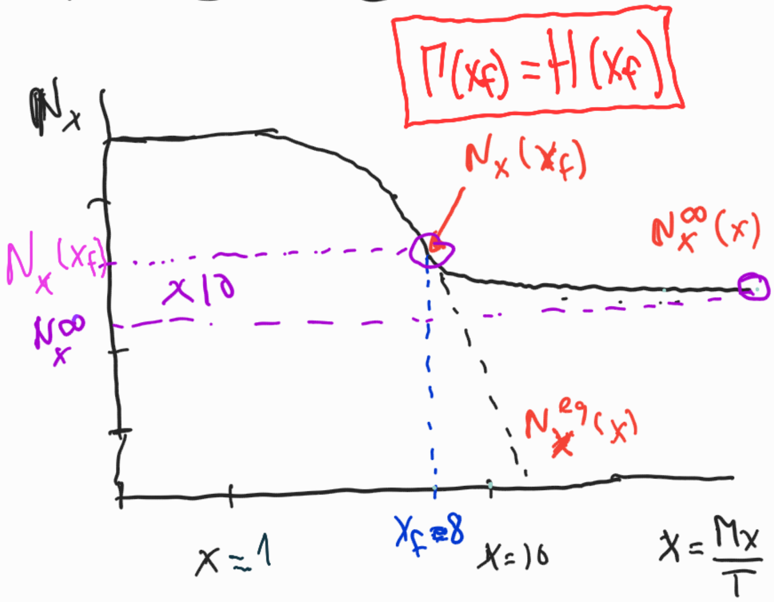
Ricatti Equation



$$\boxed{\frac{dN_x}{dx} = -\frac{\langle\sigma v\rangle s}{x\, H}\left[N_x^2 - \left(N_x^{eq}\right)^2\right]}$$

$$\Downarrow$$

$$\frac{dN_x}{dx} = -\frac{\lambda}{x^2}\left[N_x^2 - \underbrace{\overset{x \gg x_f\ ?\ 0}{\left(\cancel{N_x^{eq}}\right)^2}}_{N_x^{eq} \ll N_{x_1}}\right]$$

Lets assume the $X, \bar{X}$ decouple during the epoch of radiation domination, so:

$$H = \frac{\pi}{3}\left(\frac{g_*}{10}\right)^{1/2}\frac{T^2}{M_{Pl}} = \qquad \Bigg\}\ \text{but}\ x = \frac{M_x}{T} \Rightarrow T = \frac{M_x}{x}$$

$$= \frac{\pi}{3}\left(\frac{g_*}{10}\right)^{1/2}\left(\frac{M_x}{x}\right)^2\frac{1}{M_{Pl}} =$$

$$= H(M_x)\,\frac{1}{x^2} \quad \text{where} \quad H(M_x) = \frac{\pi}{3}\left(\frac{\overset{g_*(T \sim M_x)}{g_*}}{10}\right)^{1/2}\frac{M_x^2}{M_{Pl}}$$

Now a similar expression can be obtained for the entropy density

$$s = \frac{2\pi^2}{45}\, g_{*s}\, T^3 = \frac{2\pi^2}{45}\, g_{*s}\left(\frac{M_x}{x}\right)^3$$

So:

$$\frac{dN_x}{dx} \simeq - \frac{\langle \sigma v \rangle}{x} \frac{\frac{2\pi}{45} g_{*s} \left(\frac{M_x}{x}\right)}{H(M_x) \frac{1}{x^2}} \left[N_x^2 - \left(N_x^{eq}\right)^2 \right]$$

$$= - \overbrace{\frac{2\pi^2}{45} \frac{g_{*s}}{H(M_x)} \langle \sigma v \rangle M_x^3}^{\lambda} \frac{1}{x^2} \left[N_x^2 - \left(N_x^{eq}\right)^2 \right] =$$

$$= - \frac{\lambda}{x^2} \left[N_x^2 - \left(N_x^{eq}\right)^2 \right]$$

where

$$\lambda = \frac{2\pi^2}{45} \frac{g_{*s}}{H(M_x)} \langle \sigma v \rangle M_x^3 =$$

$$= \frac{2\pi^{\cancel{2}}}{45} \frac{g_{*s} M_x^{\cancel{3}} \langle \sigma v \rangle}{\cancel{\pi}/3 \left(\frac{g_*}{10}\right)^{1/2} \cancel{M_x^2}/M_{Pl}} =$$

$$= \frac{2\pi}{15} \left(\frac{10\, g_{*s}^2}{g_*} \right)^{1/2} \langle \sigma v \rangle M_x M_{Pl}$$

Away from mass thresholds λ can be treated as constant (including a high $x = M_x/T$).

At late times, ie $T \ll M_x \rightsquigarrow x \gg x_F$, so the Ricatti equation simplifies as:

$$\frac{dN_x}{dx} = - \frac{\lambda}{x^2} \left[N_x^2 - \cancelto{\sim 0}{{N_x^{eq}}^2} \right] \simeq - \frac{\lambda}{x^2} N_x^2$$

So integrating with respect to x:

$$dN_X = \frac{dN_X}{dx}\,dx =$$

$$= -\frac{\lambda}{x^2} N_X^2\,dx \Leftrightarrow$$

$$\Leftrightarrow \int_{N_X(x_f)}^{N_X(\infty)} \frac{dN_X}{N_X^2} = \int_{x_f}^{\infty} -\lambda \frac{dx}{x^2} \Leftrightarrow$$

assuming $\lambda \simeq$ constant

$$\Leftrightarrow \left[\frac{N_X^{-1}}{\cancel{-1}}\right]_{N_X(x_f)}^{\infty} = -\lambda \left[\frac{x^{-1}}{\cancel{-1}}\right]_{x_f}^{\infty} \Leftrightarrow$$

$$\frac{1}{N_X^{\infty}} - \frac{1}{N_X(x_f)} = -\lambda\left[\cancelto{0}{\frac{1}{\infty}} - \frac{1}{x_f}\right] \Leftrightarrow$$

$$\boxed{\frac{1}{N_X^{\infty}} - \cancelto{\sim 0}{\frac{1}{N_X(x_f)}} = +\frac{\lambda}{x_f}}$$

N_X^{∞} is typically 10 times smaller than $N_X(x_f)$

So one can neglect the term

$$\frac{1}{N_X(x_f)} << \frac{1}{N_X^{\infty}}$$

So:

$$\frac{1}{N_X^{\infty}} \simeq \frac{\lambda}{x_f} \Leftrightarrow \boxed{N_X^{\infty} \simeq \frac{x_f}{\lambda}}$$

This result gives a good way to estimate the freeze-out (relic) abundances of $X, \bar{X}$