z)}$$

Decoupling from thermal equilibrium

Particle distributions after decoupling

We therefore conclude that **the temperature of decoupled relativistic species** also scales with the inverse of the scale factor ($T = T_D\, a_D/a$) **as** it happens **for relativistic species in thermal equilibrium** ($T_\gamma \propto g_{*S}^{-1/3}\, a^{-1}$).



However when a species decouples, g_{*S}, decreases and therefore the temperature of the fluid, T_γ, decreases at a lower rate then the temperatures of the decoupled species.

Decoupling from thermal equilibrium

Particle distributions after decoupling

The ***number density*** *of a relativistic species* ***after decoupling*** scales as:

$$n_i = g_i \frac{\zeta(3) T_D^3}{\pi^2}\left(\frac{a_D}{a}\right)^3 \longrightarrow \boxed{n_i = n_{i,D}\left(\frac{a_D}{a}\right)^3}$$

- **The species decouples while it is non-relativistic** ($m \gg T$)

$$f(x,p,t \geq t_D) = f\left(p\frac{a(t)}{a_D}, t_D\right) \simeq \frac{g_i}{2\pi^3}\exp\left(-\frac{\sqrt{(pa/a_D)^2+m^2}+\mu_D}{T_D}\right)$$

$$\simeq \frac{g_i}{2\pi^3}\exp\left(-\frac{p^2}{2mT_D}\left(\frac{a}{a_D}\right)^2 - \frac{m}{T_D}+\frac{\mu_D}{T_D}\right)$$

$$\simeq \frac{g_i}{2\pi^3}\exp\left(-\frac{p^2}{2mT} - \frac{m}{T}+\frac{\mu}{T}\right)$$

this distribution has the same functional form of a distribution of *non-relativistic particles with temperature and chemical potential* given by:

$$\boxed{T(t) = T_D\left(\frac{a_D}{a(t)}\right)^2 = \frac{T_D}{(1+z_D)^2}(1+z)^2} \qquad \boxed{\mu(t) = m + (\mu_D - m)\frac{T(t)}{T_D}}$$

Decoupling from thermal equilibrium

Particle distributions after decoupling

Using these scalings in the expressions for the number density of a non-relativistic species one obtains that, after decoupling the number density scales as:

$$n_i = g_i \left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}$$

$$= g_i \left(\frac{mT_D}{2\pi}\right)^{3/2} \left(\frac{a_D}{a}\right)^3 e^{-(m-\mu_D)/T_D}$$

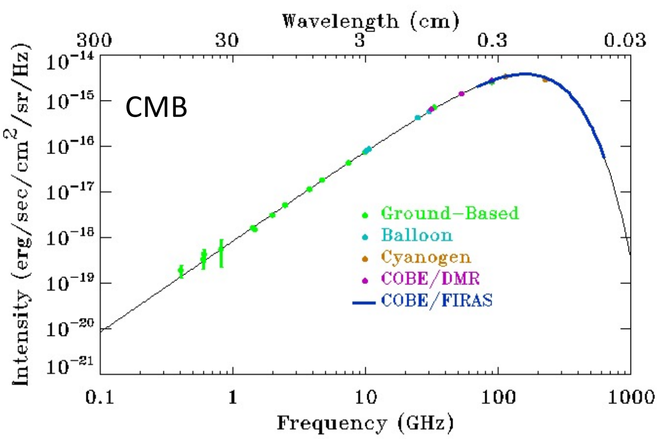
this means that if a particle species decouples when it is non-relativistic, its number density also scales as:

$$\boxed{n_i = n_{i,D}\left(\frac{a_D}{a}\right)^3}$$

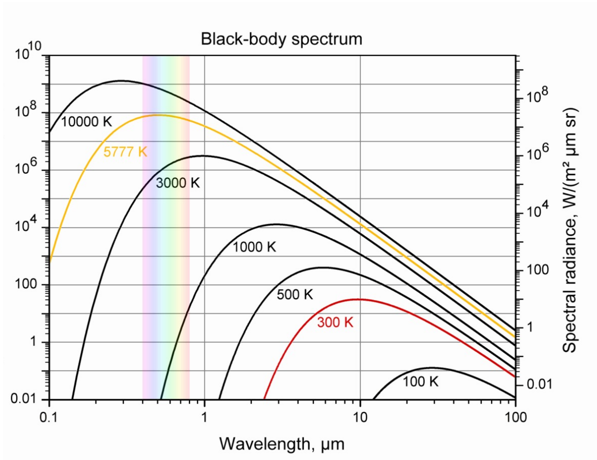
Decoupling from thermal equilibrium

Particle distributions after decoupling

Example: photons (relativistic)



$T_{\gamma,0} = 2.726$



$T_{\gamma,D} = 2.726\,(1+z_D) \sim 3000$ K

Relativistic:

$$\boxed{T(t) = T_D\frac{a_D}{a(t)} = \underbrace{\frac{T_D}{(1+z_D)}}_{T_0}(1+z)}$$

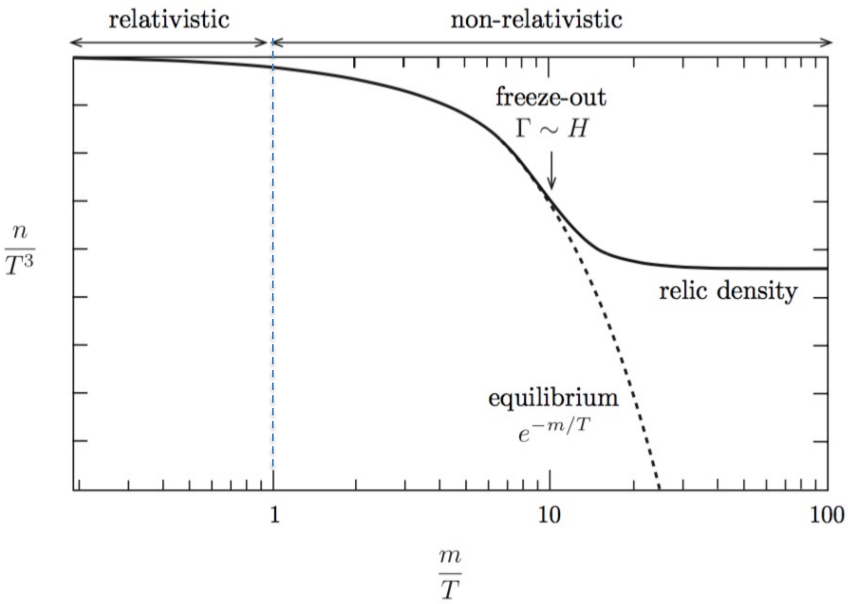
Non-Relativistic:

$$\boxed{T(t) = T_D\left(\frac{a_D}{a(t)}\right)^2 = \underbrace{\frac{T_D}{(1+z_D)^2}}_{T_0}(1+z)^2}$$

Decoupling from thermal equilibrium

Decoupling and Freeze-out

As **massive particles decouple** their abundances are **Boltzmann suppressed by $e^{-m/T}$**. While relativistic, for $T \gg m$, one should expect that n_i/T^3 is constant (because $n_i \propto \mathrm{T}^3$). However **these predictions assume that the decoupling species is always in equilibrium** as its density is being supressed. **But this hypothesis cannot hold at very low temperatures**, $T \ll m$, because particle abundances become too small to be able to achieve equilibrium.



At high enough m/T one should expect that the real number density departures from the equilibrium prediction:

$$\boxed{n_{eq} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m/T)}}$$

In fact beyond m/T larger than ~10 the ratio n_{non_eq}/T^3 becomes constant again. The density, n_{non_eq}, is the **non-equilibrium Freeze-Out density**.

Decoupling from thermal equilibrium

Neutrino decoupling:

Neutrinos are coupled to the thermal bath via weak interaction processes like:

$$\nu_e + \bar{\nu}_e \leftrightarrow e^+ + e^- \qquad e^+ + e^- \leftrightarrow \gamma + \gamma$$

$$e^- + \bar{\nu}_e \leftrightarrow e^- + \bar{\nu}_e$$

At 10 MeV, photons, **neutrinos**, **electrons** (and their antiparticles) are the only remaining particles of the relativistic fluid. Then, g_*,reads:

$$g_* = \underset{\gamma}{2} + \frac{7}{8}(2 \times \underset{e^\pm}{2} + 3 \times \underset{\nu}{2}) = 10.75$$

Using this in the Friedman equations

$$H = \sqrt{\frac{8\pi G}{3}}\rho^{1/2} = \sqrt{\frac{8\pi G}{3}}\left(g_* \frac{\pi^2}{30} T^4\right)^{1/2} \approx 5.44\sqrt{8\pi}\frac{T^2}{m_{\mathrm{Pl}}}$$

Combining with the expression for Γ one concludes that neutrinos decouple below at **about 1 MeV** (accurate calculation yields $T = 0.8$ MeV) .

$$\boxed{\frac{\Gamma}{H} \sim \frac{\alpha^2 M_{\mathrm{pl}} T^3}{M_W^4} \sim \left(\frac{T}{1\ \mathrm{MeV}}\right)^3}$$

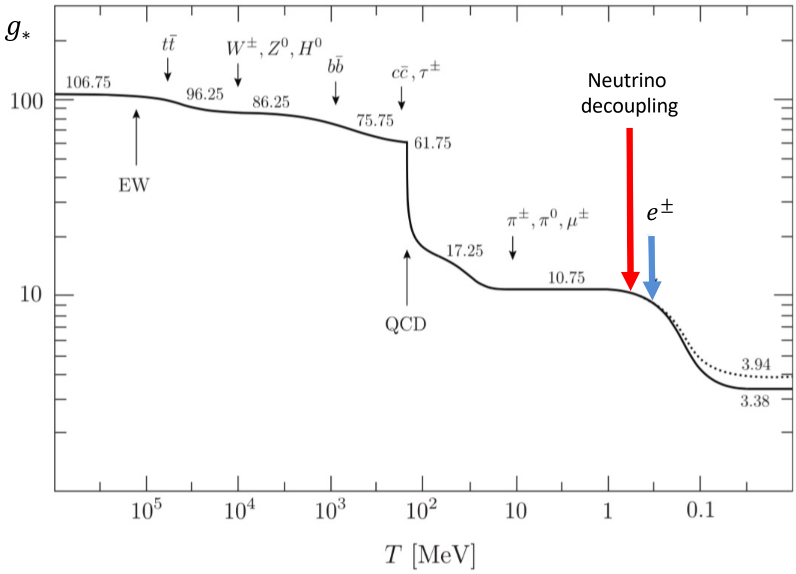
Decoupling from thermal equilibrium

Electron-positron annihilation

Electron-positron annihilation occurs soon after the neutrino decoupling. In fact, as soon as $T \lesssim 1.022$ MeV electron-positron pair creation becomes less effective, and the interaction

$$e^+ + e^- \leftrightarrow \gamma + \gamma$$

progressively moves to the right (more pairs $e^-/\,e^+$ being destroyed than created).



- Neutrino decoupling occurs around $T \sim 0.8$ MeV;
- $e^-/\,e^+$ annihilation occurs around $T \sim 0.5$ MeV, with a transition $0.1 \lesssim T/\mathrm{MeV} \lesssim 1$
- But these processes partially overlap: **neutrino decoupling is not over when electro-positron annihilation starts**

Decoupling from thermal equilibrium

Electron-positron annihilation

Let us now compute the effective degrees of freedom of relativistic particles before neutrino decoupling and after electron-positron annihilation.

Before neutrino decoupling ($T \gtrsim 1$ MeV):
Relativistic particles species are the γ, $e^\pm$ and ν_s, so:

$$g_{*S} = g_* = 2 + \frac{7}{8}(2 \times 2 + 3 \times 2) = 10.75$$

After electron-positron annihilation ($T \lesssim 0.5$ MeV):
Relativistic particles species are just the γ and ν_s (note that neutrinos are already decoupled but they remain relativistic and therefore contribute to the entropy):

$$g_{*S} = 2 + \frac{7}{8}(3 \times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}$$

$$g_* = 2 + \frac{7}{8}(3 \times 2)\left(\frac{T_\nu}{T_\gamma}\right)^4_{\text{after}}$$

Since entropy is conserved one has:

$$S = sV = \frac{2\pi}{45} g_{*S}(T_\gamma a)^3 = const.$$

Decoupling from thermal equilibrium

Electron-positron annihilation

So one can write:

$$g_{*S}(T_\gamma a)^3_{\text{before}} = g_{*S}(T_\gamma a)^3_{\text{after}} \Leftrightarrow$$

$$\Leftrightarrow \left(2+\frac{7}{8}(2\times 2+3\times 2)\right)(T_\gamma a)^3_{\text{before}} = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)(T_\gamma a)^3_{\text{after}} \Leftrightarrow$$

$$\Leftrightarrow \frac{43}{4}(T_\gamma)^3_{\text{before}} = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)(T_\gamma)^3_{\text{after}}\left(\frac{a_{\text{after}}}{a_{\text{before}}}\right)^3$$

But after decoupling neutrino temperature scales as: $T_{\nu,\text{before}}/T_{\nu,\text{after}} = a_{\text{after}}/a_{\text{before}}$.
Moreover, since $T_{\nu,\text{before}} = T_{\gamma,\text{ before}}$, one has:

$$\frac{43}{4}(T_\gamma)^3_{\text{before}} = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)(T_\gamma)^3_{\text{after}}\left(\frac{T_{\nu,\text{before}}}{T_{\nu,\text{after}}}\right)^3 \Leftrightarrow$$

$$\frac{43}{4}(T_{\gamma,\text{before}})^3 = \left(2+\frac{7}{8}(3\times 2)\left(\frac{T_\nu}{T_\gamma}\right)^3_{\text{after}}\right)\left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}}(T_{\nu,\text{before}})^3 \Leftrightarrow$$

$$\Leftrightarrow \frac{43}{4}(\cancel{T_{\gamma,\text{before}}})^3 = \left(2\left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}}+\frac{7}{8}(3\times 2)\right)(\cancel{T_{\gamma,\text{before}}})^3 \Leftrightarrow$$

Decoupling from thermal equilibrium

Electron-positron annihilation

From which one concludes that:

$$\frac{43}{4} = 2\left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}} + \frac{7}{8}(3\times 2) \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{T_\gamma}{T_\nu}\right)^3_{\text{after}} = \frac{1}{2}\left(\frac{43}{4}-\frac{21}{4}\right) = \frac{11}{4}$$

So after $e^-/\,e^+$ annihilation the neutrino temperature is somewhat smaller than the the photon temperature:

$$\boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma}$$

With this result one can estimate the relativistic degrees of freedom for $T \lesssim 0.5$ MeV:

$$g_{*s} = 2+\frac{7}{8}\times 3\times 2\left(\frac{4}{11}\right) = 3.91,$$

$$g_{*} = 2+\frac{7}{8}\times 3\times 2\left(\frac{4}{11}\right)^{4/3} = 3.36$$

Which explains the difference between g_{*S} and g_* at low temperatures

Decoupling from thermal equilibrium

Electron-positron annihilation

However, **this result is only valid if the decoupling of neutrinos was instantaneous** (and happened before the beginning of the electron-positron annihilation).
But **these processes are not instantaneous** and in fact overlap in time. Part of the energy and entropy of the electron-positron annihilation leaks into the remaining relativistic species, increasing their temperature, via an decrease of g_{*S} and g_* (as discussed in slide 13).

A more accurate computation (taking into account the variation of $g_{*S}(T)$ and $g_*(T)$) gives:

$$\boxed{g_* = 3.38\,;\, g_{*S} = 3.94}$$

In order to keep the calculation of the effective degrees of freedom simple, it is usual to define a quantity, $\boldsymbol{N_{\mathrm{eff}}}$, known as **effective number of neutrino species**, so that:

$$g_* = 2 + \frac{7}{8} \times 3 \times 2\left(\frac{4}{11}\right)^{4/3} = 3.36$$

$$g_{*s} = 2 + \frac{7}{8} \times 3 \times 2\left(\frac{4}{11}\right) = 3.91$$

$$\boxed{\begin{aligned} g_\star &= 2 + \frac{7}{8} \times 2N_{\mathrm{eff}}\left(\frac{4}{11}\right)^{4/3} = 3.38 \\ g_{\star S} &= 2 + \frac{7}{8} \times 2N_{\mathrm{eff}}\left(\frac{4}{11}\right) = 3.94\,, \end{aligned}}$$

Where $N_{\mathrm{eff}} = 3.046$ (N_{eff} is by itself a parameter that can be fit by CMB observations). If neutrino decoupling was instantaneous $N_{\mathrm{eff}} = 3$.

Decoupling from thermal equilibrium

Cosmic Neutrino Background

A **Cosmic Neutrino background** ($\mathrm{C}_\nu\mathrm{B}$) should be present in the universe since decoupling. Its temperature should scale with with the inverse of scale factor and it is related to the CMB temperature (which also scales with the inverse of the scale factor) as:

$$T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,0} = \left(\frac{4}{11}\right)^{1/3} 2.73 = 1.95\,\mathrm{K}$$

(which corresponds to $T_{\nu,0} = 0.17\ \mathrm{meV}$).

Plugging this result in the expression of the neutrino number and energy densities one obtains:

$$\boxed{n_\nu = \frac{3}{4}N_{\mathrm{eff}} \times \frac{4}{11}n_\gamma}$$

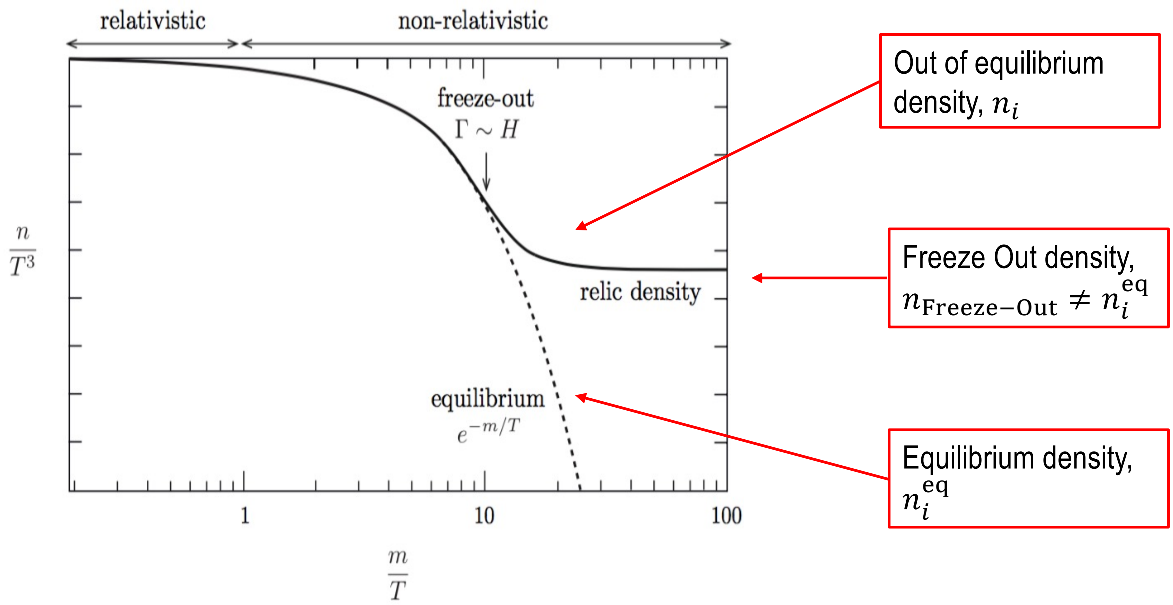
$$\boxed{\rho_\nu = \frac{7}{8}N_{\mathrm{eff}}\left(\frac{4}{11}\right)^{4/3}\rho_\gamma}$$

Assuming $N_{\mathrm{eff}} = 3.046$ and the observed values of the CMB densities, one obtains $n_{\nu,0} = 112\ \mathrm{cm}^{-3}$ and $\Omega_{\nu,0} = 0.00014$ (assuming massless neutrinos).

Decoupling from thermal equilibrium

Beyond thermal equilibrium: the Boltzmann Equation

Equilibrium quantities such as density expression derived in the previous chapter **assume that the decoupling species is always in equilibrium** as its density is being supressed. In reality **this hypothesis cannot hold at very low temperatures**, $T \ll m$, because particle abundances become too small to be able to achieve equilibrium. The formal way of computing **out of equilibrium densities** is through the use of the **Boltzmann equation** (see next slides)



Decoupling from thermal equilibrium

Boltzmann Equation

In the **absence of interactions** the number density of the **decoupled species, *i***, obeys to the conservation equation:

$$\frac{dn_i}{dt} + 3\frac{\dot{a}}{a}n_i = 0$$

This follows from the fact that the number of particles in a physical volume remains constant after decoupling $n_i = n_{0,i}\, a^{-3}$.

One can **generalise** this expression to **include interactions**:

$$\boxed{\frac{1}{a^3}\frac{d(n_i a^3)}{dt} = C_i[\{n_j\}]}$$

Where the left hand side is equivalent to the above equation and in the right hand side one adds a collision term, $C_i[\{n_j\}]$, that accounts for sinks / sources of the density of the species n_i due to interaction (collisions) with other species n_j.

The latter equation is known as the **collisional Boltzmann equation**.

When $C_i[\{n_j\}] = 0$ one obtains the **collisionless Boltzmann equation** (on the top)

Decoupling from thermal equilibrium

Boltzmann Equation

The form of the collision term $\boldsymbol{C_i[\{n_j\}]}$ **depends on the type of interaction**. For interactions of 2 particles species (3 body interactions are in principle much less likely):

$$1 + 2 \rightleftarrows 3 + 4$$

(this means that species 1 annihilates with species 2, giving rise to species 3 and 4. Conversely species 3 and 4 annihilate back to species 1 and 2).

To follow the out of equilibrium evolution of, for example, n_1 one needs to take into account the balance of efficiency of the reaction between 1 and 2, that originates a sink of density, and the (reverse) reaction between 3 and 4, that originates a source of n_1. This can be translated into the collisional Boltzmann equation by replacing the collision term $C_i[\{n_j\}]$ with 2 terms:

$$\boxed{\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\alpha\, n_1 n_2 + \beta\, n_3 n_4}$$

where $\boldsymbol{\alpha n_1 n_2}$ **is a sink term describing the destruction of particles** (due to the reaction to the right) and $\boldsymbol{\beta n_3 n_4}$ **is a source term describing the creation of particles** of type 1 (due to the reaction to the left).
Naturally, each term should be proportional to the densities of each pair.

Decoupling from thermal equilibrium

Boltzmann Equation

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\alpha\, n_1 n_2 + \beta\, n_3 n_4$$

Now, the parameters α and β can be written as:

- $\alpha = \langle \sigma v \rangle$ is the **thermally averaged cross section** ($\alpha n_1 n_2 = \Gamma_1 n_1$)

- β needs to be related to α so that the right hand side of the equation vanishes when particles are in equilibrium.

$$\beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}} \alpha$$

Where the densities inside the parenthesis are equilibrium densities n_i^{eq}. Thus:

$$\boxed{\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = -\langle \sigma v \rangle \left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{\mathrm{eq}} n_3 n_4 \right]}$$

Decoupling from thermal equilibrium

Boltzmann Equation

It is instructive to write this equation in terms of the number of particles in a commoving volume, defined in chapter 3:

$$N_i \equiv \frac{n_i}{s}$$

which is a conserved quantity (i.e. as long as the average number of particles is conserved) resulting from the entropy conservation equation. Using $n_i = N_i s$ one finds:

$$\boxed{\frac{d\ln N_1}{d\ln a} = -\frac{\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{\rm eq}\frac{N_3 N_4}{N_1 N_2}\right]}$$

Note that:

- The factor Γ_1/H describes the **interaction efficiency**.
- The 2nd term in the parenthesis characterises **deviations from equilibrium**.
- **If $\boldsymbol{\Gamma_1 \ll H}$**, the r.h.s of the equation is supressed and N_1 **freezes out**.
- **If $\boldsymbol{\Gamma_1 \gg H}$ equilibrium is rapidly established**. For example, if $N_1 > N_1^{\rm eq}$ the r.h.s becomes negative (more particles are destroyed). If $N_1 < N_1^{\rm eq}$ the r.h.s becomes positive (more particles are created). Both of these effects push $N_1 \to N_1^{\rm eq}$.

Decoupling from thermal equilibrium

Boltzmann Equation (see, Peter & Uzan Section 4.2.2.1)

Evolution of the distribution function

The evolution of the distribution function is obtained from the Boltzmann equation

$$L[f] = C[f], \tag{4.49}$$

where C describes the collisions and $L = \mathrm{d}/\mathrm{d}s$ is the Liouville operator, with s the length along a worldline. The operator L is a function of eight variables taking the explicit form

$$L[f] = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}. \tag{4.50}$$

In a homogeneous and isotropic space-time, f is only a function of the energy and time, $f(E,t)$, so that

$$L[f] = E\frac{\partial f}{\partial t} - Hp^2 \frac{\partial f}{\partial E}. \tag{4.51}$$

Using the definition (4.11) of the particle density, and integrating this equation with respect to the momentum $\boldsymbol{p}$, we obtain[7]

$$\dot{n}_i + 3Hn_i = C_i, \qquad C_i = \frac{g_i}{(2\pi)^3}\int C\left[f_i(\boldsymbol{p}_i, t)\right]\frac{\mathrm{d}^3 \boldsymbol{p}_i}{E_i}. \tag{4.52}$$

Decoupling from thermal equilibrium

Boltzmann Equation (see, Peter & Uzan Section 4.2.2.1)

The difficult part lies in the modelling and the evaluation of the collision term. Here, we restrain ourselves to the simple case of an interaction of the form

$$i + j \longleftrightarrow k + l, \tag{4.53}$$

for which the collision term can be decomposed as $C_i = C_{kl\to ij} - C_{ij\to kl}$ with

$$C_{ij\to kl} = (2\pi)^4 \int \frac{g_i \mathrm{d}^4 p_i}{(2\pi)^3}\delta(E_i - p_i^2 - m_i^2)\cdots\frac{g_l \mathrm{d}^4 p_l}{(2\pi)^3}\delta(E_l - p_l^2 - m_l^2)$$
$$\times\delta^{(4)}(p_i + p_j - p_k - p_l)|\mathcal{M}|^2_{ij\to kl} f_i f_j (1 \pm f_k)(1 \pm f_l), \tag{4.54}$$

with a + sign for bosons and a − sign for fermions. $|\mathcal{M}|_{ij\to kl}$ are the matrix elements describing the interaction. The Dirac delta function imposes the conservation of momentum and of energy. This form also shows that the probability for i to disappear is proportional to $f_i f_j$, i.e. roughly to the density of the interacting species.[8] The factors $(1 \pm f_k)$ arise from quantum mechanics and are related to the Pauli exclusion principle for fermions and to stimulated emission for bosons.

If $\hat{C}\hat{P}$ invariance holds, as we assume here, then $C_{kl\to ij}$ and $C_{ij\to kl}$ involve a unique matrix element, $|\mathcal{M}|^2$, determined by the physical process. Indeed, this invariance implies that the process we consider is reversible and thus that $i + j \to k + l$ and $k + l \to i + j$ have the same matrix elements. It follows that

$$C_i = (2\pi)^4 \int \delta^{(4)}(p_i + p_j - p_k - p_l)\frac{g_i \mathrm{d}^3 p_i}{2(2\pi)^3 E_i}\cdots\frac{g_l \mathrm{d}^3 p_l}{2(2\pi)^3 E_l}\times$$
$$\times|\mathcal{M}|^2\left[f_k f_l (1 \pm f_i)(1 \pm f_j) - f_i f_j (1 \pm f_k)(1 \pm f_l)\right]. \tag{4.55}$$

Decoupling from thermal equilibrium

Boltzmann Equation (see, Peter & Uzan Section 4.2.2.1)

In cosmologically interesting situations, $E - \mu \gg T$. Quantum effects can thus be neglected and $1 \pm f \simeq 1$.

$$\boxed{\dot{n}_i + 3Hn_i = \frac{g_i\cdots g_l}{(2\pi)^8}\int\frac{\mathrm{d}^3 p_i}{2E_i}\cdots\frac{\mathrm{d}^3 p_l}{2E_l}\delta^{(4)}(p_i + p_j - p_k - p_l)|\mathcal{M}|^2 (f_k f_l - f_i f_j).} \tag{4.56}$$

In this limit, the distribution functions are of the form $f \propto \exp[(\mu - E)/T]$ so that the particle density (4.11) can be expressed as a function of that at $\mu = 0$ as

$$n_i = e^{\mu_i/T}\bar{n}_i, \qquad \bar{n}_i \equiv n_i[\mu_i = 0]. \tag{4.57}$$

Furthermore, the conservation of energy implies that $E_k + E_l = E_i + E_j$ such that the term $f_k f_l - f_i f_j$ takes the form

$$e^{-(E_k+E_l)/T}\left[e^{(\mu_k+\mu_l)/T} - e^{(\mu_i+\mu_j)/T}\right] = e^{-(E_k+E_l)/T}\left(\frac{n_k n_l}{\bar{n}_k\bar{n}_l} - \frac{n_i n_j}{\bar{n}_i\bar{n}_j}\right).$$

The Boltzmann equation (4.56) can thus be written as

$$\dot{n}_i + 3Hn_i = -\langle\sigma v\rangle\left(n_i n_j - \frac{\bar{n}_i\bar{n}_j}{\bar{n}_k\bar{n}_l}n_k n_l\right), \tag{4.58}$$

where $\langle\sigma v\rangle$ is defined as

$$\bar{n}_i\bar{n}_j\langle\sigma v\rangle \equiv \int\frac{\mathrm{d}^3 p_i}{2E_i}\cdots\frac{\mathrm{d}^3 p_l}{2E_l}\delta^{(4)}(p_i + p_j - p_k - p_l)|\mathcal{M}|^2\frac{e^{-(E_i+E_j)/T}}{(2\pi)^8}. \tag{4.59}$$