WIMP Miracle

Let us now relate the freeze-out abundance of dark matter relic particles to the DM density today.

$$\Omega_{X,0} = \frac{\rho_{X,0}}{\rho_{crit,0}} \tag{1}$$

the critical density $\boxed{\rho_{crit,0} = 3H_0^2/8\pi G}$ can be writen as a function of the Planck Mass:

$$M_P = \sqrt{\frac{\hbar c}{8\pi G}} \xRightarrow[c=1]{\hbar=1} M_P^2 = \frac{1}{8\pi G} \tag{2}$$

as:

$$\rho_{crit,0} = \frac{3H_0^2}{1/M_P^2} = 3H_0^2 M_P^2 \tag{3}$$

Since the DM wimp particles are already decoupled

$$\rho_{X,0} = M_X\, n_{X,0} \tag{4}$$

combining (3) and (4) in (1) one has:

$N_{X,0}\, s_0$

$$\Omega_{X,0} = \frac{M_X\, n_{X,0}}{3H_0^2 M_P^2} \tag{5}$$

Since the wimp particles are already decoupled at present time $n_{X,0}$ and s_0 scale as a_0^{-3} and ... N^{∞} obtained with

$N_{x,0}$ should be equal to N_X^{∞} obtained with the Ricotti equation:

$$N_{X,0} \equiv N_X^{\infty} \simeq \frac{x_f}{\lambda} \qquad (6)$$

So:

$$\Omega_{x,0} = \frac{M_x N_{x,0} s_0}{3 H_0^2 M_P^2} = M_X N_X^{\infty} \frac{s_0}{3 H_0^2 M_P^2}$$

$$= M_X \left(\frac{x_f}{\lambda}\right) \frac{s_0}{3 H_0^2 M_P^2} \qquad (7)$$

Recal that:

$$\lambda = \frac{2\pi^2}{45} g_{*s}(M_X) \frac{M_X^3 \langle\sigma v\rangle}{H(M_X)} \qquad (8)$$

$$s_0 = \frac{2\pi^2}{45} g_{*s}(T_0)\, T_0^3 \qquad (9)$$

Using these in $\Omega_{x,0}$, gives:

$$\Omega_{x,0} = M_x x_f \left(\frac{45}{2\pi^2} \frac{H(M_x)}{g_{*s}(M_x) M_x^3 \langle\sigma v\rangle}\right) \times$$

s_0 $\qquad$ $\frac{1}{\lambda}$

$$\frac{\left(2\pi/45\; g_{*s}(T_0)\, T_0^3\right)}{3 H_0^2 M_p^2} \qquad (10)$$

After simplification gives:

$$\boxed{\Omega_{X,D} = \frac{H_1(M_X)}{M_X^2} \frac{x_f}{\langle \sigma v \rangle} \frac{g_{*s}(T_0)}{g_{*s}(M_X)} \frac{T_0^3}{3 H_0^2 M_P^2}} \quad (11)$$

We can further simplify this, recalling that

$$H(x) = \overbrace{\boxed{\frac{\pi}{3}\left(\frac{g_*(M_X)}{10}\right)^{1/2} \frac{M_X^2}{M_P}}}^{H(M_X)} \frac{1}{x^2} = H(M_X) \frac{1}{x^2} \quad (12)$$

Where

$$\boxed{H(M_X) = \frac{\pi}{3}\left(\frac{g_*(M_X)}{10}\right)^{1/2} \frac{M_X^2}{M_P}} \quad (13)$$

So:

$$\Omega_{X,D} = \frac{\frac{\pi}{3}\left(\frac{g_*(M_X)}{10}\right)^{1/2} \frac{\cancel{M_X^2}}{M_P}}{\cancel{M_X^2}} \frac{x_f}{\langle \sigma v \rangle} \frac{g_{*s}(T_0)}{g_{*s}(M_X)} \frac{T_0}{3 H_0^2 M_P^2} \Leftrightarrow$$

$$\boxed{\Omega_{X,D} = \frac{\pi}{9} \frac{x_f}{\langle \sigma v \rangle} \left(\frac{g_*(M_X)}{10}\right)^{1/2} \frac{g_{*s}(T_0)}{g_{*s}(M_X)} \frac{T_0}{M_P^2 H_0^2}} \quad (14)$$

This can be estimated from cosmological observations and our WIMP model hipothesis

- $H_0 = 100\, h\ \mathrm{km\, s^{-1} Mpc}$
- $T_0 \simeq 2.726\ \mathrm{K}$
- $g_{*s}(T_0) = 3.91$
- $x_f \simeq 10$
- $g_{*s}(M_x) \simeq g_*(M_x)$
- $h \simeq 0.7$
- $\Omega_{DM} \simeq 0.25$
- $\Omega_{DM} h^2 \simeq 0.1225$

After converting units into Natural units one can obtain an approximate expression of (14) that reads:

$$\Omega_x h^2 \simeq 0.1 \left(\frac{\overset{\sim 10}{x_f}}{10}\right)\left(\frac{10}{g_*(M_x)}\right)^{1/2} \frac{10^{-8}\ \mathrm{GeV}^{-2}}{\langle \sigma v \rangle} \quad (15)$$

But the weak interaction has typical thermally averaged cross-sections of the order

$$\langle \sigma v \rangle \sim 10^{-8}\ \mathrm{GeV}^{-2} \quad (16)$$

so the righ-hand-side of (15) matches the observational value of $\Omega_{DM} h^2 \simeq 0.1225$ If $g_*(T \sim M_x) \simeq 10$.

This calculation shows that our WIMP model hypothesis can explain the present abundance of Dark matter.

This is Know as the WIMP miracle.